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КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

Series
Theory of Functions
of a Complex Variable

методические рекомендации на английском языке
по дисциплине **«Математика»**

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Данные методические рекомендации адресованы преподавателям и студентам технических ВУЗов для проведения аудиторных занятий и организации самостоятельной работы студентов при изучении материала из рассматриваемых разделов. Методические рекомендации на английском языке «Series Theory of Functions of a Complex Variable» содержат необходимый материал по темам «Числовые и функциональные ряды», «Ряды Фурье», «Функции комплексной переменной», изучаемые студентами БрГТУ технических специальностей в курсе дисциплины «Математика». Теоретический материал сопровождается рассмотрением достаточного количества примеров и задач, при необходимости приводятся соответствующие иллюстрации. Для удобства пользования каждая тема разделена на три части: краткие теоретические сведения (определения, основные теоремы, формулы для расчетов); задания для аудиторной работы и задания для индивидуальной работы.

Данные методические рекомендации являются продолжением серии методических разработок на английском языке коллектива авторов [1]–[12]. Практика использования разработок данной серии показала целесообразность её применения в процессе обучения студентов не только технических, но и экономических специальностей. Также были получены положительные отзывы об упомянутой серии от иностранных студентов.

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CONTENT

I INFINITE SEQUENCES AND SERIES

1.1 Series. Test for Divergence

Infinite sequences and series were introduced briefly in *A Preview of Calculus* in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series.

Sequences

A *sequence* can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the n-th term. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}. Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

The sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ is also denoted by $\{a_n\}$.

Some sequences can be defined by giving a formula for the n-th term. There are three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and the third by writing out the terms of the sequence (the *Fibonacci sequence*). Notice that it does not have to start at 1.

Definition A sequence $\{a_n\}$ has the limit L and we write $\lim\limits_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence *converges* (or is *convergent*). Otherwise, we say the sequence *diverges* (or is *divergent*).

If we try to add the terms of an infinite sequence $\{a_n\}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \qquad (1)$$

which is called an *infinite series* (or just a *series*) and is denoted, for short, by the symbol $\sum\limits_{n=1}^{\infty} a_n$ or $\sum a_n$.

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the *partial sums*

$$S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3,$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \qquad (2)$$

These partial sums form a new sequence $\{S_n\}$, which may or may not have a limit. If $\lim\limits_{n\to\infty} S_n = S$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\sum a_n$.

Definition If the sequence $\{S_n\}$ is convergent and $\lim\limits_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called *convergent* and we write $S = a_1 + a_2 + a_3 + ... + a_n + ...$.

The number S is called the *sum* of the series. Otherwise, the series is called *divergent*. So when we write $\sum\limits_{n=1}^{\infty} a_n = S$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S.

Definition A series $a_{n+1} + a_{n+2} + a_{n+3} + ... = \sum\limits_{k=n+1}^{\infty} a_k$ is called the *remainder* of the series and denoted by R_n. The remainder R_n is the error made when the sum of the first *n* terms, is used as an approximation to the total sum.

Example 1 An important example of an infinite series is the *geometric series*

$$a + aq + aq^2 + aq^3 + aq^4 + ... + aq^{n-1} + ... = \sum_{n=1}^{\infty} aq^{n-1}.$$

Solution Each term is obtained from the preceding one by multiplying it by the common ratio q.

If $q = 1$, then $S_n = a + a + a + ... + a = na \to \pm\infty$. Since $\lim\limits_{n\to\infty} S_n$ does not exist, the geometric series diverges in this case.

If $q \neq 1$, we have

$$S_n = a + aq + aq^2 + aq^3 + aq^4 + ... + aq^{n-1}$$
$$qS_n = aq + aq^2 + aq^3 + aq^4 + ... + aq^n.$$

Subtracting these equations, we get $S_n - qS_n = a - aq^n$,

$$S_n = \frac{a(1-q^n)}{1-q} \tag{3}$$

If $-1 < q < 1$, we know from that $q^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \frac{a}{1-q}$$

Thus when $|q| < 1$ the geometric series is convergent and its sum is $S = \dfrac{a}{1-q}$.

If $|q| > 1$ or $q = -1$, the sequence $\{q^n\}$ is divergent and so, by Equation 3, $\lim\limits_{n\to\infty} S_n$ does not exist. Therefore the geometric series diverges in those cases.

We summarize the results of Example 1 as follows.

The geometric series is convergent if $|q|<1$ *and its sum is* $S=\dfrac{a}{1-q}$. *If* $|q|\ge 1$, *the geometric series is divergent.*

Example 2 Find the sum of the geometric series $\sum_{n=1}^{\infty}\left(\dfrac{3}{5}\right)^n$.

Solution The first term is $a_1=\dfrac{3}{5}$ and the common ratio is $q=\dfrac{3}{5}$. Since $q=\dfrac{3}{5}<1$, the series is convergent and its sum is $S=\dfrac{a}{1-q}=\dfrac{\frac{3}{5}}{1-\frac{3}{5}}=\dfrac{3}{2}$.

Example 3 Show that the series $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$ is convergent, and find its sum.

Solution This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}.$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Thus we have

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right);$$

$$S_n=\sum_{i=1}^{n}\frac{1}{i(i+1)}=1-\frac{1}{n+1}$$

and so

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1.$$

Therefore the given series is convergent and $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}=1$.

Example 4 Show that the *harmonic series* $\sum\dfrac{1}{n}=1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}+\ldots$ is divergent.

Solution For this particular series it is convenient to consider the partial sums $S_2,S_4,S_8,\ldots,S_{2^n},\ldots$ and show that they become large.

$S_2 = 1 + \frac{1}{2}$, $S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$,

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$,

$S_{16} > 1 + \frac{4}{2}$, $S_{32} > 1 + \frac{5}{2}$, $S_{64} > 1 + \frac{6}{2}$, … $S_{2^n} > 1 + \frac{n}{2}$.

This shows that $S_{2^n} \to \infty$ as $n \to \infty$ and so the harmonic series is divergent. Therefore, the harmonic series diverges.

In general, it is difficult to find the exact sum of a series. We develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum.

Theorem 1 If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Note 1 The converse of Theorem 1 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that a series is convergent. Observe that for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ we have $\lim_{n\to\infty} a_n = 0$, but we showed in Example 4 that it is divergent.

The test for divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5 Show that the series $\sum_{n=1}^{\infty} \frac{2n^2+3}{n^2+5}$ diverges.

Solution $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2+3}{n^2+5} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 \neq 0$. So the series diverges by the Test for Divergence.

Example 6 Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \left(\frac{3}{5}\right)^n \right)$.

Solution The series $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ is a geometric series with $a_1 = \frac{3}{5}$ and $q = \frac{3}{5}$, so $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{3}{2}$. In Example 3 we found that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

So, the given series is convergent and

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\left(\frac{3}{5}\right)^n\right)=3\sum\frac{1}{n(n+1)}+\sum\left(\frac{3}{5}\right)^n=3\cdot 1+\frac{3}{2}=\frac{9}{2}.$$

Exercise Set 1.1

In Exercise 1 to 6, write down first five terms using a_n:

1. $a_n=\dfrac{3n}{2n^3+1}$ *2.* $a_n=\dfrac{2+(-1)^n}{n!}$

3. $a_n=\dfrac{n+1}{n!}$ *4.* $a_n=\dfrac{\left(2+\sin\frac{n\pi}{2}\right)\cos n\pi}{n!}$

5. $a_n=\dfrac{\ln(n+1)}{n^2}$ *6.* $a_n=\dfrac{1}{\left(3+(-1)^n\right)^n}$

In Exercise 7 to 12, find a_n of the series:

7. $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+\ldots$ *8.* $1-\frac{2}{2!}+\frac{4}{3!}-\frac{8}{4!}+\ldots$

9. $\frac{\ln 2}{4}+\frac{\ln 3}{9}+\frac{\ln 4}{16}+\frac{\ln 5}{25}+\ldots$ *10.* $arctg\frac{1}{2}+arctg\frac{1}{8}+arctg\frac{1}{18}+arctg\frac{1}{32}+\ldots$

11. $\frac{\sin\frac{\pi}{2}}{\sqrt{2}}+\frac{\sin\frac{\pi}{4}}{\sqrt{4}}+\frac{\sin\frac{\pi}{6}}{\sqrt{6}}+\frac{\sin\frac{\pi}{8}}{\sqrt{8}}+\ldots$ *12.* $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\frac{8}{14}+\ldots$

In Exercise 13 to 27, determine whether the series $\sum a_n$ is convergent or divergent. If it is convergent, find its sum

13. $\sum_{n=1}^{\infty}\frac{2n+3}{n+5}$ *14.* $\sum_{n=1}^{\infty}\frac{3n^2-1}{2n^2+5}$ *15.* $\sum_{n=1}^{\infty}\frac{2n^3+6}{5n^3+5n}$

16. $\sum_{n=1}^{\infty}\frac{n^3+3}{n^2+9}$ *17.* $\sum_{n=1}^{\infty}\frac{2n+3}{4n+5}$ *18.* $\sum_{n=1}^{\infty}\frac{n^2+3n-1}{n^2+5n+9}$

19. $\sum_{n=1}^{\infty}\left(\frac{7}{5}\right)^n$ *20.* $\sum_{n=1}^{\infty}\left(\frac{3}{25}\right)^n$ *21.* $\sum_{n=1}^{\infty}4\left(\frac{13}{5}\right)^n$

22. $\sum_{n=1}^{\infty}\frac{5^n+2^n}{10^n}$ 23. $\sum_{n=1}^{\infty}\frac{3^n+4^n}{12^n}$ 24. $\sum_{n=1}^{\infty}\frac{5^n-15^n}{25^n}$

25. $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ 26. $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$ 27. $\sum_{n=1}^{\infty}\frac{5}{n(n+4)}$

Individual Tasks 1.1

1-5. Determine whether the series $\sum a_n$ is convergent or divergent. If it is convergent, find its sum.

I.	II.
1. $\sum_{n=1}^{\infty}\frac{n+1}{2n+3}$	*1.* $\sum_{n=1}^{\infty}\frac{7n+1}{4n+3}$
2. $\sum_{n=1}^{\infty}\frac{4n^4}{3n+2n^3}$	*2.* $\sum_{n=1}^{\infty}\frac{n^3}{5+n^2}$
3. $\sum_{n=1}^{\infty}\frac{5^n+3^n}{15^n}$	*3.* $\sum_{n=1}^{\infty}\frac{7^n+3^n}{21^n}$
4. $\sum_{n=1}^{\infty}\frac{1}{(6n-1)(6n+5)}$	*4.* $\sum_{n=1}^{\infty}\frac{1}{(2n+3)(2n+5)}$
5. $\sum_{n=1}^{\infty}\frac{1}{3n+n^2}$	*5.* $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$

1.2 Tests of Convergence of Positive Series

Theorem (Integral Test) Suppose f is a continuous, positive, decreasing function on $[1,+\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(b) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Example 1 For what values of α is the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}=1+\frac{1}{2^{\alpha}}+\frac{1}{3^{\alpha}}+...+\frac{1}{n^{\alpha}}+...$ convergent?

Solution If $\alpha<0$, then $\lim_{n\to\infty}\frac{1}{n^{\alpha}}=\infty$. If $\alpha=0$, then $\lim_{n\to\infty}\frac{1}{n^{\alpha}}=1$. In either case $\lim_{n\to\infty}\frac{1}{n^{\alpha}}\neq 0$, so the given series diverges by the Test for Divergence.

If $\alpha<0$, then the function $y=\frac{1}{x^{\alpha}}$ is clearly continuous, positive, and decreasing on $\left[1,+\infty\right)$. We found that $\int_1^{\infty}\frac{1}{x^{\alpha}}dx$ converges if $\alpha>1$ and diverges if $\alpha\le 1$.

It follows from the Integral Test that the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ *converges if* $\alpha>1$ *and diverges if* $\alpha\le 1$.

Example 2 Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^4(n+1)}$ converges or diverges.

Solution The function $f(x)=\frac{1}{(x+1)\ln^4(x+1)}$ is positive and continuous for $x\ge 1$ because the logarithm function is continuous.

So we can apply the Integral Test:

$$\int_1^{\infty}f(x)\,dx=\int_1^{\infty}\frac{dx}{(x+1)\ \ln^4(x+1)}dx=\int_1^{\infty}\frac{d(\ln(x+1))}{\ln^4(x+1)}dx=\left|\begin{array}{l}\ln(x+1)=t\\ d(\ln(x+1))=dt\\ x=1\Rightarrow t=\ln 2\\ x=\infty\Rightarrow t=\infty\end{array}\right|=\int_{\ln 2}^{+\infty}\frac{dt}{t^4}=$$

$$=\lim_{N\to\infty}\int_{\ln 2}^{N}t^{-4}dt=\lim_{N\to\infty}\left.\left(\frac{t^{-3}}{-3}\right)\right|_{\ln 2}^{N}=\lim_{N\to\infty}\left(-\frac{1}{3N^3}+\frac{1}{3\ln^3 2}\right)=0+\frac{1}{3\ln^3 2}=\frac{1}{3\ln^3 2}=1{,}001$$

.

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^4(n+1)}$ is also convergent by the Integral Test.

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent.

Theorem (Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms:

(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, the $\sum a_n$ is also convergent;

(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Example 3 Determine whether the series $\sum \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution The largest of the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum \frac{5}{2n^2}$.

Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know that $\sum \frac{5}{2n^2} = \frac{5}{2}\sum \frac{1}{n^2}$ is convergent because it is a constant times α-series with $\alpha = 2 > 1$. Therefore $\sum \frac{5}{2n^2+4n+3}$ is convergent by part (a) of the Comparison Test.

Theorem (Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ is a finite number, then either both series converge or both diverge.

Example 4 Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{3n^2+4n+7}{n^5+6}$ converges or diverges.

Solution The dominant part of the numerator is $3n^2$ and the dominant part of the denominator is n^5. This suggests taking

$$a_n = \frac{3n^2+4n+7}{n^5+6}, \qquad b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^2+4n+7)n^3}{n^5+6} = \lim_{n\to\infty} \frac{3 + \frac{4}{n^2} + \frac{7}{n^3}}{1 + \frac{6}{n^5}} = 3.$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ is convergent α-series with $\alpha = 3 > 1$, the given series converges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series by keeping only the highest powers in the numerator and denominator.

Example 5 Use the sum of the first 100 terms to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$. Estimate the error involved in this approximation.

Solution Since

$$\frac{1}{n^3+1}<\frac{1}{n^3},$$

the given series is convergent by the Comparison Test. There we found that

$$T_n \le \int_n^{+\infty}\frac{1}{x^3}dx=\frac{1}{2n^2}$$

Therefore, the remainder R_n for the given series satisfies $R_n \le T_n \le \frac{1}{2n^2}$. With $n=100$ we have $R_{100} \le \frac{1}{2\cdot 100^2}=0.00005$.

The following tests are very useful in determining whether a given series is convergent.

Theorem (Ratio Test)

(a) If $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=A<1$, then the series $\sum a_n$ is convergent.

(b) If $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=A>1$ or $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=A=\infty$, then the series $\sum a_n$ is divergent.

(c) If $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=A=1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty}a_n$.

Example 6 Determine whether the series $\sum_{n=1}^{\infty}\frac{3^n}{(n^2+1)\cdot n!}$ converges or diverges.

Solution We use the Ratio Test with

$$a_n=\frac{3^n}{(n^2+1)\cdot n!},\ a_{n+1}=\frac{3^{n+1}}{((n+1)^2+1)\cdot(n+1)!}=\frac{3^{n+1}}{(n^2+2n+2)\cdot(n+1)!},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}\cdot(n^2+1)\cdot n!}{(n^2+2n+2)\cdot n!(n+1)\cdot 3^n}=\lim_{n\to\infty}\frac{3\cdot(n^2+1)}{(n^2+2n+2)(n+1)}=$$

$$=3\cdot\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{n^3}}{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)}=3\cdot 0=0.$$

Since $A = 0 < 1$, the given series is convergent by the Ratio Test.

The following test is convenient to apply when n-th powers occur. Its proof is similar to the proof of the Ratio Test.

Theorem (Root Test)

(a) If $\lim_{n\to\infty} \sqrt[n]{a_n} = A < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\lim_{n\to\infty} \sqrt[n]{a_n} = A > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = A = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty} \sqrt[n]{a_n} = A = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty} \sqrt[n]{a_n} = A = 1$, then part (c) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge. (If $A = 1$ in the Ratio Test, do not try the Root Test because A will again be 1. And if $A = 1$ in the Root Test, do not try the Ratio Test because it will fail too.)

***Example** 7* Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Solution We use the Root Test with $a_n = \left(\frac{2n+3}{3n+2}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{2n}{3n} = \frac{2}{3}$$

Since $A = \frac{2}{3} < 1$, the given series is convergent by the Root Test.

Exercise Set 1.2

In Exercise 1 to 42, determine whether the series $\sum a_n$ is convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{3n-1}{n^2+1}$ *2.* $\sum_{n=1}^{\infty} \frac{n+1}{3n^2}$ *3.* $\sum_{n=1}^{\infty} \frac{n}{(n+1)^3}$

4. $\sum_{n=1}^{\infty} \frac{n+4}{\sqrt{n^3+1}}$ *5.* $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ *6.* $\sum_{n=1}^{\infty} \frac{n+4}{\sqrt{n^2+4n}}$

7. $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^n$ *8.* $\sum_{n=1}^{\infty} \left(\frac{4n}{4n+2}\right)^{2n}$ *9.* $\sum_{n=1}^{\infty} \left(\frac{5n}{5n+3}\right)^n$

10. $\sum_{n=1}^{\infty}\left(\frac{2n-1}{5n+3}\right)^{n}$ 11. $\sum_{n=1}^{\infty}\left(\frac{3n-1}{7n+2}\right)^{n}$ 12. $\sum_{n=1}^{\infty}\left(\frac{4n-1}{9n+5}\right)^{n}$

13. $\sum_{n=1}^{\infty}n^{6}\left(\frac{8n+1}{9n+5}\right)^{n}$ 14. $\sum_{n=1}^{\infty}\left(\frac{2n^{2}-3}{2n^{2}+1}\right)^{n^{2}}$ 15. $\sum_{n=1}^{\infty}\arcsin^{n}\frac{3}{n}$

16. $\sum_{n=1}^{\infty}\frac{2^{n}}{n^{4}}$ 17. $\sum_{n=1}^{\infty}\frac{(n+1)^{3}}{4^{n}}$ 18. $\sum_{n=1}^{\infty}\frac{3^{n-1}}{n(n+1)}$

19. $\sum_{n=1}^{\infty}\frac{n\cdot 5^{n}}{(n+1)^{3}}$ 20. $\sum_{n=1}^{\infty}\frac{n\cdot 3^{n-1}}{(n+1)^{3}}$ 21. $\sum_{n=1}^{\infty}\frac{n(n+1)}{5^{n}}$

22. $\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{n!}$ 23. $\sum_{n=1}^{\infty}\frac{n\cdot 2^{n-1}}{(n+1)!}$ 24. $\sum_{n=1}^{\infty}\frac{n!}{(n+1)\cdot 3^{n}}$

25. $\sum_{n=1}^{\infty} tg\frac{\pi}{2^{n+1}}$ 26. $\sum_{n=1}^{\infty}2^{n}\cdot\sin\frac{1}{6^{n}}$ 27. $\sum_{n=1}^{\infty}\arcsin\frac{\pi}{2^{n}}$

28. $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{4\cdot 8\cdot 12\cdot\ldots(4n)}$ 29. $\sum_{n=1}^{\infty}\frac{3\cdot 5\cdot\ldots\cdot(2n+1)}{1\cdot 4\cdot\ldots\cdot(3n-2)}$ 30. $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{5\cdot 9\cdot 13\cdot\ldots(4n+1)}$

31. $\sum_{n=1}^{\infty}\frac{n^{n}}{n!\,(2^{n}+1)}$ 32. $\sum_{n=1}^{\infty}n^{n}\cdot tg^{n}\frac{1}{3n}$ 33. $\sum_{n=1}^{\infty}\frac{2^{n}\cdot n!}{n^{3}}$

34. $\sum_{n=1}^{\infty}\left(\frac{3n}{n+\sqrt{n}+5}\right)$ 35. $\sum_{n=1}^{\infty}\frac{1}{5^{n}}\left(\frac{n+1}{n}\right)^{n^{2}}$ 36. $\sum_{n=1}^{\infty}arctg^{n}\frac{n}{3}$

37. $\sum_{n=1}^{\infty}\arcsin^{n}\frac{3}{n}$ 38. $\sum_{n=1}^{\infty}\frac{n!}{n^{\sqrt{n}}}$ 39. $\sum_{n=1}^{\infty}\frac{n!a^{n}}{n^{n}},\ a\geq 0$

40. $\sum_{n=1}^{\infty}\frac{1}{(n+1)\cdot\ln(n+1)}$ 41. $\sum_{n=1}^{\infty}\frac{1}{(7n-1)\ln(7n-1)}$ 42. $\sum_{n=1}^{\infty}\frac{1}{\ln^{n}(n+1)}$

Individual Tasks 1.2

1-6. Determine whether the series $\sum a_n$ is convergent or divergent. If it is convergent, find its sum.

I.	II.
1. $\sum_{n=1}^{\infty}\frac{1}{(3n+1)n}$	*1.* $\sum_{n=1}^{\infty}\frac{n}{(n+5)^2}$
2. $\sum_{n=1}^{\infty}\frac{2n+1}{n\sqrt{n}+3}$	*2.* $\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sqrt{2n-1}}$
3. $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^2(n+1)}$	*3.* $\sum_{n=1}^{\infty}\frac{1}{(2n+1)\ln^3(2n+1)}$
4. $\sum_{n=1}^{\infty}\frac{3n+2}{(n+1)!}$	*4.* $\sum_{n=1}^{\infty}\frac{5^n\sqrt{n+1}}{(n+2)!}$
5. $\sum_{n=1}^{\infty}\frac{5^n(n+1)!}{(2n)!}$	*5.* $\sum_{n=1}^{\infty}\frac{n!(2n+1)!}{(3n)!}$
6. $\sum_{n=1}^{\infty}\left(\frac{4n-1}{4n+2}\right)^{n^2-n}$	*6.* $\sum_{n=1}^{\infty}\left(\frac{3n-2}{3n+2}\right)^{n^2+n}$

1.3 Alternating Series

The convergence tests that we have looked at so far are applied only to series with positive terms. In this section we learn how to deal with the series whose terms are not necessarily positive. The *alternating series,* whose terms alternate in sign are of particular importance.

Definition An *alternating series* is a series whose terms are alternately positive and negative.

Theorem (Alternating Series Test) If the alternating series

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1}a_n + \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

satisfies the following conditions:

(a) $a_1 > a_2 > a_3 > \ldots > a_n > a_{n+1} > \ldots$

(b) $\lim_{n\to\infty} a_n = 0$

then the series is convergent.

Example 1 Test the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ for convergence or divergence.

Solution The alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n-1}\frac{1}{n}+...$$

satisfies the following conditions:

(a) $a_1 > a_2 > a_3 > ... > a_n > a_{n+1} > ...$, because $\frac{1}{n+1} < \frac{1}{n}$;

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0$.

So the series is convergent by the Alternating Series Test.

Example 2 Test the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+2}$ for convergence or divergence.

Solution The given series is alternating, so we try to verify conditions (a) and (b) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $a_n = \frac{n}{n^2+2}$ is decreasing. However, if we consider the related function $f(x) = \frac{x}{x^2+2}$, we find that

$$f'(x) = \frac{2-x^2}{(x^2+2)^2}.$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $x > \sqrt{2}$. Thus f is decreasing on the interval $(\sqrt{2};+\infty)$. This means that $f(n+1) < f(n)$ and, therefore, $a_n > a_{n+1}$ when $n \geq 2$. (The inequality $a_1 > a_2$ can be verified directly but all that really matters is that the sequence $\{a_n\}$ is eventually decreasing.)

Condition (b) is readily verified:

$$\lim_{n\to\infty}\frac{n}{n^2+2} = \lim_{n\to\infty}\frac{n}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Thus the given series is convergent by the Alternating Series Test.

Estimating Sums

A partial sum S_n of any convergent series can be used as an approximation to the total sum S, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $S \approx S_n$ is the remainder $R_n = S - S_n$.

Theorem (Alternating Series Estimation Theorem) If $S = \sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$ is the sum of an alternating series that satisfies the following conditions:

(a) $a_1 > a_2 > a_3 > ... > a_n > a_{n+1} > ...$

(b) $\lim\limits_{n\to\infty} a_n = 0$

then $|S - S_n| = |R_n| \leq a_{n+1}$.

Example 3 Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}$ correct to three decimal places.

Solution We first observe that the series is convergent by the Alternating Series Test because

a) $\frac{1}{(n+1)!}=\frac{1}{n!(n+1)}<\frac{1}{n!}$

b) $0<\frac{1}{n!}<\frac{1}{n}\to 0 \quad so \quad \frac{1}{n!}\to 0 \ as\ n\to\infty$.

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$S=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+...=1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}-\frac{1}{5040}+...$$

Notice that $a_7=\frac{1}{5040}<\frac{1}{5000}=0.0002$

and $S=1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\approx 0.368056$.

By the Alternating Series Estimation Theorem we know that $|S-S_6|\le a_7<0.0002$.

This error of less than $0,0002$ does not affect the third decimal place, so we have $S\approx 0.368$ correct to three decimal places.

Example 4 Find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4n+1}{6^n}$ correct to three decimal places.

Solution We first observe that the series is convergent by the Alternating Series Test

(a) If we consider the related function $f(x)=\frac{4x+1}{6^x}$, we find that

$$f'(x)=\frac{4\cdot 6^x-6^x\ln 6\cdot(4x-1)}{6^{2x}}=\frac{4-\ln 6\cdot(4x-1)}{6^x}.$$

Since we are considering only positive x, we see that $f'(x)<0$ if $x>\frac{1}{4}+\frac{1}{\ln 6}\approx 0.808$. Thus f is decreasing on the interval $[1;+\infty)$. This means that $f(n+1)<f(n)$ and therefore $a_n>a_{n+1}$ when $n\ge 1$.

(b) $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{4n+1}{6^n}=[L'Hopitals\ Rule]=\lim_{n\to\infty}\frac{4}{6^n\ln 6}=0$.

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series: $S=\frac{5}{6}-\frac{9}{36}+\frac{13}{216}-\frac{17}{6^4}+\frac{21}{6^5}-\frac{25}{6^6}+\cdots$.

Notice that $a_6 = \frac{25}{6^6} = 0.0005358 < 0.001$ and

$$S \approx S_5 = \frac{5}{6} - \frac{9}{36} + \frac{13}{216} - \frac{17}{1296} + \frac{21}{7776} =$$

$$= 0.8333 - 0.2500 + 0.0602 - 0.0131 + 0.0003 = 0.6307 \approx 0.631$$

Absolute Convergence

Definition A series $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Example 5 Test the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+3)}$ for an absolute convergence.

Solution We use the Limit Comparison Test with $a_n = \frac{1}{n(n+3)}$, where a_n is an absolute value of the $n-$th term. The dominant part of the numerator is 0 and the dominant part of the denominator is n^2. This suggests taking

$$a_n = \frac{1}{n(n+3)} \text{ and } b_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty}\left(\frac{1}{n(n+3)} : \frac{1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n(n+3)} \cdot \frac{n^2}{1}\right) = \lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{3}{n}\right)} = \lim_{n\to\infty}\frac{1}{1+\frac{3}{n}} = 1\cdot$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (α -series with $\alpha = 2 > 1$), the given series converges by the Limit Comparison Test. Thus, the given series is absolutely convergent and, therefore, convergent.

Definition A series $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ is called *conditionally convergent* if it is convergent, but not absolutely convergent.

Theorem If a series $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ is absolutely convergent, then it is convergent.

Example 6 Determine whether the series

$$\sum_{n=1}^{\infty}\frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \ldots + \frac{\cos n}{n^2} + \ldots$$

is convergent or divergent.

Solution This series has both positive and negative terms, but it is not alternating. The first term is positive, the next three are negative, and the following three are positive. The signs change irregularly. We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$$

Since $|\cos n|\le 1$ for all n, we have $\dfrac{|\cos n|}{n^2}\le\dfrac{1}{n^2}$.

We know that $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is convergent (α-series with $\alpha=2>1$) and therefore $\sum\limits_{n=1}^{\infty}\dfrac{|\cos n|}{n^2}$ is convergent by the Comparison Test. Thus the given series $\sum\limits_{n=1}^{\infty}\dfrac{\cos n}{n^2}$ is absolutely convergent and therefore convergent by Theorem.

***Example** 7* Determine whether the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+2}$ is absolutely convergent, conditionally convergent or divergent.

Solution We use the Limit Comparison Test with $a_n=\dfrac{n}{n^2+2}$.

The dominant part of the numerator is n and the dominant part of the denominator is n^2. This suggests taking

$$a_n=\frac{n}{n^2+2}\ \text{ and }\ b_n=\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{n^2+1}:\frac{1}{n}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=\lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=1.$$

Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ is divergent (α-series with $\alpha=1\le1$), the given series diverges by the Limit Comparison Test.

We try to verify conditions (a) and (b) of the Alternating Series Test:

(a) $a_1>a_2>a_3>a_4>a_5>a_6>\ldots$ (see Example 2);

(b) $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\dfrac{n}{n^2+2}=\lim\limits_{n\to\infty}\dfrac{n}{n^2\left(1+\frac{2}{n^2}\right)}=\lim\limits_{n\to\infty}\dfrac{1}{n\cdot\left(1+\frac{2}{n^2}\right)}=0$.

Exercise Set 1.3

In Exercise 1 to 18, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n+1}$ *2.* $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2n^2+1}$ *3.* $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\sqrt{n+1}}$

4. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}3^n}{n+2}$ 5. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{4^n(2n+3)}$ 6. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2(n+1)}$

7. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}5^n}{(n+2)^2}$ 8. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n\sqrt{n+1}}$ 9. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(2n-1)}$

10. $\sum\limits_{n=1}^{\infty}(-1)^n\cdot\sin\frac{\pi}{2^n}$ 11. $\sum\limits_{n=1}^{\infty}(-1)^n\cdot\frac{\sin 3^n}{3^n}$ 12. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n\cdot tg\frac{\pi}{4\sqrt{n}}}{\sqrt{5n-1}}$

13. $\sum\limits_{n=3}^{\infty}\frac{(-1)^n}{n\cdot\ln(2n)}$ 14. $\sum\limits_{n=3}^{\infty}\frac{(-1)^n}{(n+1)\cdot\ln n}$ 15. $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{2n+1}\right)^n$

16. $\sum\limits_{n=1}^{\infty}\frac{\cos 2\pi\alpha}{n^2+1}$ 17. $\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n-\ln n}$ 18. $\sum\limits_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$

In Exercise 19 to 24, approximate the sum of the series correct to three decimal places.

19. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(0,2)^n}{(n+1)\cdot(4n+3)}$ 20. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2^n\cdot n^2}$ 21. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n\cdot n^3}$

22. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot n\cdot 3^n}{(2n+1)\cdot 7^n}$ 23. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n^3+}{7^n}$ 24. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{(0,7)^n}{(n+1)!}$

Individual Tasks 1.3

1-3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

4. Approximate the sum of the series correct to three decimal places.

I.		**II.**	
1.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n}{6n^2+7}$	1.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n^6+3}$
2.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4n+1}{6^n}$	2.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{3n+2}{5^n}$
3.	$\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{3^n\cdot n!}$	3.	$\sum\limits_{n=1}^{\infty}\frac{(-1)^n\sqrt{n+1}}{(2n)!}$
4.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{3n+2}{n!}$	4.	$\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n+6}{n!}$

1.4 Power Series

Definition A ***power series*** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \tag{1}$$

where x is a variable and the a_n's are constants called the *coefficients* of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The *sum* of the series is a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

More generally, a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n + \ldots \tag{2}$$

is called a *power series in* $(x - x_0)$ or a *power series centered at* x_0 or a *power series about* x_0. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - x_0)^n = 1$, even when $x = x_0$. Notice also that when $x = x_0$, all of the terms are 0 for $n \ge 1$, and so the power series (2) always converges when $x = x_0$.

Example 1 For what values of x is the series $\sum_{n=1}^{\infty} \frac{3n+2}{(n+1)\cdot 7^n} \cdot (x+4)^n$ convergent?

Solution We use the Ratio Test. We apply the Ratio Test for the absolute value of the n –th trrm of the series $u_n(x)$: $|u_n(x)| = \frac{3n+2}{(n+1)\cdot 7^n} \cdot |x+4|^n$.

If $x \ne -4$, we have $|u_{n+1}(x)| = \frac{3n+5}{(n+2)\cdot 7^{n+1}} \cdot |x+4|^{n+1}$:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \frac{(3n+5)\cdot |x+4|^{n+1} \cdot (n+1)\cdot 7^n}{(n+2)\cdot 7^{n+1} \cdot (3n+2)\cdot |x+4|^n} = \frac{|x+4|}{7} \cdot \lim_{n\to\infty} \frac{(3n+5)(n+1)}{(n+2)(3n+2)} =$$

$$= \frac{|x+4|}{7} \lim_{n\to\infty} \frac{3n^2}{3n^2} = \frac{|x+4|}{7}.$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $\frac{|x+4|}{7}<1$ and divergent when $\frac{|x+4|}{7}>1$. Now

$$\frac{|x+4|}{7}<1 \Leftrightarrow |x+4|<7 \Leftrightarrow -7<x+4<7 \Leftrightarrow -11<x<3,$$

so the series converges when $x\in(-11;3)$ and diverges when $x\in(-\infty;-11)\cup(3;+\infty)$.

The Ratio Test gives no information when $\frac{|x+4|}{7}=1$ so we must consider $x=-11$ and $x=3$ separately.

If we put $x=3$ in the series, it becomes $\sum_{n=1}^{\infty}\frac{(3n+2)\cdot(3+4)^n}{(n+1)\cdot 7^n}=\sum_{n=1}^{\infty}\frac{3n+2}{n+1}$, which is divergent by the test for divergence.

If $x=-11$, the series is

$$\sum_{n=1}^{\infty}\frac{(3n+2)\cdot(-11+4)^n}{(n+1)\cdot 7^n}=\sum_{n=1}^{\infty}\frac{(3n+2)}{(n+1)}\cdot(-\frac{7}{7})^n=\sum_{n=1}^{\infty}(-1)^n\frac{3n+2}{n+1},$$

which diverges by the Alternating Series Test ($\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{3n+2}{n+1}=3\neq 0$). Thus the given power series converges for $-11<x<3$.

Theorem For a given power series $\sum_{n=0}^{\infty}a_n(x-x_0)^n$, there are only three possibilities:

(a) The series converges only when $x=x_0$;

(b) The series converges for all x;

(c) There is a positive number R such that the series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

The number in case (c) is called the *radius of convergence* of the power series. By convention, the radius of convergence is $R=0$ in case (a) and $R=\infty$ in case (b). The *interval of convergence* of a power series is the interval that consists of all values of x for which the series converges. In case (a) the interval consists of just a single point x_0. In case (b) the interval is $(-\infty;+\infty)$. In case (c) note that the inequality $|x-x_0|<R$ can be rewritten as $x_0-R<x<x_0+R$.

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Note The following formulas to finding radius of convergence can be used:

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|,\ R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}}$$

Example 2 Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2\cdot 3^{n-1}}$$

Solution Let $a_n=\dfrac{1}{n^2\cdot 3^{n-1}}$. Then $a_{n+1}=\dfrac{1}{(n+1)^2\cdot 3^n}$:

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2\cdot 3^n}{n^2\cdot 3^{n-1}}\right|=3\cdot\lim_{n\to\infty}\frac{(n+1)^2}{n^2}=3$$

So it converges if $|x|<3$ and diverges if $|x|>3$. Thus the radius of convergence is $R=3$.

The inequality $|x|<3$ can be written as $-3<x<3$, so we test the series at the endpoints $x=-3$ and $x=3$. When $x=3$, the series is $\sum_{n=1}^{\infty}\frac{3}{n^2}=3\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (α-series with $\alpha=2>1$), the given series converges by the Limit Comparison Test.

When $x=-3$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 3}{n^2}=3\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$,which is absolutely convergent and therefore convergent.

Thus the series converges only when $-3\le x\le 3$, so the interval of convergence is $x\in[-3;3]$.

Exercise Set 1.4

In Exercise 1 to 18, find the interval of convergence of the series.

1. $\sum_{n=1}^{\infty}\frac{(2n-1)^n}{2^n\cdot n^n}(x+1)^n$ *2.* $\sum_{n=1}^{\infty}\frac{(x+6)^n}{\sqrt{n^2+1}}$ *3.* $\sum_{n=1}^{\infty}\frac{(x-3)^n}{5^n\cdot(n+1)}$

4. $\sum_{n=1}^{\infty}(n+2)(x+3)^n$ *5.* $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n(2n+3)}$ *6.* $\sum_{n=1}^{\infty}\frac{n!}{6^n}(x-6)^{2n}$

7. $\sum_{n=1}^{\infty}\frac{4^n}{n!}(x+1)^n$ 8. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot(x-3)^n}{3^n\cdot n^n}$ 9. $\sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{9^n\cdot n^n}$

10. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}(x+1)^n$ 11. $\sum_{n=1}^{\infty}\frac{2^n\cdot(x-1)^n}{n(n+1)}$ 12. $\sum_{n=1}^{\infty}\frac{2^n\cdot(x+2)^n}{\sqrt{n+2}}$

13. $\sum_{n=1}^{\infty}(-1)^n n^{-x}$ 14. $\sum_{n=1}^{\infty}\frac{\sin nx}{e^{nx}}$ 15. $\sum_{n=1}^{\infty}\frac{n!}{x^n}$

16. $\sum_{n=1}^{\infty}\frac{ln^n x}{n}$ 17. $\sum_{n=1}^{\infty}\left(\frac{x-2}{1-2x}\right)^n$ 18. $\sum_{n=1}^{\infty}\frac{n\cdot 3^n}{(x+2)^n}$

Individual Tasks 1.4

1-4. Find the interval of convergence of the given series.

I.	II.
1. $\sum_{n=1}^{\infty}\frac{n+1}{4^n}x^n$	1. $\sum_{n=1}^{\infty}\frac{2n-1}{3^n(n+1)}\cdot x^n$
2. $\sum_{n=1}^{\infty}\frac{n\cdot(x-3)^n}{(6n+1)^2}$	2. $\sum_{n=1}^{\infty}\frac{n}{4n^2-3}(x-2)^n$
3. $\sum_{n=1}^{\infty}\frac{n+4}{n(n^3+1)}(x+2)^n$	3. $\sum_{n=1}^{\infty}\frac{(2n+1)}{n^3+4}(x+4)^n$
4. $\sum_{n=1}^{\infty}2^n\cdot\sin\frac{x}{3^n}$	4. $\sum_{n=1}^{\infty}\left(\frac{x-3}{1-3x}\right)^n$

1.5 Representations of Functions as Power Series

We start with an equation:

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^{n-1}+\ldots=\sum_{n=0}^{\infty}x^n,\quad -1<x<1 \tag{1}$$

We now regard Equation 1 as expressing the function as a sum of a power series.

Example 1 Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1+(-x^2)+(-x^2)^2+(-x^2)^3+\ldots=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Because this is a geometric series, it converges when $\left|-x^2\right|<1$, that is, $x^2<1$, or $-1<x<1$. Therefore the interval of convergence is $(-1;1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

Example 2 Find a power series representation for $\dfrac{1}{x+2}$.

Solution In order to put this function in the form of the left side of Equation 1 we first factor 2 from the denominator:

$$\frac{1}{x+2}=\frac{1}{2\cdot\left(1+x/2\right)}=\frac{1}{2}\cdot\frac{1}{1-\left(-x/2\right)}=\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}\frac{(-1)^n x^n}{2^{n+1}}\cdot$$

This series converges when $\left|-\dfrac{x}{2}\right|<1$, that is, $|x|<2$. So the interval of convergence is $(-2;2)$.

Example 3 Find a power series representation for $\ln\left(1-x\right)$ and its radius of convergence.

Solution We notice that, except for a factor of -1, the derivative of this function is $1/\left(1-x\right)$. So we integrate both sides of Equation (1):

$$-\ln\left(1-x\right)=\int\frac{1}{1-x}dx=\int\left(1+x+x^2+x^3+\ldots\right)dx=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots+C=$$

$$=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+C=\sum_{n=1}^{\infty}\frac{x^n}{n}+C \qquad |x|<1\cdot$$

To determine the value of C, we put $x=0$ in this equation and obtain $-\ln\left(1-0\right)=C$. Thus $C=0$ and

$$\ln\left(1-x\right)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots=-\sum_{n=1}^{\infty}\frac{x^n}{n} \qquad |x|<1.$$

The radius of convergence is the same as for the original series: $R=1$.

Taylor and Maclaurin Series

We start by supposing that f is any function that can be represented by a power series

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots,\ \left|x-x_0\right|<R \qquad (2)$$

Let's try to determine what the coefficients a_n must be in terms of f.

Theorem 1 If f has a power series representation (expansion) at x_0, that is, if

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\ \left|x-x_0\right|<R,$$

then its coefficients are given by the formula $a_n=\dfrac{f^{(n)}(x_0)}{n!}$.

Substituting this formula for a_n back into the series, we see that if f has a power series expansion at x_0, then it must be of the following form:

$$f(x)=f(x_0)+f'(x_0)\cdot(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+... \qquad (3)$$

The series in Equation 3 is called the *Taylor series of the function f at* x_0 (or *about* x_0 or *centered at* x_0). For the special case the Taylor series becomes

$$f(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+... \qquad (4)$$

This case arises frequently enough that it is given the special name *Maclaurin series.*

Note 1 We have shown that if f can be represented as a power series about x_0, then it isequal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series.

We collected some important Maclaurin series that we have derived in this section and the in preceding one and organized them in the following table.

Table 1

$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$	$-\infty<x<\infty$
$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-...+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+...$	$-\infty<x<\infty$
$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-...+(-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!}+...$	$-\infty<x<\infty$
$\frac{1}{1-x}=1+x+x^2+...+x^{n-1}+...$	$-1<x<1$
$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...$	$-1<x<1$
$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+...+\frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n+...$	$-1<x<1$

Exercise Set 1.5

In Exercise 1 to 15, find a power series representation for the function and determine the interval of convergence.

1. $f(x)=\frac{1}{3+x}$ *2.* $f(x)=\frac{2x-5}{x^2-4x+3}$ *3.* $f(x)=\frac{x^2}{1+x}$

4. $f(x)=\dfrac{1}{x+6}$ *5.* $f(x)=\dfrac{3x+5}{x^2-3x+2}$ *6.* $f(x)=\dfrac{2}{1-3x^2}$

7. $f(x)=\dfrac{1}{\sqrt{e^x}}$ *8.* $f(x)=\sin\dfrac{x^2}{3}$ *9.* $f(x)=\cos\dfrac{2x^3}{3}$

10. $f(x)=\mathrm{e}^{4x}$ *11.* $f(x)=x\cos\sqrt{x}$ *12.* $f(x)=\sqrt{1-x^2}\arcsin x$

13. $f(x)=\mathrm{e}^{-2x^3}$ *14.* $f(x)=x\sin 2\sqrt{x}$ *15.* $f(x)=\ln(1-4x)$

Individual Tasks 1.5

1-4. Find a power series representation for the function and determine the interval of convergence.

I.	**II.**
1. $f(x)=\dfrac{1}{\sqrt{1+x^2}}$	*1.* $f(x)=\sqrt[3]{1+x}$
2. $f(x)=x^2\cdot e^{-x}$	*2.* $f(x)=\dfrac{e^x-1}{x}$
3. $f(x)=\ln\left(1-5x+4x^2\right)$	*3.* $f(x)=\ln\left(1+3x+2x^2\right)$
4. $f(x)=\sin\dfrac{2x^4}{3}$	*4.* $f(x)=\sin^2\left(\dfrac{x}{2}\right)$

1.6Aplications of Representations of Functions as Power Series

One reason why Taylor series are important is that they enable us to integrate functions that we could not previously handle. The function $f(x)=e^{-x^2}$ can not be integrated by the techniques discussed so far because its antiderivative is not an elementary function. In the following example we use Newton's idea to integrate this function.

Example 1 Evaluate $\int\limits_0^1 e^{-x^2}dx$ correct to within an error of 0.001.

Solution First, we find the Maclaurin series for $f(x)=e^{-x^2}$. Although it is possible to use the direct method, let's find it simply by replacing x with x^2 in the series for e^x given in Table 1. Thus, for all values of x, $e^{-x^2}=\sum\limits_{n=0}^{\infty}\dfrac{(-x^2)^n}{n!}=1-\dfrac{x^2}{1!}+\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+...$

Now we integrate term by term:

$$\int_0^1 e^{-x^2}dx = \int_0^1\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots\right)dx = \left(x-\frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\frac{x^9}{9\cdot 4!}\ldots\right)\Bigg|_0^1 =$$

$$=1-\frac{1^3}{3}+\frac{1^5}{10}-\frac{1^7}{42}+\frac{1}{216}..\approx 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}\approx 0.748.$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than $\frac{1^9}{9\cdot 4!}=\frac{1}{216}<0.01$.

Example 2 Evaluate $\lim\limits_{x\to 0}\frac{e^x-1-x}{x^2}$.

Solution Using the Maclaurin series for e^x , we have

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)-1-x}{x^2}=\lim_{x\to 0}\frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots}{x^2}=$$

$$=\lim_{x\to 0}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\ldots\right)=\frac{1}{2}$$

because power series are continuous functions.

Example 3 Use power series to solve the initial-value problem

$$y'=4xy^2-x^3,\ y(0)=2.$$

Solution We assume there is a solution of the form

$$y(x)=y(0)+\frac{y'(0)}{1!}\cdot x+\frac{y''(0)}{2!}\cdot x^2+\frac{y'''(0)}{3!}\cdot x^3+\frac{y^{(4)}(0)}{4!}\cdot x^4+\ldots.$$

$y'(0)=4\cdot 0\cdot 2^2-0^3=0$.

We can differentiate power series term by term, so

$$y''(x)=\left(4xy^2-x^3\right)'=\left(4xy^2\right)'-\left(x^3\right)'=4y^2\cdot(x)'+4x\cdot\left(y^2\right)'-3x^2=$$

$$=4y^2\cdot 1+4x\cdot 2y\cdot y'-3x^2=4y^2+8xyy'-3x^2.$$

Let $x=0,\ \ y=2,\ \ y'(0)=0$, then

$$y''(0)=4\cdot 2^2+8\cdot 0\cdot 2\cdot 0-3\cdot 0^2=16.$$

$$y'''(x)=\left(4y^2+8xyy'-3x^2\right)'=\left(4y^2\right)'+\left(8xyy'\right)'-\left(3x^2\right)'=$$

$$=8y\cdot y'+8yy'+8x\cdot\left(yy'\right)'-6x=16yy'+8xyy''+8x\cdot(y')^2-6x.$$

Let $x=0,\ y=2,\ y'(0)=0,\ y''(0)=16$, then

$$y'''(0)=16\cdot 2\cdot 0+8\cdot 0\cdot 2\cdot 16+8\cdot 0\cdot (0)^2-6\cdot 0=0.$$

$$y^{(4)}(x)=\left(16y\,y'+8x\,y\,y''+8x\cdot (y')^2-6x\right)'=$$

$$=(16y\,y')'+\left(8x\,y\,y''\right)'+\left(8x\cdot (y')^2\right)'-(6x)'=$$

$$=16y'\,y'+16y\,y''+8y\,y''+8x\cdot\left(y\,y''\right)'+8(y')^2+8x\cdot 2y'\cdot y''-6=$$

$$=24(y')^2+24y\,y''+24x\,y'\,y''+8xy\,y'''-6.$$

Let $x=0,\ y=2,\ y'(0)=0,\ y''(0)=16,\ y'''(0)=0$, then

$$y^{(4)}(0)=24\cdot 0^2+24\cdot 2\cdot 16+24\cdot 0\cdot 0\cdot 16+8\cdot 0\cdot 2\cdot 0-6=768-6=762.$$

Substituting the obtained coefficients in the Maclaurin series, we will obtain the solution of the initial differential equation

$$y(x)=2+\frac{0}{1!}\cdot x+\frac{16}{2!}\cdot x^2+\frac{0}{3!}\cdot x^3+\frac{762}{4!}\cdot x^4+...=2+\frac{16}{2}\cdot x^2+\frac{762}{24}\cdot x^4+...=$$
$$=2+8x^2+31.75x^4+....$$

Exercise Set 1.6

In Exercise 1 to 12, use a power series to approximate the definite integral to three decimal places.

1. $\int_0^{0,5}\frac{dx}{1+x^6}$ *2.* $\int_0^1\cos\sqrt{x}\,dx$ *3.* $\int_{0.1}^{0.5}\frac{e^x-1}{x}dx$

4. $\int_0^{0,5}\sqrt{1+x^3}$ *5.* $\int_0^{0.25}\sin x^2dx$ *6.* $\int_0^1 x^2\cdot e^{-x^2}dx$

7. $\int_0^{1/3}\sqrt{1+x^4}$ *8.* $\int_0^{0,5}\frac{arctgx}{x}dx$ *9.* $\int_5^{10}\frac{\ln\left(1+x^2\right)}{x^2}dx$

10. $\int_0^{0.25}\frac{dx}{\sqrt[3]{1+x^3}}$ *11.* $\int_0^{0,4}\cos\left(\frac{5x}{2}\right)^2 d$ *12.* $\int_{0.01}^{0.1}\frac{\ln(1+x)}{x}dx$

In Exercise 13 to 22, use power series to solve the initial-value problem.

13. $y'=2y+y^2,\ y(0)=3$ *14.* $y'=3\cos x+y^2, y(0)=1$

15. $y'=3xy-e^x+4,\ y(0)=0$ *16.* $y'=2\sin x-x^2y,\ y(0)=1$

17. $y'=e^y+xy,\ y(0)=0$ *18.* $y'=2x^2+y^3,\ y(1)=1$

19. $y'' + \frac{2}{x} y' + y = 0,\ y(0) = 1,\ y'(0) = 0$

20. $y'' = y \cdot \cos y' + x,\ y(0) = 1,\ y'(0) = \pi / 3$

21. $y'' = x^2 + y,\ y(0) = -1,\ y'(0) = 1$

22. $y' = 1 + x + x^2 - 2y^2,\ y(1) = 1$

Individual Tasks 1.6

1-2. Use a power series to approximate the definite integral to three decimal places.

3-4. Use power series to solve the initial-value problem.

I.	**II.**
1. $\int_0^{0,25} e^{-x^2} dx$	*1.* $\int_0^{0,5} \sin x^3 dx$
2. $\int_{0.1}^{1} \frac{\sin x - x - x^3}{x^2} dx$	*2.* $\int_{0.2}^{1} \frac{\cos x - 1}{x^2} dx$
3. $y' = 4y + 2xy^2 - e^{3x},\ y(0) = 2$	*3.* $y' = y \cos x + x^2,\quad y(0) = 2$
4. $y'' = y^3 + 3xy', y(1) = -1, y'(1) = 1$	*4.* $y'' = 2yy', y(-1) = 1, y'(-1) = 0.5$

1.7 Fourier Series

Many phenomena in the applications of the natural and engineering sciences are periodic in nature. Examples are the vibrations of strings, springs and other objects, rotating parts in machines, the movement of the planets around the sun, the tides of the sea, etc. *The central problem* of the theory of Fourier series is *how arbitrary periodic functions or signals might be written as a series of sine and cosine functions*.

Definition 1 (Fourier coefficients) Let $f(x)$ be a periodic function with period T and fundamental frequency $\omega_0 = 2\pi / T$, then the *Fourier coefficients* a_n, b_n of $f(x)$, if they exist, are defined by

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \omega_0 n x \, dx \qquad (n = 0, 1, 2, ...) \tag{1}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \omega_0 n x \, dx \qquad (n = 1, 2, 3, ...) \tag{2}$$

In fact, in Definition1 a mapping or transformation is defined from functions to number sequences. This is also denoted as a transformation pair:

$$f(x) \leftrightarrow a_n, b_n .$$

One should pronounce this as: "to the function $f(x)$ belong the Fourier coefficients a_n, b_n". This mapping is the Fourier transform for periodic functions. The function $f(x)$ can be complex-valued. In that case, the coefficients a_n, b_n will also be complex. Using definition 1 one can now define the Fourier series associated with a function $f(x)$.

Definition 2 (Fourier series) When a_n, b_n are the Fourier coefficients of the periodic function $f(x)$ with period T and fundamental frequency $\omega_0 = 2\pi / T$, then the Fourier series of $f(x)$ is defined by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x\right) \tag{3}$$

Example 1 Determine the Fourier coefficients of the sawtooth function given by $f(x) = x$ on the interval $(-\pi, \pi)$ and extended periodically elsewhere, and sketch the graph.

Solution In the present situation we have $T = 2\pi$, so $\omega_0 = 2\pi / T = 1$. The definition of the Fourier coefficients can immediately be applied to the function $f(x)$. Using integration by parts it follows for $n \geq 1$ that Fourier series

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos n x\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = \frac{1}{\pi n}\left[x \sin nx\right]_{x=-\pi}^{x=\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi}\sin nx dx =$$

$$= \frac{1}{n^2\pi}\left[\cos nx\right]_{x=-\pi}^{x=\pi} = 0.$$

For $n = 0$ we have

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_{x=-\pi}^{x=\pi} = 0$$

For the coefficients b_n we have that

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin n x\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = -\frac{1}{\pi n}\left[x \cos nx\right]_{x=-\pi}^{x=\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi}\cos nx dx =$$

$$= -\frac{1}{\pi n}\left(\pi \cos \pi n - (-\pi)\cos(-\pi n)\right) - \frac{1}{n^2\pi}\left[\sin nx\right]_{x=-\pi}^{x=\pi} = -\frac{2\pi}{\pi n}\cos n\pi = \left(-1\right)^{n-1}\frac{2}{n}.$$

Here we used that $\cos \pi n = (-1)^n$ for $n \in N$. Hence, the Fourier coefficients are all equal to zero, while the coefficients b_n are equal to $2\frac{(-1)^{n-1}}{n}$. The Fourier series of the sawtooth function is thus indeed equal to

$$\sum_{n=1}^{\infty} 2\frac{(-1)^{n-1}}{n}\sin nx$$

Theorem (Fundamental theorem of Fourier series) Let $f(x)$ be a piecewise smooth periodic function on R with Fourier coefficients a_n, b_n , with period T and fundamental frequency $\omega_0 = 2\pi / T$. Then for any $x \in R$ one has:

1. $S(x) = f(x)$ at each point of continuity of a piecewise smooth periodic function;

2. $S(x_0)=\dfrac{f(x_0-0)+f(x_0+0)}{2}$ at a point x_0 where the function is discontinuous;

3. $S(-T/2)=S(T/2)=\dfrac{f(T/2-0)+f(-T/2+0)}{2}$ at the endpoints of the interval $\left(-T/2,T/2\right)$.

According to the fundamental theorem, the Fourier series converges to the function at each point of continuity of a piecewise smooth periodic function. At a point where the function is discontinuous, the Fourier series converges to the average of the left- and right-hand limits at that point. Hence, both at the points of continuity and at the points of discontinuity the series converges to $\left(f(x-0)+f(x+0)\right)/2$.

Exercise Set 1.7

In Exercises 1 to 10 determine the Fourier coefficients of the given functions on the given intervals:

1. $f(x)=\begin{cases}1+\dfrac{x}{\pi}, & if\ -\pi\le x<0,\\ 1-\dfrac{x}{\pi}, & if\ \ 0\le x<\pi.\end{cases}$

2. $f(x)=\begin{cases}0, & if\ -\pi<x\le 0,\\ \dfrac{\pi x}{4}, & if\ \ 0<x<\pi.\end{cases}$

3. $f(x)=x^2,\ \ x\in[-\pi;\pi].$

4. $f(x)=\begin{cases}a x, & if\ -\pi<x<0,\\ b x, & if\ \ 0\le x<\pi.\end{cases}$

5. $f(x)=1-x,\ if\ x\in(-2;2)$

6. $f(x)=x(1-x),\ if\ x\in(-1;1)$

7. $f(x)=\begin{cases}x, & if\ \ 0\le x\le 1,\\ 0, & if\ \ 1<x<3/2.\end{cases}$

8. $f(x)=\begin{cases}1, & if\ \ 0\le x\le 1,\\ x, & if\ \ 1<x<3.\end{cases}$

9. $f(x)=\begin{cases}0, & if\ -3<x\le 0;\\ x, & if\ \ 0<x<3.\end{cases}$

10. $f(x)=\begin{cases}0, & if\ -\pi<x<0;\\ 2, & if\ \ 0<x<\pi.\end{cases}$

Individual Tasks 1.7

1-2. Determine the Fourier coefficients of the given functions on the given intervals:

I.	**II.**
1. $f(x)=\begin{cases}\pi, & if\ -\pi\le x<0,\\ \pi-x, & if\ \ 0<x<\pi.\end{cases}$	*1.* $f(x)=\begin{cases}0, & if\ -\pi\le x<0,\\ 1-x, & if\ \ 0\le x<\pi\end{cases}$
2. $f(x)=1-3x,\ if\ x\in(-1;1)$	*2.* $f(x)=2x+1,\ if\ x\in(-3;3)$

1.8 Fourier Cosine and Fourier Sine Series

The ordinary Fourier series of an even periodic function contains only cosine terms and the Fourier series of an odd periodic function contains only sine terms. For the standard functions we have seen that the periodic block function and the periodic triangle function, which are even, do indeed contain cosine terms only, and that the sawtooth function, which is odd, contains sine terms only. Sometimes it is desirable to obtain what for an arbitrary function on the interval $(0,T)$ a Fourier series containing only sine terms or containing only cosine terms. Such series are called *Fourier sine series and Fourier cosine series*. In order to find a Fourier cosine series for a function defined on the interval $(0,T)$, we extend the function to an even function on the interval $(-T,T)$ by defining $f(-x)=f(x)$ for $-T<x<0$ and subsequently extending the function periodically with period $2T$.The function thus created is now an even function and its ordinary Fourier series will contain only cosine terms, while $f(x)$ is equal to the original function on the interval $(0,T)$.

In a similar way one can construct a Fourier sine series for a function by extending the function defined on the interval $(0,T)$ to an odd function on the interval $(-T,T)$ and subsequently extending it periodically with period $2T$. Such an odd function will have an ordinary Fourier series containing only sine terms. Determining a Fourier sine series or a Fourier cosine series in the way described above is called a forced series development.

Example 1 Determine the Fourier coefficients of the sawtooth function given by $f(x)=x^2$ on the interval $(-1,1)$.

Solution

Let the function $f(x)$ be given by $f(x)=x^2$ on the interval (0, 1). We wish to obtain a Fourier sine series for this function. We then first extend it to an odd function on the interval (−1, 1) and subsequently extend it periodically with period 2. The function and its odd and periodic extension are drawn in Figure 1.

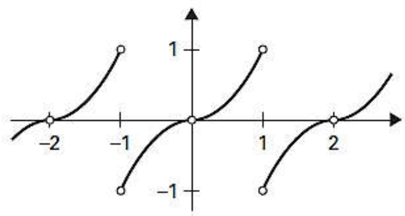

Figure 1

The ordinary Fourier coefficients of the function thus created can be calculated using (1) and (2). Since the function is odd, all coefficients a_n will equal 0. For b_n we have

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin nx\,dx=\int_{-1}^{0}(-x^2)\sin nxdx+\int_{0}^{1}x^2\sin nxdx=2\int_{0}^{1}x^2\sin nxdx.$$

Applying the integration by parts twice, it follows that

$$b_n = \frac{-2}{\pi n}\left(\left[x^2 \cos \pi nx\right]_0^1 - \frac{2}{\pi n}\left[x \sin \pi nx\right]_0^1 - \frac{2}{\pi^2 n^2}\left[\cos \pi nx\right]_0^1\right) =$$

$$= \frac{2}{\pi n}\left(\frac{2((-1)^n - 1)}{\pi^2 n^2} - (-1)^n\right).$$

The Fourier sine series of $f(x) = x^2$ on the interval (0, 1) is thus equal to

$$\sum_{n=0}^{\infty} \frac{2}{\pi n}\left(\frac{2((-1)^n - 1)}{\pi^2 n^2} - (-1)^n\right) \sin \pi nx$$

Example 2 Determine the Fourier coefficients of the function given by $f(x) = \sin x$ on the interval $(0, \pi)$.

Solution In this final example we will show that one can even obtain a Fourier cosine series for the sine function on the interval $(0, \pi)$. To this end we first extend $f(x) = \sin x$ to an even function on the interval $(-\pi, \pi)$ and then extend it periodically with period $T = 2\pi$; see Figure 2. The ordinary Fourier coefficients of the function thus created can be calculated using (1) and (2). Since the function is even, all coefficients will be equal to 0.

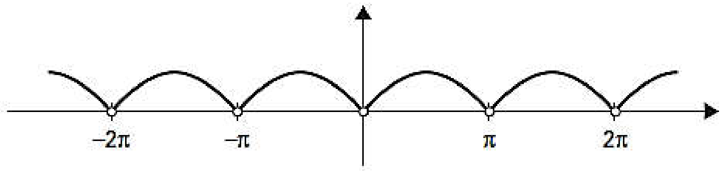

Figure 2

For a_n one has

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-\sin x) \cos nx dx + \int_{0}^{\pi} \sin x \cos nx dx\right) = \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx dx.$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} \left(\sin(1+n)x + \sin(1+n)x\right) dx = \frac{1}{\pi}\left[\frac{-1}{1+n}\cos(1+n)x + \frac{-1}{1-n}\cos(1-n)x\right]_0^{\pi} =$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{1+n} + \frac{1-(-1)^{n-1}}{1-n}\right) = \frac{2(1-(-1)^{n-1})}{\pi(1-n^2)}.$$

If $n = 0$, then $a_0 = \frac{2(1-(-1)^{0-1})}{\pi(1-0^2)} = \frac{4}{\pi}$.

The Fourier cosine series of the function $f(x) = \sin x$ on the interval $(0, \pi)$ is thus equal to

$$\frac{2}{\pi} + \sum_{n=0}^{\infty} \frac{2(1-(-1)^{n-1})}{\pi(1-n^2)} \cos nx$$

Exercise Set 1.8

In Exercises 1 to 4 determine the Fourier sine series of the given functions on the given intervals:

1. $y = 1 - 0.5x, x \in [0;2]$.

2. $f(x) = 1 - x,\ x \in (0;2)$.

3. $f(x) = x(1-x),\ x \in (0;1)$.

4. $f(x) = x,\ x \in (0;2)$.

In Exercises 5 to 8 determine the Fourier cosine series of the given functions on the given intervals:

5. $f(x) = 2 - x,\ x \in (0;2)$

6. $f(x) = x(2-x),\ x \in (0;2)$.

7. $f(x) = \begin{cases} x, & if\ \ 0 < x \le 2, \\ 2, & if\ \ 2 < x < 4. \end{cases}$

8. $y = \cos x, x \in [0;\pi]$

Individual Tasks 1.8

1. Determine the Fourier sine series of the given functions on the given intervals.

2. Determine the Fourier cosine series of the given functions on the given intervals.

I.	**II.**
1. $f(x) = \begin{cases} 1, if\ \ 0 < x < \pi/2; \\ 0, if\ \ \pi/2 < x < \pi. \end{cases}$	*1.* $f(x) = x - 4\pi, x \in (4\pi;5\pi)$
2. $f(x) = x \cdot \sin x,\ x \in (0;\pi)$	*2.* $f(x) = \begin{cases} x, & if\ \ 0 \le x \le \pi/2, \\ 0, & if\ \ \pi/2 < x < \pi \end{cases}$

II FUNCTIONS OF A COMPLEX VARIABLE

2.1 The Complex Number System

We can consider a complex number as having the form $a+ib$ where a and b are real numbers and i, which is called *the imaginary unit*, has the property that $i^2=-1$. If $z=a+ib$, then a is called *the real part of* z and b is called *the imaginary part of* z and are denoted by $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$, respectively. The symbol z, which can stand for any complex number, is called *a complex variable*.

Definition Two complex numbers $a+ib$ and $c+di$ are equal if and only if $a=c$ and $b=d$.

Definition *The complex conjugate*, or briefly *conjugate*, of a complex number $a+ib$ is $a-ib$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

Note In algebraic operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

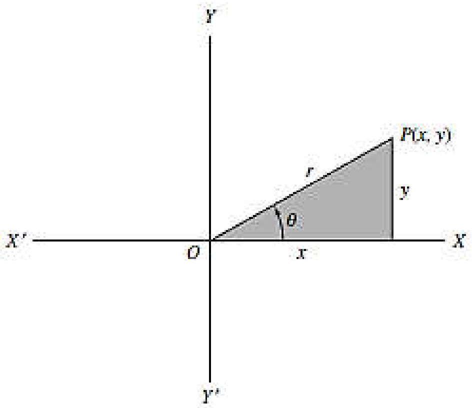

Figure 3

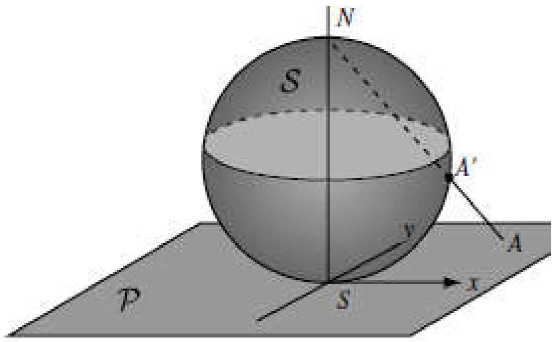

Figure 4

Definition *The absolute value* or *modulus* of a complex number $a+ib$ is defined as $|a+bi|=\sqrt{a^2+b^2}$.

Example 1 $|-4+2i|=\sqrt{(-4)^2+2^2}=\sqrt{20}=2\sqrt{5}$.

Polar Form of Complex Numbers

Let P be a point in the complex plane corresponding to the complex number (x,y) or $x+iy$. Then we see from Figure 3 that $x=r\cos\varphi, y=r\sin\varphi$, where $r=\sqrt{x^2+y^2}=|x+iy|$ is called *the modulus* or *absolute value of* $z=x+iy$ (denoted by $\mathrm{mod}\, z$ or $|z|$) and φ, called *the amplitude* or *argument of* $z=x+iy$ (denoted by $\arg z$), is the angle that line OP makes with the positive x axis.

It follows that

$$z=x+iy=r(\cos\varphi+i\sin\varphi) \qquad (1)$$

which is called *the polar form of the complex number*, where r and φ are called *the polar coordinates.*

For any complex number $z \neq 0$ only one value of φ in $0 \leq \varphi < 2\pi$ corresponds. However, any other interval of length 2π, for example $-\pi \leq \varphi < \pi$, can be used. Any particular choice, decided upon in advance, is called *the principal range*, and the value of φ is called its *principal value*.

Let $z_1 = x_1 + iy_1 = r_1\left(\cos\varphi_1 + i\sin\varphi_1\right)$ and $z_2 = x_2 + iy_2 = r_2\left(\cos\varphi_2 + i\sin\varphi_2\right)$, then we can show that

$$z_1 z_2 = r_1 r_2\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right) \quad (2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right) \quad (3)$$

A generalization of (2) leads to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n\left(\cos(\varphi_1 + \varphi_2 + ... + \varphi_n) + i\sin(\varphi_1 + \varphi_2 + ... + \varphi_n)\right) \quad (4)$$

$$z^n = r^n\left(\cos n\varphi + i\sin n\varphi\right) \quad (5)$$

Euler's Formula. Polynomial Equations. Roots of Complex Numbers

A number w is called an n-th root of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. From Equattion 5 we can show that if n is a positive integer

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\varphi + 2k\pi}{n}\right) + i\sin\left(\frac{\varphi + 2k\pi}{n}\right)\right), k = 0,1,2,...,n-1 \quad (6)$$

from which it follows that there are n different values for $z^{1/n}$, i.e., n different n-th roots of z, provided $z \neq 0$.

By assuming that the infinite series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ holds when $x = i\varphi$, we can arrive at the result

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \quad (7)$$

which is called *Euler's formula*. It is more convenient, however, simply to take (7) as a definition of $e^{i\varphi}$. In general, we define

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) \quad (8)$$

Let P (Figure 4) be a complex plane and consider that a sphere $\underline{S}$ tangents to P at $z = 0$. The diameter NS is perpendicular to P and we call points N and S the north and south poles of $\underline{S}$. Corresponding to any point A on P we can construct line NA intersecting $\underline{S}$ at point A'. Thus to each point of the complex plane P there corresponds one and only one point of the sphere $\underline{S}$, and we can represent any complex number by a point on the sphere. To complete at all we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called *the entire complex plane*, *the entire z plane*, or *the extended complex plane.*

Example 2 Perform each of the indicated operations.
Solution
(a) $(3+2i)+(6-7i)=3+6+2i-7i=9-5i$.
(b) $(-4-3i)-(5-7i)=-4-5-3i+7i=-9+4i$.
(c) $(2-3i)\cdot(5+2i)=2\cdot 5+2\cdot 2i-5\cdot 3i-3\cdot 2i^2=10+6+4i-15i=16-11i$.
(d) $\dfrac{3-2i}{1+6i}=\dfrac{3-2i}{1+6i}\cdot\dfrac{1-6i}{1-6i}=\dfrac{3-2i-18i+12i^2}{1-36i^2}=\dfrac{-9-20i}{37}=-\dfrac{9}{37}-\dfrac{20}{37}i$.
Example 3 Suppose, $z_1=2+i$, $z_2=3-2i$. Evaluate each of the following.
Solution
(a) $|3z_1-4z_2|=|3(2+i)-4(3-2i)|=|-6+11i|=\sqrt{(-6)^2+11^2}=\sqrt{157}$.
(b) $z_1^3-3z_1^2+4z_1-8=(2+i)^3-3(2+i)^2+4(2+i)-8=$
$=2^3+3\cdot 2^2 i+3\cdot 2i^2+i^3-3(2^2+4i+i^2)+8+4i-8=$
$=8+12i-6-i-12-12i+3+8+4i-8=-7+3i$.
Example 4 Express each of the following complex numbers in a polar form.
Solution *(a)* $2+2\sqrt{3}i$ (See Figure 5)

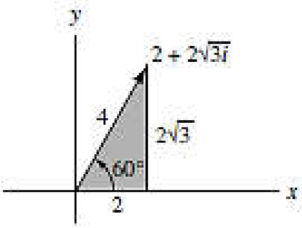

Figure 5

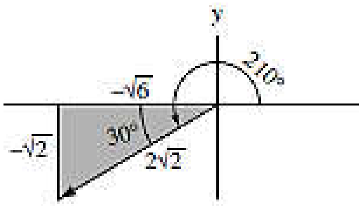

Figure 6

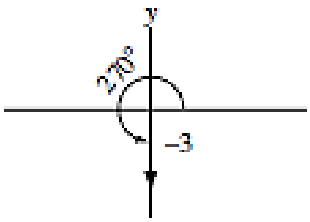

Figure 7

Modulus or absolute value equals $r=\left|2+2\sqrt{3}i\right|=\sqrt{4+12}=4$.

Amplitude or argument $\varphi=\arcsin\dfrac{\sqrt{3}}{2}=60^\circ=\dfrac{\pi}{3}$ (radians).

Then $2+2\sqrt{3}i=r(\cos\varphi+i\sin\varphi)=4(\cos 60^\circ+i\sin 60^\circ)=4\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$.

The result can also be written as, using Euler's formula, $4e^{\frac{\pi i}{3}}$.
(b) $-\sqrt{6}-\sqrt{2}i$ (See Figure 6)

$$r=\left|-\sqrt{6}-\sqrt{2}i\right|=\sqrt{6+2}=2\sqrt{2},\ \varphi=180^\circ+30^\circ=210^\circ=7\pi/6.$$

Then $-\sqrt{6}-\sqrt{2}i=2\sqrt{2}\left(\cos\dfrac{7\pi}{6}+i\sin\dfrac{7\pi}{6}\right)=2\sqrt{2}e^{\frac{7\pi i}{6}}$.
(c) $-3i$ (See Figure 7)

$$r=|-3i|=|0-3i|=\sqrt{0+9}=3,\ \varphi=270^\circ=3\pi/2.$$

Then $-3i=3\left(\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}\right)=3e^{\frac{3\pi i}{2}}$.

Example 5 Find all values of z for which.

Solution In polar form, $-32 = 32\left(\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k)\right), k = 0, \pm 1, \pm 2\ldots$.
Let $z = r(\cos\varphi + i\sin\varphi)$. Then, by De Moivre's theorem,

$$z^5 = r^5\left(\cos 5\varphi + i\sin 5\varphi\right) = 32\left(\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k)\right).$$

and so $r^5 = 32, 5\varphi = \pi + 2\pi k$, from which $r = 2, \varphi = \dfrac{(\pi + 2\pi k)}{5}$. Hence

$$z = 2\left(\cos\left(\frac{\pi + 2\pi k}{5}\right) + i\sin\left(\frac{\pi + 2\pi k}{5}\right)\right).$$

If

$$k = 0, z = z_1 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$$

$$k = 1, z = z_2 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

$$k = 2, z = z_3 = 2\left(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right) =$$

$$k = 3, z = z_4 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$$

$$k = 4, z = z_5 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$$

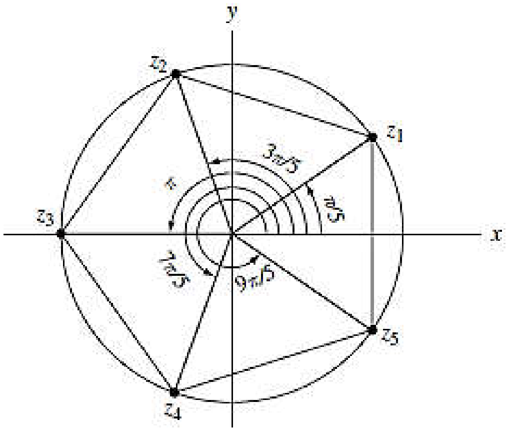

Figure 8

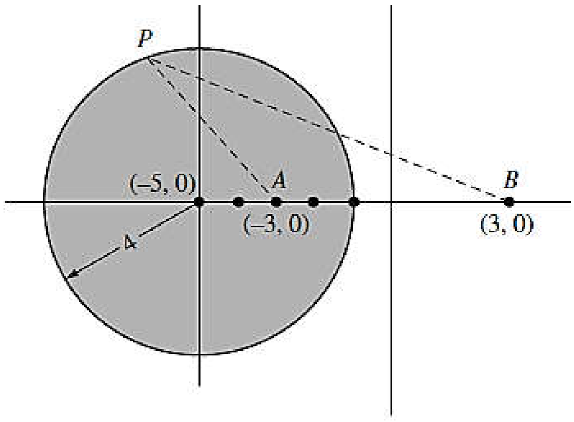

Figure 9

By considering $k = 5, 6, ..$ as well as negative values, $-1, -2, ...$, repetitions of the above five values of z are obtained. Hence, these are the only solutions or roots of the given equation. These five roots are called the fifth roots of -32 and are collectively denoted by $(-32)^{1/5}$. In general, $a^{1/n}$ represents the n-th roots of a and there are n such roots. The values of z are indicated in Figure 8.

Example 6 Represent graphically the set of values of z for which

$$\text{(a) } \left|\frac{z-3}{z+3}\right| = 2, \text{ (b) } \left|\frac{z-3}{z+3}\right| < 2.$$

Solution The given equation is equivalent to $|z-3| = 2|z+3|$ or, if $z = x + iy$, $|x + iy - 3| = 2|x + iy + 3|$, i.e., $\sqrt{(x-3)^2 + y^2} = 2\sqrt{(x+3)^2 + y^2}$.

Squaring and simplifying, this becomes $x^2+y^2+10x+9=0$ or $(x+5)^2+y^2=16$

i.e., $|z+5|=4$, a circle of radius 4 with the center at $(-5,0)$ as shown in Figure 9.

(b) The given inequality is equivalent to $|z-3|<2|z+3|$ or $\sqrt{(x-3)^2+y^2}<2\sqrt{(x+3)^2+y^2}$. Squaring and simplifying, this becomes $x^2+y^2+10x+9>0$ or $(x+5)^2+y^2>16$, i.e. $|z+5|>4$.

The required set thus consists of all points external to the circle of Figure 9.

Exercise Set 2.1

In Exercises 1 to 9 perform each of the indicated operations:

1. $(2-3i)+(5+8i)$ *2.* $(i-2)\cdot((4-i)+3(7+6i))$ *3.* $i(2-i)(4+3i)$

4. $4i^5-3i^4+2i^3+7i-1$ *5.* $(i^4-5i)(3i^3+2i+1)$ *6.* $(2-i)^2(3+i)$

7. $\frac{3-i}{5+i}$ *8.* $\frac{(4-6i)(i-2)}{1+i}$ *9.* $\frac{i^4+i^9+i^{16}}{2-i^5+i^{10}}$

In Exercises 10 to 12, suppose $z_1=2+i,\ z_2=3-2i$. Evaluate each of the following expressions:

10. $z_1^2+2z_1-3i+5$ *11.* $|2z_2-3z_1|^2$ *12.* $|z_1\bar{z}_2-4\bar{z}_1z_2|$

In Exercises 13 to 18, describe and graph the locus represented by each of the following expressions:

13. $|z-i|=2$ *14.* $|z+2i|+|z-2i|=6$ *15.* $|z-3|-|z+3|=4$

16. $z(\bar{z}+2)=3$ *17.* $z=3e^{it}-\frac{1}{2e^{it}}$ *18.* $z=\frac{1+i}{1-t}+\frac{t(2-4i)}{1-t}$

In Exercises 19 to 24, describe graphically the region represented by each of the following inequalities:

19. $1<|z+i|\le 2$ *20.* $-\pi/4\le\arg(z-i)<\pi/2$ *21.* $|z+3i|>4$

22. $|z+2|+|z-2|<1$ *23.* $0\le\arg z<5\pi/6$ *24.* $\operatorname{Re} z^2>1$

In Exercises 25 to 30, express each of the following complex numbers in a polar form:

25. $2-2i$ *26.* $-1+\sqrt{3}i$ *27.* $\sqrt{2}i$

28. $-2\sqrt{3}-2i$ *29.* $\frac{\sqrt{3}}{2}-\frac{1}{2}i$ *30.* -5

In Exercises 31 to 36, solve the following equations, obtaining all roots:

31. $z^2+4=0$ *32.* $z^4-81=0$ *33.* $z^3-27=0$

34. $z^2+6z+25=0$ *35.* $z^4+5z^2+4=0$ *36.* $z^2-2z+5=0$

Individual Tasks 2.1

1. Perform each of the indicated operations.

2-3. Describe and graph the locus represented by the following expression.

4. Describe graphically the region represented by the following expression.

5. Solve the following equations, obtaining all roots.

I.		**II.**	
1.	$\dfrac{2i^3-i^9+4i^{18}}{3-2i^5+i^{10}}$	*1.*	$\dfrac{4i^3-3i^8+i^{15}}{3-4i^5+i^{12}}$
2.	$\operatorname{Im} z^2=4$	*2.*	$\operatorname{Im} z\cdot\operatorname{Re} z=1$
3.	$z=\dfrac{1}{\cos t}+i\,2tg\,t$	*3.*	$z=\dfrac{1+t}{1-t}+i\dfrac{2+t}{2-t}$
4.	$\lvert z-i\rvert<1,\ arg\,z\geq\pi/4$	*4.*	$\lvert 2z\rvert>\lvert 1+z^2\rvert$
5.	$z^4+16=0$	*5.*	$z^3+64=0$

2.2 Functions of a Complex Variable

A symbol such as z which can stand for any one of a set of complex numbers is called *a complex variable*. Let two sets D and E be given, whose elements are complex numbers. The numbers $z=x+iy$ of the set D will represent the points of the complex plane z, and the numbers $w=u+iv$ of the set E are the points of the complex plane w (see Figure 10).

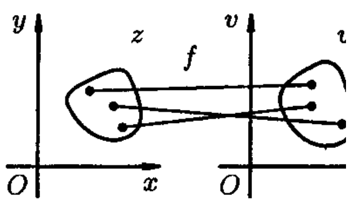

Figure 10

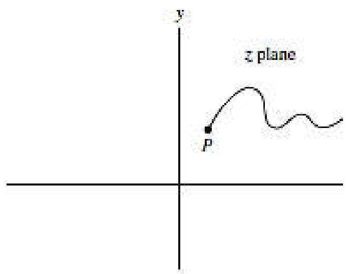

Figure 11

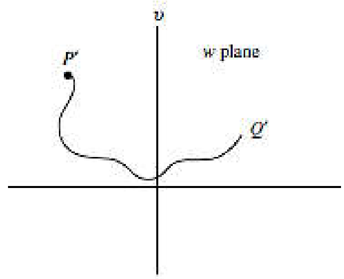

Figure 12

Suppose to each value of a complex variable z can assume, one or more values of a complex variable w corresponds. Then we say that w is *a function of* z and write $w=f(z)$ or $w=G(z)$, etc. The variable z is sometimes called *an independent* *variable*, while w is called *a dependent variable*. The value of a function at $z=a$ is often written as $f(a)$. Thus, if $f(z)=z^2$, then $f(2i)=(2i)^2=-4$.

The set D is called *a domain of function* $w=f(z)$; the set E is called *a domain of the values of this function* (if each point of the set E is the value of the function, then E is *the range of values functions*; in this case, the function $w=f(z)$ maps D to E).

If only one value of w corresponds to each value of z, we say that w is *a single-valued function of* z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z, we say that w is *a multiple-valued* or *many-valued function of* z.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called *a branch of the function*. It is customary to consider one particular member as a principal branch of the multiple-valued function and the value of the function corresponding to this branch as the principal value.

Example 1

(a) If $w=z^2$, then to each value of z there is only one value of w. Hence, $w=f(z)=z^2$ is a single-valued function of z.

(b) If $w^2=z$, then to each value of z there are two values of w. Hence, $w^2=z$ defines a multiple-valued (in this case two-valued) function of z.

Whenever we speak of a function, we shall, unless otherwise stated, assume a single-valued function. If $w=f(z)$, then we can also consider z as a function, possibly multiple-valued, of w, written as $z=g(w)=f^{-1}(w)$. The function f^{-1} is often called *the inverse function corresponding to* f. Thus, $w=f(z)$ and $w=f^{-1}(z)$ are inverse functions of each other.

If $w=u+iv$ (where u and v are real) is a single-valued function of $z=x+iy$ (where x and y are real), we can write $u+iv=f(x+iy)$. By equating real and imaginary parts, this is seen to be equivalent to

$$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases} \tag{1}$$

Thus given a point (x,y) in the z plane, such as P in Figure 11, there corresponds a point (u,v) in the w plane, say P' in Figure 12. The set of equations (1) (or the equivalent, $w=f(z)$) is called *a transformation*. We say that point P is mapped or transformed into point P' by means of the transformation and call P' *the image* of P.

Example 2 If $w = z^2$, then $w = u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi$ and the transformation is $u = x^2 - y^2$, $v = 2xy$. The image of a point $(1,2)$ in the $z-$plane is the point $(-3,4)$ in the w plane.

The Elementary Functions

1. Polynomial Functions are defined by

$$w = a_0 z^n + a_1 z^{n-1} + ... + a_n = P(z) \tag{2}$$

where $a_0 \neq 0, a_1, ..., a_n$ are complex constants and n is a positive integer called *the degree of the polynomial* $P(z)$. The transformation $w = az + b$ is called *a linear transformation*.

2. Rational Algebraic Functions are defined by

$$w = \frac{P(z)}{Q(z)} \tag{3}$$

where $P(z)$ and $Q(z)$ are polynomials. We sometimes call (3) *a rational transformation*. The special case $w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ is often called *a bilinear or fractional linear transformation*.

3. Exponential Functions are defined by

$$w = \mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y) \tag{4}$$

where e is the natural base of logarithms. If a is real and positive, we define

$$a^z = \mathrm{e}^{z \ln a} \tag{5}$$

where $\ln a$ a is the natural logarithm of a. This reduces to (4) if $a = e$.

4. Trigonometric Functions. We define the trigonometric or circular functions $\sin z, \cos z$, etc., in terms of exponential functions as follows:

$$\sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}, \cos z = \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}, \mathrm{tg}\, z = \frac{\sin z}{\cos z}, \mathrm{ctg}\, z = \frac{\cos z}{\sin z} \tag{6}$$

Note that the trigonometric functions $\sin z$ and $\cos z$ in the complex plane are *unbounded*: $\lim\limits_{\mathrm{Im} z \to \pm\infty} \sin z = \infty$, $\lim\limits_{\mathrm{Im} z \to \pm\infty} \cos z = \infty$. For example, $\cos i = \dfrac{\mathrm{e}^1 + \mathrm{e}^{-1}}{2} \approx 1{,}54 > 1, \cos 3i > 10$.

5. Hyperbolic Functions are defined as follows:

$$\mathrm{sh}\, z = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}, \mathrm{ch}\, z = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}, \mathrm{th}\, z = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}}, \mathrm{cth}\, z = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{\mathrm{e}^z - \mathrm{e}^{-z}} \tag{7}$$

The following relations exist between the trigonometric or circular functions and the hyperbolic functions:

$$\operatorname{sh} z = -i\sin(iz),\ \operatorname{ch} z = \cos(iz),$$
$$\sin z = \sin x\cdot\operatorname{ch} y + i\cos x\cdot\operatorname{sh} y,$$
$$\cos z = \cos x\cdot\operatorname{ch} y - i\sin x\cdot\operatorname{sh} y.$$

6. Logarithmic Functions. If $z = \mathrm{e}^{w}$, then we write $w = \operatorname{Ln} z$ and it is called *the natural logarithm of* z.

The natural logarithmic function can be defined by

$$w = \operatorname{Ln} z = \ln|z| + i\arg z + i2k\pi,\ k\in Z, \tag{8}$$

where $z = r\mathrm{e}^{i\varphi} = r\mathrm{e}^{i(\varphi+2k\pi n)}$. Note that $\operatorname{Ln} z$ is a multiple-valued (in this case, *infinitely many-valued*) function. The principal-value or *principal branch of* $\operatorname{Ln} z$ is sometimes defined as $\ln|z| + i\varphi$, where $0\le\varphi<2\pi$. However, any other interval of length 2π can be used, e.g., $-\pi<\varphi\le\pi$, etc.

7. Inverse Trigonometric Functions. If $z = \sin w$, then $w = \sin^{-1} z$ is called *the inverse sine of* z or arcsin of z.

Using the definition of $\sin z$, we have

$$w = -i\operatorname{Ln}\left(iz + \sqrt{1-z^2}\right) \qquad 9)$$

Similarly, we define other inverse trigonometric or circular functions $\cos^{-1} z$, $\operatorname{tg}^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i, k\in Z$ in the logarithm:

$$w = \operatorname{Arcsin} z = -i\operatorname{Ln}\left(iz + \sqrt{1-z^2}\right) \tag{9a}$$

$$w = \operatorname{Arccos} z = -i\operatorname{Ln}\left(z + \sqrt{z^2-1}\right) \tag{9b}$$

$$w = \operatorname{Arctg} z = \frac{1}{2i}\operatorname{Ln}\frac{1+iz}{1-iz} \tag{9c}$$

$$w = \operatorname{Arcctg} z = -\frac{1}{2i}\operatorname{Ln}\frac{1+iz}{iz-1} \tag{9d}$$

8. The Function z^{α}, where α may be complex, is defined as $e^{\alpha\ln z}$. Similarly, if $f(z)$ and $g(z)$ are two given functions of z, we can define $f^{g(z)}(z) = \mathrm{e}^{g(z)\ln f(z)}$. In general, such functions are multiple-valued.

9. Algebraic and Transcendental Functions. If w is a solution of the polynomial equation

$$P_0(z)w^n + P_1(z)w^{n-1} + \ldots + P_{n-1}(z)w + P_n(z) = 0 \tag{10}$$

where $P_0 \neq 0, P_1(z), ..., P_n(z)$ are polynomials in $z-$plane and n is a positive integer, then $w = f(z)$ is called *an algebraic function of* z.

Any function that cannot be expressed as a solution of (9) is called ***a transcendental function***. The logarithmic, trigonometric, and hyperbolic functions and their corresponding inverses are examples of transcendental functions.

The functions considered in 1–9 above, together with the functions derived from them by a finite number of operations involving addition, subtraction, multiplication, division and roots are called ***elementary functions***.

Example 3 Determine the values of (a) i^{1+i}, (b) $\mathrm{Arcsin}\,3$, (c) $\mathrm{Ln}(12+5i)$.

Solution

(a) $i^{1+i} = \mathrm{e}^{(1+i)\,\mathrm{Ln}\,i} = \mathrm{e}^{(1+i)(\ln|i|+i\arg i+2k\pi i)} = \mathrm{e}^{(1+i)\left(i\frac{\pi}{2}+2k\pi i\right)} = \mathrm{e}^{(i-1)\left(\frac{\pi}{2}+2k\pi\right)} =$

$$= \mathrm{e}^{-\left(\frac{\pi}{2}+2k\pi\right)} \cdot \mathrm{e}^{i\left(\frac{\pi}{2}+2k\pi\right)} = \mathrm{e}^{-\left(\frac{\pi}{2}+2k\pi\right)}\left(\cos\left(\frac{\pi}{2}+2k\pi\right)+i\sin\left(\frac{\pi}{2}+2k\pi\right)\right) =$$

$$= i\,\mathrm{e}^{-\left(\frac{\pi}{2}+2k\pi\right)}, k \in Z.$$

(b) $\mathrm{Arcsin}\,3 = -i\,\mathrm{Ln}\,(3i \pm i\sqrt{8}) = -i\,\mathrm{Ln}\left(\left(3 \pm 2\sqrt{2}\right)i\right) =$

$$= -i\left(\ln\left(3 \pm 2\sqrt{2}\right) + i\frac{\pi}{2} + 2k\pi i\right) = \frac{\pi}{2} + 2k\pi - i\ln\left(3 \pm 2\sqrt{2}\right),\ k \in Z.$$

(c) $\mathrm{Ln}\,(12+5i) = \ln|12+5i| + i\arg(12+5i) + i\,2k\pi =$

$$= \left| \begin{array}{l} |12+5i| = \sqrt{144+25} = 13, \\ 12>0,\ 5>0,\ \arg(12+5i) = \mathrm{arctg}\dfrac{5}{12} \end{array} \right| = \ln 13 + i\left(\mathrm{arctg}\frac{5}{12} + 2k\pi\right), k \in Z.$$

Example 4 Show that the line joining the points $P(-2,1)$ and $Q(1,-3)$ in the z plane is mapped by $w = z^2$ into the curve joining points $P'Q'$ (Figure 13) and determine the equation of this curve.

Solution Points P and Q have coordinates $(-2,1)$ and $(1,-3)$ respectively. Then, the parametric equations of the line joining these points are given by

$$\frac{x-(-2)}{1-(-2)} = \frac{y-1}{-3-1} = t \text{ or } \begin{cases} x = 3t-2, \\ y = 1-4t. \end{cases}$$

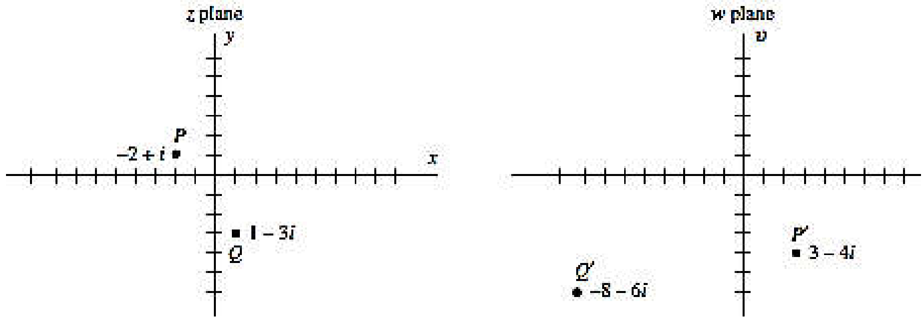

Figure 13

The equation of the line PQ can be represented by $z = 3t - 2 + i(1-4t)$. The curve in the w plane into which this line is mapped has the equation

$$w = z^2 = \left(3t - 2 + i(1-4t)\right)^2 = (3t-2)^2 - (1-4t)^2 + 2(3t-2)(1-4t)i =$$
$$= 3 - 4t - 7t^2 + (-4 + 22t - 24t^2)i.$$

Then, since $w = u + iv$, the parametric equations of the image curve are given by

$$u = 3 - 4t - 7t^2 \text{ and } v = -4 + 22t - 24t^2.$$

By assigning various values to the parameter t, this curve may be graphed.

Limits. Continuity

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z = z_0$ itself (i.e., in a deleted δ neighborhood of z_0).

Definition The number L is *the limit* of $f(z)$ as z approaches z_0 and write $\lim\limits_{z\to z_0} f(z) = L$, if for any positive number ε (however small), we can find some positive number δ (usually depending on ε) such that $\left|f(z) - L\right| < \varepsilon$, whenever $0 < \left|z - z_0\right| < \delta$.

There are three conditions that must be met in order that $f(z)$ be *continuous* at $z = z_0$: (1) $\lim\limits_{z\to z_0} f(z) = L$ must exist; (2) $f(z_0)$ must exist, i.e., $f(z)$ is defined at z_0; (3) $L = f(z_0)$.

Points in the $z-$plane, where $f(z)$ fails to be continuous, are called *discontinuities* of $f(z)$, and $f(z)$ is said to be discontinuous at these points. If $\lim\limits_{z\to z_0} f(z)$ exists, but is not equal to $f(z_0)$, we call z_0 *a removable discontinuity,* since by redefining $f(z_0)$ to be the same as $\lim\limits_{z\to z_0} f(z)$, the function becomes continuous.

Note To examine the continuity of $f(z)$ at $z = \infty$, we let $z = 1/w$ and examine the continuity of $f\left(1/w\right)$ at $w = 0$.

Exercise Set 2.2

1. Let $w = f(z) = z(2 - z)$. Find the values of w corresponding to *(a)* $z = 1 + i$, *(b)* $z = 2 - 2i$ and graph corresponding values in the w and z planes.

2. Let $w=f(z)=\dfrac{1+z}{1-z}$. Find: *(a)* $f(i)$, *(b)* $f(1-i)$ and represent them graphically.

In Exercises 3 to 8 separate each of the following expressions into real and imaginary parts, i.e., find $u(x,y)$ and $v(x,y)$ such that $f(z)=u+iv$:

3. $w=(2-3i)z^2-iz-i$	*4.* $w=\lvert z\rvert\cdot\mathrm{Re}\,z$	*5.* $w=\dfrac{z-i}{z+i}$
6. $w=5iz-iz^2-1$	*7.* $w=\bar{z}\cdot\mathrm{Im}\,z$	*8.* $w=3z^2-2iz+8$

9. Find all values of z for which *(a)* $e^{3z}=1$, *(b)* $e^{4z}=i$.

In Exercises 10 to 21, find the value of the given numbers:

10. $\mathrm{Ln}\left(\sqrt{3}-i\right)$	*11.* $\mathrm{Ln}\left(1+\sqrt{3}i\right)$	*12.* $\mathrm{Ln}(-1-i)$
13. $\sin\left(\dfrac{3\pi}{4}+i\right)$	*14.* $\cos\left(\dfrac{\pi}{6}-i\right)$	*15.* $\mathrm{tg}\dfrac{\pi}{2}i$
16. $\mathrm{sh}\left(1-\dfrac{\pi}{2}i\right)$	*17.* $\mathrm{ch}\left(2+\dfrac{\pi}{4}i\right)$	*18.* $\mathrm{Arctg}1$
19. $\mathrm{Arcsin}\,i$	*20.* $\mathrm{Arccos}1$	*21.* $\left(\sqrt{3}+i\right)^{6i}$

Individual Tasks 2.2

1-2. Separate each of the following expressions into real and imaginary parts.

3-4. Find the value of the given numbers.

5. Solve the following equations, obtaining all roots.

I.		**II.**	
1.	$w=(2+5i)z-iz^2+3i$	*1.*	$w=(3+4i)z^2+7iz+6$
2.	$w=z\cdot\mathrm{Re}(z^2-2z)$	*2.*	$w=z\cdot\mathrm{Im}(3z-z^2)$
3.	$\sin\pi i$	*3.*	$\sin\left(\pi/4-i\right)$
4.	$(-1-i)^{4i}$	*4.*	$(4-3i)^{i}$
5.	$3^{4z}=-i$	*5.*	$5^{3z}=i$

2.3 Derivatives. Analytic Functions. Cauchy–Riemann Equations.

Definition If $f(z)$ is single-valued in some region $\mathfrak{R}$ of the z plane, the derivative of $f(z)$ is defined as

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \qquad (1)$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is *differentiable at* z. In the definition (1), we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true.

If the derivative $f'(z)$ exists at all points z of a region $\Re$, then $f(z)$ is said to be *analytic in* $\Re$ and is referred to as an analytic function in $\Re$ or *a function analytic in* $\Re$. The terms ***regular*** and ***holomorphic*** are sometimes used as synonyms for analytic.

Definition A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

Theorem A necessary and sufficient conditions that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\Re$ is that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

where the partial derivatives in (2) are continuous in $\Re$.

Using the Cauchy–Riemann conditions the derivative $f'(z)$ can be evaluated by one of the following formulas

$$f'(z) = u'_x + i v'_x = v'_y - i u'_y = u'_x - i u'_y = v'_y + i v'_x \tag{3}$$

Note The Cauchy–Riemann equations in the polar coordinates $f(x, y) = u(r\cos\varphi, r\sin\varphi) + iv(r\cos\varphi, r\sin\varphi)$ take the following form

$$\begin{cases} u'_r = \dfrac{1}{r} \cdot v'_\varphi \\ u'_\varphi = -r \cdot v'_r \end{cases} \tag{3a}$$

The functions $u(x, y)$ and $v(x, y)$ are sometimes called *conjugate functions*. Given $u(x, y)$ ($v(x, y)$) having continuous first partials on a simply connected region $\Re$, we can find $v(x, y)$ ($u(x, y)$) [within an arbitrary additive constant] so that $u + iv = w = f(z)$ is analytic.

$$v = \int\limits_{(x_0, y_0)}^{(x, y)} \left(-u'_y(x, y)\,dx + u'_x(x, y)\,dy\right) \tag{4a}$$

$$u = \int\limits_{(x_0, y_0)}^{(x, y)} \left(v'_y(x, y)\,dx - v'_x(x, y)\,dy\right) \tag{4b}$$

If the second partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y exist and are continuous in a region $\Re$, then we find from (2) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{5}$$

Definition Functions such as $u(x,y)$ and $v(x,y)$ which satisfy Laplace's equation in a region $\Re$ are called *harmonic functions* and are said to be harmonic in $\Re$.

Let z_0 be a point P in the z plane and let w_0 be its image P' in the w plane under the transformation $w=f(z)$. Since we suppose that $f(z)$ is single-valued, the point z_0 maps into only one point w_0.

Then $|f'(z_0)|$ is equal to *the coefficient of expansion* at the point z_0 of the $z-$ plane in the w plane under the transformation $w=f(z)$. If $|f'(z_0)|>1$, then *stretching* takes place, and if $|f'(z_0)|<1$, then *compression* occurs.

Let an arbitrary point $z=z_0+\Delta z$ from a neighborhood of the point z_0 moves to the point z_0 along some continuous curve l. Then in the plane w the corresponding point $w=w_0+\Delta w$ will move to the point w_0 along some curve L, which is a map of curve l in the plane w.

The argument $f'(z_0)$ is geometrically equal to the angle at which you need to turn the tangent line at the point z_0 to a smooth curve l in the z plane passing through the point z_0 to get the direction of the tangent line at the point $w_0=f(z_0)$ to the image L of this curve in the w plane under the transformation $w=f(z)$. This angle is called *the rotation angle* at the point z_0 under the transformation $w=f(z)$.

Example 1 Find out which of the following functions are analytic at least at one point

(a) $w=(2+5i)z-iz^2+3i$ *(b)* $w=z^2\cdot\bar{z}$.

Solution

(a) If $z=x+iy$, then

$$w=(2+5i)z-iz^2+3i=(2+5i)(x+iy)-i(x+iy)^2+3i=2x-5y+$$
$$+i(5x+2y)+2xy-i(x^2-y^2)+3i=(2x-5y+2xy)+i(-x^2+y^2+5x+2y+3)$$

$$u=2x-5y+2xy \qquad v=y^2-x^2+5x+2y+3$$

$$u'_x=(2x-5y+2xy)'_x=2+2y \qquad u'_y=(2x-5y+2xy)'_y=-5+2x$$

$$v'_x=(y^2-x^2+5x+2y+3)'_x=-2x+ \qquad v'_y=(y^2-x^2+5x+2y+3)'_y=2y+2$$

Using the Cauchy–Riemann equations we have:

$$\begin{cases} u'_x=v'_y \\ u'_y=-v'_x \end{cases}, \qquad \begin{cases} 2+2y=2y+2 \\ -5+2x=2x-5 \end{cases}, \begin{cases} 2=2 \\ 0=0 \end{cases}.$$

The system has infinity set of solutions, therefore the function is analytic at any points of the complex plane.

(b) If $z=x+iy$, then

$$w = z^2 \cdot \bar{z} = (x+iy)^2(x-iy) = (x^2 - y^2 + 2xyi)(x-iy) = (x^3 + xy^2) + i(x^2y + y^3).$$

$$u = x^3 + xy^2 \qquad\qquad v = x^2y + y^3$$

$$u'_x = (x^3 + xy^2)'_x = 3x^2 + y^2 \qquad\qquad u'_y = (x^3 + xy^2)'_y = 2xy$$

$$v'_x = (x^2y + y^3)'_x = 2xy \qquad\qquad v'_y = (x^2y + y^3)'_y = x^2 + 3y^2$$

Using the Cauchy–Riemann equations we have:

$$\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases}, \quad \begin{cases} 3x^2 + y^2 = x^2 + 3y^2 \\ 2xy = -2xy \end{cases}, \quad \begin{cases} x^2 = y^2 \\ 4xy = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y = 0 \end{cases}.$$

The given function is analytic at origin.

Example 2 (a) Prove that $v = x^2 - y^2 + 2x + 1$ is harmonic. (b) Find $u(x,y)$ such that $f(z) = u + iv$ is analytic.

Solution

(a)

$$v'_x = 2x + 2 \qquad\qquad v'_y = -2y$$

$$v''_{xx} = 2 \qquad\qquad v''_{yy} = -2$$

Adding v''_{xx} and v''_{yy} yields $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ and v is harmonic.

(b) Using the Cauchy–Riemann equations we have:

$$\begin{cases} u'_x = v'_y = -2y, \\ u'_y = -v'_x = -2x - 2. \end{cases}$$

Integrate u'_y with respect to y, keeping x constant. Then

$$u = \int (-2x - 2)dy = -2xy - 2y + F(x),$$

where $F(x)$ is an arbitrary real function of x.

Substitute $-2xy - 2y + F(x)$ into $u'_x = -2y$ and obtain $-2y + F'(x) = -2y$ or $F'(x) = 0$ and $F(x) = c$ is a constant. Then, $u(x,y) = -2xy - 2y + c$.

Example 3 Find a coefficient of expansion and the rotation angle at a given point when mapping $w = u(x,y) + i\,v(x,y)$ is given by:

$$u(x,y) = 3x^2y - y^3, \quad v(x,y) = 3xy^2 - x^3, \quad z_0 = 1 - i.$$

Solution Using the Cauchy–Riemann equations we have:

$$\begin{cases} u'_x = v'_y = 6xy, \\ u'_y = -v'_x = 3x^2 - 3y^2 \end{cases}$$

for all points of the complex plane. Then

$$f'(z) = u'_x + i\,v'_x = v'_y(x,y) - i\,u'_y(x,y),$$

$$f'(z) = u'_x + i\,v'_x = 6xy + i(3y^2 - 3x^2)$$

and find the value of the set point $z_0 = 1 - i$

$$f'(1-i) = \left(6xy + i(3y^2 - 3x^2)\right)\Big|_{\substack{x=1\\ y=-1}} = -6.$$

A coefficient of expansion equals the modulus of a complex number $f'(1-i) = -6 + 0i$,

$$|-6+0i| = \sqrt{36+0} = 6.$$

The rotation angle equals the argument of $f'(1-i) = -6 + 0i$

$$\arg z = \varphi = arctg\frac{0}{-6} = \pi.$$

Differentials. Rules for Differentiation. Derivatives of Elementary Functions

Definition The expression

$$dw = f'(z)dz \tag{6}$$

is called *the differential of* w or $f(z)$, or *the principal part of* Δw. Note that $\Delta w \neq dw$ in general. We call dz the differential of z.

Suppose $f(z)$, $g(z)$ are analytic functions of z. Then the following ***differentiation rules*** (identical with those of elementary calculus) are valid.

1.	$(f \pm g)' = f' \pm g'$.
2.	$(cf)' = c \cdot f'$ where c is any constant.
3.	$(fg)' = f' \cdot g + g' \cdot f$.
4.	$\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - g' \cdot f}{g^2}$ if $g(z) \neq 0$.
5.	$\left(f(g(z))\right)' = f'_g \cdot g'_z$.

If $z = f(t)$ and $w = g(t)$ where t is a parameter, then $w'_z = \dfrac{g'(t)}{f'(t)}$.

In the following we assume that the functions are defined in the similar way as in previous chapter. In the cases where functions have branches, i.e., they are multi-valued, the branch of the function on the right is chosen so as to correspond to the branch of the function on the left. Note that the results are identical with those of elementary calculus.

1.	$(c)' = 0$.	*2.*	$(z^n)' = nz^{n-1}$.
3.	$(e^z)' = e^z$.	*4.*	$(a^z)' = a^z \ln a$.
5.	$(\ln z)' = \dfrac{1}{z}$.	*6.*	$(\log_a z)' = \dfrac{1}{z \ln a}$.

7.	$(\sin z)' = \cos z$.	*8.*	$(\cos z)' = -\sin z$.
9.	$(\mathrm{tg}\, z)' = \dfrac{1}{\cos^2 z}$.	*10.*	$(\mathrm{ctg}\, z)' = -\dfrac{1}{\sin^2 z}$.
11.	$(\arcsin z)' = \dfrac{1}{\sqrt{1-z^2}}$.	*12.*	$(\arccos z)' = -\dfrac{1}{\sqrt{1-z^2}}$.
13.	$(\mathrm{arctg}\, z)' = \dfrac{1}{1+z^2}$.	*14.*	$(\mathrm{arcctg}\, z)' = -\dfrac{1}{1+z^2}$.
15.	$(\mathrm{sh}\, z)' = \mathrm{ch}\, z$.	*16.*	$(\mathrm{ch}\, z)' = \mathrm{sh}\, z$.
17.	$(\mathrm{th}\, z)' = \dfrac{1}{\mathrm{ch}^2 z}$.	*18.*	$(\mathrm{cth}\, z)' = -\dfrac{1}{\mathrm{sh}^2 z}$.

Higher Order Derivatives. L'Hospital's Rule. Singular Points

If $w = f(z)$ is analytic in a region $\Re$, its derivative is given by $f'(z)$, w' or $\dfrac{dw}{dz}$. If $f'(z)$ is also analytic in the region $\Re$, its derivative is denoted by $f''(z)$, w'', or $\dfrac{d^2w}{dz^2}$. Similarly, the n - th derivative of $f(z)$, if it exists, is denoted by $f^{(n)}(z)$, $w^{(n)}$, or $\dfrac{d^n w}{dz^n}$ where n is called *the order of the derivative*. Thus the derivatives of the first, second, third, etc. orders are given by $f'(z)$, $f''(z)$, ... Computations of these higher order derivatives follow the repeated application of the above differentiation rules.

Theorem 1 Suppose $f(z)$ is analytic in a region $\Re$. Then so also are $f'(z)$, $f''(z)$, ... analytic in $\Re$, i.e., all higher derivatives exist in $\Re$.

Let $f(z)$ and $g(z)$ be analytic in a region R containing the point z_0 and suppose that $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Then, ***L'Hospital's rule*** states that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \qquad (7)$$

In the case of $f'(z_0) = g'(z_0) = 0$, the rule may be extended.

Definition The point $z = z_0$ is called *a zero of* $f(z)$ if $f(z_0) = 0$. If $f(z_0) = f'(z_0) = ... = f^{(k-1)}(z_0) = 0$, but $f^{(k)}(z_0) \neq 0$ then $z = z_0$ is called *a zero of* $f(z)$ *of order k.*

Definition A point at which $f(z)$ fails to be analytic is called *a singular point* or *singularity of* $f(z)$. Various types of singularities exist.

1. Isolated Singularities

Definition The point $z = z_0$ is called *an isolated singularity* or *an isolated singular point of* $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e., there exists a deleted δ neighborhood of z_0 containing no singularity). If no such δ can be found, we call z_0 *a non-isolated singularity.*

Definition If z_0 is not a singular point and we can find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then we call z_0 *an ordinary point of* $f(z)$.

2. Poles

Definition If z_0 is an isolated singularity and we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called *a pole of order n*. If $n = 1$, z_0 is called *a simple pole*.

3. Branch Points

Branch Points of multiple-valued functions, already considered in the previous chapter, are non-isolated singular points since a multiple-valued function is not continuous and, therefore, not analytic in a deleted neighborhood of a branch point.

4. Removable Singularities

Definition An isolated singular point z_0 is called *a removable singularity of* $f(z)$ if $\lim_{z \to z_0} f(z)$ exists. By defining $f(z_0) = \lim_{z \to z_0} f(z)$, it can then be shown that $f(z)$ is not only continuous at z_0 but is also analytic at z_0.

5. Essential Singularities

Definition If $f(z)$ does not have the limit at the point z_0 then it is called *an essential singularity*.

If a function has an isolated singularity, then the singularity is either a removable one, a pole, or an essential singularity. For this reason, a pole is sometimes called *a non-essential singularity*. Equivalently, $z = z_0$ is an essential singularity if we cannot find any positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

6. Singularities at Infinity

The type of singularity of $f(z)$ at $z = \infty$ (the point at infinity) is the same as that of $f\left(1/w\right)$ at $w = 0$.

For methods of classifying singularities using infinite series, see next chapter.

Example 4 Using rules of differentiation, find the derivatives of each of the following functions:

(a) $\cos^2(2z + 3i)$ *(b)* $z\,\mathrm{tg}(\ln z)$ *(c)* $\left(z - 3i\right)^{4z+2}$

Solution Using the chain rule, we have

(a) $\left(\cos^2(2z + 3i)\right)' = -2\cos(2z + 3i)\sin(2z + 3i)2 = -4\sin(4z + 6i)$.

(b) $\left(z \cdot \mathrm{tg}(\ln z)\right)' = \mathrm{tg}(\ln z) + \left(\mathrm{tg}(\ln z)\right)' \cdot z = \mathrm{tg}(\ln z) + \dfrac{z}{\cos^2 \ln z} \cdot \dfrac{1}{z}$.

(c) $\left((z - 3i)^{4z+2}\right)' = \left(\mathrm{e}^{(4z+2)\ln(z-3i)}\right)' = \mathrm{e}^{(4z+2)\ln(z-3i)} \cdot \left(4\ln(z - 3i) + \dfrac{4z + 2}{z - 3i}\right)$.

Example 5 Suppose $w^3 - 3z^2w + 4\ln z = 0$. Find $\dfrac{dw}{dz}$.

Solution Differentiating with respect to z, considering w as an implicit function of z, we have $3w^2w' - 3z^2w' - 6zw + \dfrac{4}{z} = 0$. Then, solving for $\dfrac{dw}{dz}$, we obtain

$$\frac{dw}{dz} = \frac{4zw - \frac{4}{z}}{3w^2 - 3z^2}.$$

Example 6 Evaluate

(a) $\lim\limits_{z \to i} \dfrac{z^{10} + 1}{z^6 + 1}$ *(b)* $\lim\limits_{z \to 0} \dfrac{1 - \cos z}{z^2}$

Solution

(a) Let $f(z) = z^{10} + 1$ and $g(z) = z^6 + 1$. Then $f(i) = g(i) = 0$. Also, $f(z), g(z)$ are analytic at $z = i$.

Hence, by L'Hospital's rule

$$\lim_{z \to i} \frac{z^{10} + 1}{z^6 + 1} = \left(\frac{0}{0}\right) = \lim_{z \to i} \frac{10z^9}{6z^5} = \lim_{z \to i} \frac{5}{3} z^4 = \frac{5}{3}$$

(b) Let $f(z) = 1 - \cos z$ and $g(z) = z^2$. Then $f(0) = g(0) = 0$. Also, $f(z), g(z)$ are analytic at $z = 0$.

Hence, by L'Hospital's rule

$$\lim_{z \to 0} \frac{1 - \cos z}{z^2} = \left(\frac{0}{0}\right) = \lim_{z \to 0} \frac{\sin z}{2z} = \frac{1}{2} \lim_{z \to 0} \frac{\sin z}{z} = \frac{1}{2}$$

***Example** 7* Classify all the singularities of the functions.

(a) The function $f(z) = \dfrac{1}{(z-3)^4}$ has a pole of order 4 at $z = 3$.

(b) The function $f(z) = \dfrac{3z - 2}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z = 1$, and simple poles at $z = -1$ and $z = 4$.

(c) The function $f(z) = (z-3)^{\frac{1}{2}}$ has a branch point at $z = 3$.

(d) The function $f(z) = \ln(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$, i.e., at $z = 1$ and $z = -2$.

(e) The singular point $z = 0$ is a removable singularity of $f(z) = \dfrac{\sin z}{z}$ since $\lim\limits_{z \to 0} \dfrac{\sin z}{z} = 1$.

(f) The function $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z = 2$.

(g) The function $f(z)=z^3$ has a pole of order 3 at $z=\infty$, since $f\left(1/w\right)=1/w^3$ has a pole of order 3 at $w=0$.

Exercise Set 2.3

In Exercises 1 to 4, find out which of the following functions are analytic at least at one point.

1. $w=i\cos z$ *2.* $w=\left|z\right|\cdot \operatorname{Re} z$ *3.* $w=\overline{z}\cdot \operatorname{Im} z$

4. $w=(3+4i)z^2+7iz+6$

In Exercises 5 to 10, prove that given function is harmonic. Find $u(x,y)$ or $v(x,y)$ such that $f(z)=u+iv$ is analytic.

5. $u=x^2-y^2+x,\ f(0)=0$. *6.* $u=1-\mathrm{e}^x\sin y,\ f(0)=1+i$.

7. $v=3x^2y-y^3,\ f(0)=1$. *8.* $u=\mathrm{e}^{-y}\cos x+x,\ f(0)=1$.

9. $u=3x^2y-y^3,\ f(1-i)=0$. *10.* $v=-\mathrm{e}^{1+y}\sin x,\ f\left(\pi/4+i\right)=0$.

In Exercises 11 to 14, find a coefficient of expansion and the rotation angle at this point when the mapping given by the following transformation

11. $u(x,y)=x^2+2x-y^2$, $v(x,y)=2xy+2y$, $z_0=i$.

12. $u(x,y)=x^3-3xy^2+x^2-y^2$, $v(x,y)=3x^2y-y^3+2xy$, $z_0=2i/3$.

13. $u(x,y)=x^3-3xy^2+3x,\quad v(x,y)=3x^2y-y^3+3y-1,\quad z_0=-1+i$.

14. $u(x,y)=\mathrm{e}^{1+y}\cos x,\quad v(x,y)=-\mathrm{e}^{1+y}\sin x,\quad z_0=\pi/4+i$.

In Exercises 15 to 23, using rules of differentiation, find the derivatives of each of the following functions:

15. $\sin^3(5z+7i)$ *16.* $\ln\left(\operatorname{tg}5z\right)$ *17.* $z\mathrm{e}^{z-i}$

18. $(z^2-3z)\cos 4z$ *19.* $\left(z+4i\right)^{i-2z}$ *20.* $\left(z^2+2z\right)^{\cos z}$

21. $\dfrac{z^2-3}{\operatorname{sh}2z}$ *22.* $\dfrac{\operatorname{ctg}^5 7z}{\ln(z+3)}$ *23.* $\arcsin\left(\dfrac{3z-7i}{z^2+i}\right)$

24. Suppose $w^4-5z^2w^2+4\sin z=0$. Find $\dfrac{dw}{dz}$.

In Exercises 25 to 33, evaluate the following limits.

25. $\lim\limits_{z\to 1}\dfrac{z^{10}-1}{z^6-1}$ *26.* $\lim\limits_{z\to 2}\dfrac{z^2-4}{z^2+z-6}$ *27.* $\lim\limits_{z\to 3i}\dfrac{z^2+9}{z-3i}$

28. $\lim\limits_{z\to 0}\dfrac{\cos 4z-\cos z}{3z^2}$ *29.* $\lim\limits_{z\to 0}\dfrac{1-\cos z}{z\operatorname{tg}z}$. *30.* $\lim\limits_{z\to \infty}\dfrac{\mathrm{e}^z}{z^2}$

31. $\lim\limits_{z\to \infty}\dfrac{5z^2+z-4}{z^2+z-6}$ *32.* $\lim\limits_{z\to \infty}\left(z-2i\right)^{z+i}$ *33.* $\lim\limits_{z\to \pi/2}\left(\operatorname{ctg}z\right)^{\operatorname{tg}z}$

Individual Tasks 2.3

1. Find out which of the following functions are analytic at least at one point.

2. Prove that the given function is harmonic. Find $u(x,y)$ or $v(x,y)$ such that $f(z)=u+iv$ is analytic.

3. Find a coefficient of expansion and the rotation angle at this point when mapping given by the following transformation.

4-5. Differentiate the following functions.

I.		**II.**	
1.	$w=i(1-z^2)-2z$	*1.*	$w=2z-iz^2$
2.	$u(x,y)=-2e^x\sin y+x+y$	*2.*	$v(x,y)=2e^x\cos y+y-x$
3.	$u(x,y)=e^{-x}\sin y$, $v(x,y)=e^{-x}\cos y$, $z=\pi i$	*3.*	$u(x,y)=3xy^2-x^3$, $v(x,y)=y^3-3x^2y$, $z=-1+i$
4.	$w=(z-i)^{4iz}$	*4.*	$w=(z-3i)^{iz}$
5.	$w=\dfrac{\mathrm{tg}^4 3z}{\ln(z+3)}$	*5.*	$w=\dfrac{\ln^2 6z}{arctg(z+3)}$

2.4 Complex Integration

Complex Line Integrals

Let $f(z)$ be continuous at all points of a curve C (Figure 14), which we shall assume has a finite length, i.e., C is a rectifiable curve.

Subdivide C into n parts by means of points $z_0, z_1, z_2, ..., z_{n-1}, z_n$, chosen arbitrarily, and call $z_0=a, b=z_n$. On each arc joining z_{k-1} to z_k [where k goes from 1 to n], choose a point ξ_k. Form the sum

$$S_n = f(\xi_1)(z_1-a)+...+f(\xi_n)(b-z_{n-1}) \tag{1}$$

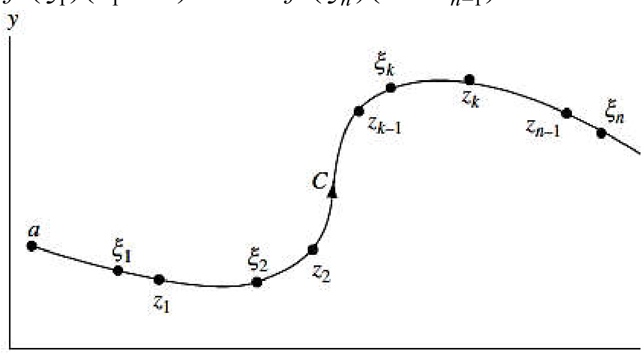

Figure 14

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \tag{2}$$

Definition Let the number of subdivisions n increases in such a way that the largest of the chord lengths Δz_k approaches zero. Then, since $f(z)$ is continuous, the sum S_n approaches a limit that does not depend on the mode of subdivision and we denote this limit by

$$\lim_{\substack{n\to\infty \\ \Delta z_k\to 0}} S_n = \lim_{\substack{n\to\infty \\ \Delta z_k\to 0}} \sum_{k=1}^{n} f(\xi_k)\Delta z_k = \int_a^b f(z)dz = \int_C f(z)dz \qquad (3)$$

and it is called *the complex line integral* or simply *line integral of* $f(z)$ *along curve* C, or the definite integral of $f(z)$ from a to b along curve C.

Suppose $f(z) = u(x,y) + i\,v(x,y)$. Then the complex line integral (3) can be expressed in terms of real line integrals as follows

$$\begin{aligned}\int_C f(z)dz &= \int_C \big(u(x,y) + iv(x,y)\big)(dx + i\,dy) = \\ &= \int_C \big(u(x,y)dx - v(x,y)dy\big) + i\int_C \big(v(x,y)dx - u(x,y)dy\big)\end{aligned} \qquad (4)$$

For this reason, (4) is sometimes taken as a definition of a complex line integral.

Let $z = z(t), \begin{cases} x = x(t), \\ y = y(t), \end{cases}$ $\alpha \le t \le \beta$ be a continuous function of a complex variable $t = u + iv$. Suppose that curve C in the z plane corresponds to curve C' in the z plane and that the derivative $z'(t)$ is continuous on C'.

Then

$$\int_C f(z)dz = \int_\alpha^\beta f(z(t))\cdot z'(t)\,dt = \int_{C'} f(z(t))\cdot z'(t)\,dt \qquad (5)$$

These conditions are certainly satisfied if z is analytic in a region containing curve C'.

Example 1 Evaluate $\int_C \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by:

(a) $z = t^2 + i\,t$, *(b)* the line from $z = 0$ to $z = 2i$

Solution

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$, respectively. Then, the line integral equals

$$\int_0^2 \left(t^2 - it\right)d\left(t^2 + it\right) = \int_0^2 \left(t^2 - it\right)(2t + i)\,dt = \int_0^2 \left(2t^3 - it^2 + t\right)dt = 10 - \frac{8i}{3}$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + idy) = \int_C xdx + ydy + i\int_C xdy - ydx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from $(0;0)$ to $(0;2)$ for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_0^2 ydy + i\int_0^2 0dy = \int_0^2 ydy = 2$$

Definition A region $\Re$ is called *simply-connected* if any simple closed curve, which lies in $\Re$, can be shrunk to a point without leaving $\Re$. A region $\Re$, which is not simply-connected, is called *multiply-connected*.

Definition Any continuous, closed curve that does not intersect itself and may or may not have a finite length, is called a *Jordan curve*.

Definition The boundary C of a region is said to be traversed in the *positive sense* or *direction* if an observer travelling in this direction (and perpendicular to the plane) has the region to the left. This convention leads to the directions indicated by the arrows in Figures 15, 16, and 17.

We use a special symbol $\oint\limits_C f(z)dz$ to denote integration of $f(z)$ around the boundary C in the positive sense. In the case of a circle (Figure 15), the positive direction is the *counterclockwise direction*. The integral around C is often called a *contour integral*.

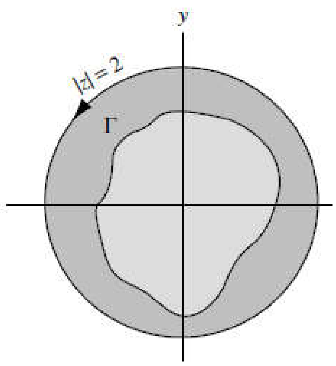

Figure 15

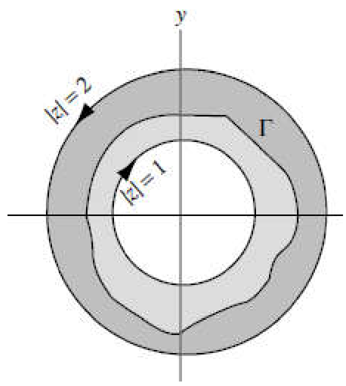

Figure 16

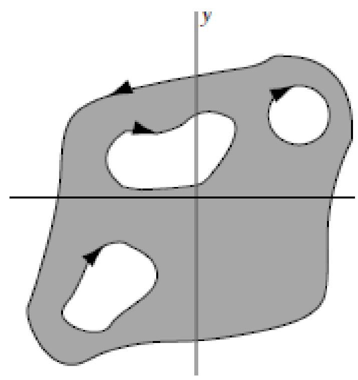

Figure 17

Theorem Let $f(z)$ be analytic in a region $\Re$ and on its boundary C. Then

$$\oint\limits_C f(z)dz = 0 \qquad (6)$$

This fundamental theorem, often called *Cauchy's integral theorem* or simply *Cauchy's theorem*, is valid for both simply- and multiply-connected regions.

Theorem Let $f(z)$ be continuous in a simply-connected region $\Re$ and suppose that $\oint\limits_C f(z)dz = 0$ around every simple closed curve C in $\Re$. Then $f(z)$ is an analytic function in $\Re$.

Example 2 Evaluate $\oint\limits_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z = a$ is *(a)* outside C, *(b)* inside C.

Solution

(a) If a is outside C, then $f(z)=\dfrac{1}{z-a}$ is analytic everywhere inside and on C. Hence, by Cauchy's theorem, $\oint\limits_C \dfrac{dz}{z-a}=0$.

(b) Suppose a is inside C and let Γ be a circle of radius e with center at $z=a$ so that Γ is inside C (this can be done since $z=a$ is an interior point).

$$\oint\limits_C \frac{dz}{z-a}=\oint\limits_\Gamma \frac{dz}{z-a} \tag{7}$$

Now on Γ, $|z-a|=\varepsilon$ or $z-a=\varepsilon e^{i\varphi}$, i.e., $z=a+\varepsilon e^{i\varphi}$, $0\le\varphi<2\pi$. Thus, since $dz=i\varepsilon e^{i\varphi}d\varphi$, the right side of (7) becomes

$$\int\limits_0^{2\pi}\frac{i\varepsilon e^{i\varphi}d\varphi}{\varepsilon e^{i\varphi}}=i\int\limits_0^{2\pi}d\varphi=2\pi i$$

which is the required value.

Definition Suppose $f(z)$ and $F(z)$ are analytic in a region $\Re$ and such that $F'(z)=f(z)$. Then $F(z)$ is called *an indefinite integral* or *anti-derivative of* $f(z)$ denoted by

$$F(z)=\int f(z)dz+A,\ A-const \tag{8}$$

Just as in real variables, any two indefinite integrals differ by a constant. For this reason, an arbitrary constant A is often added to the right of (8).

Table of Indefinite Integrals

$\int z^n dz=\dfrac{z^{n+1}}{n+1}, n\ne -1$.	$\int \dfrac{dz}{z}=\ln z$.
$\int a^z dz=\dfrac{a^z}{\ln a}$.	$\int \mathrm{e}^z\, dz=\mathrm{e}^z$.
$\int \cos z\, dz=\sin z$.	$\int \sin z\, dz=-\cos z$.
$\int \dfrac{dz}{\cos^2 z}=\mathrm{tg}\, z$.	$\int \dfrac{dz}{\sin^2 z}=-\mathrm{ctg}\, z$.
$\int \dfrac{dz}{z^2-a^2}=\dfrac{1}{2a}\ln\left\|\dfrac{z-a}{z+a}\right\|$.	$\int \dfrac{dz}{\sqrt{z^2\pm a^2}}=\ln\left\|z+\sqrt{z^2\pm a^2}\right\|$.
$\int \dfrac{dz}{z^2+a^2}=\dfrac{1}{a}\mathrm{arctg}\dfrac{z}{a}$.	$\int \dfrac{dz}{\sqrt{a^2-z^2}}=\arcsin\dfrac{z}{a}$.
$\int \mathrm{sh}\, z\, dz=\mathrm{ch}\, z$.	$\int \mathrm{ch}\, z\, dz=\mathrm{sh}\, z$.

Example 3 Determine

(a) $\int \sin z \,\sin 3z\, \sin 2z\, dz$ *(b)* $\int z^2\sqrt{4+z^3}\,dz$

Solution

(a) $\int \sin z \sin 3z \sin 2z\, dz = \frac{1}{2}\int (\cos 2z - \cos 4z)\sin 2z\, dz =$

$= \frac{1}{2}\int \cos 2z \sin 2z dz - \frac{1}{2}\int \cos 4z \sin 2z dz = \frac{1}{4}\int \sin 4z dz - \frac{1}{4}\int (\sin 6z - \sin 2z) dz =$

$= -\frac{\cos 4z}{16} + \frac{\cos 6z}{24} - \frac{\cos 2z}{8} + A.$

(b) $\int z^2\sqrt{4+z^3}\,dz = \frac{1}{3}\int \left(4+z^3\right)^{\frac{1}{2}}\left(4+z^3\right)' dz = \frac{1}{3}\int \left(4+z^3\right)^{\frac{1}{2}} d(4+z^3) =$

$= \frac{2}{9}\left(4+z^3\right)^{\frac{3}{2}} + C = \frac{2}{9}\sqrt{(4+z^3)^3} + A\,.$

Theorem Suppose a and b are any two points in $\Re$ and $F'(z) = f(z)$. Then

$$\int_a^b f(z)dz = F(b) - F(a) \qquad (9)$$

This can also be written in the form familiar from elementary calculus

$$\int_a^b F'(z)dz = \left(F(z)\right)\Big|_a^b = F(b) - F(a) \qquad (10)$$

Example 4 Calculate $I = \int_{\mathrm{e}}^{\mathrm{e}^2} z\ln z dz$.

Solution Using the formula for integration by parts we get

$$I = \begin{vmatrix} u = \ln z, & du = \dfrac{dz}{z} \\ dv = z\,dz, & v = \dfrac{z^2}{2} \end{vmatrix} = \left(\frac{z^2}{2}\ln z\right)\Bigg|_{\mathrm{e}}^{\mathrm{e}^2} - \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{z^2dz}{2z} = \frac{\mathrm{e}^4}{2}\cdot 2 - \frac{\mathrm{e}^2}{2} - \left(\frac{z^2}{4}\right)\Bigg|_{\mathrm{e}}^{\mathrm{e}^2} =$$

$$= \mathrm{e}^4 - \frac{\mathrm{e}^2}{2} - \frac{\mathrm{e}^4}{4} + \frac{\mathrm{e}^2}{4} = \frac{1}{4}\left(3\mathrm{e}^4 - \mathrm{e}^2\right) \approx 39{,}10\,.$$

Exercise Set 2.4

In Exercises 1 to 14, evaluate

1. $\int_{1+i}^{2i} \left(z^3 + z\right)\mathrm{e}^{\frac{z^2}{2}}\, dz$

2. $\int_0^i z \sin z\, dz$

3. $\int\limits_{\gamma}(3-2z^2+\sin z)\,dz,\quad \gamma:|z|=2,\ \mathrm{Im}\,z\ge 0$

4. $\int\limits_{\gamma}(z^3+\cos z)\,dz,\quad \gamma:|z|=1,\ \mathrm{Re}\,z\ge 0$

5. $\int\limits_{0}^{i} z\cos z\,dz$

6. $\int\limits_{\gamma}(3z^2+2z)\,dz$, γ is the line from $z=1-i$ to $z=2i$

7. $\int\limits_{l}\mathrm{Re}\,z\cdot\mathrm{Re}(z^2)\,dz$, l is the arc of the parabola $y=1-x^2$ from $z_1=-1$ to $z_2=i$

8. $\int\limits_{l}\mathrm{Im}\,z\cdot\mathrm{Re}(z^2)\,dz$ γ is the line from $z_1=-1+i$ to $z_2=1+3i$

9. $\int\limits_{\gamma}\bar{z}^2\,dz,\quad \gamma$ is the line from $z=0$ to $z=2i$

10. $\int\limits_{\gamma}\bar{z}dz$, γ is the broken line OAB, where $O(0;0)$, $A(1;1)$, $B(1;0)$

11. $\int\limits_{\gamma}\bar{z}dz$, $\gamma:|z|=2,\ \mathrm{Im}\,z<0$

12. $\oint\limits_{\gamma} z\,\mathrm{Re}\,z\,dz,\quad \gamma:|z|=1$

13. $\int\limits_{\gamma} z\,\mathrm{Im}\,z^2dz,\quad \gamma:\ |z|=1,-\pi\le\arg z\le 0$

14. $\int\limits_{1}^{i}\frac{\ln(z+1)}{z+1}dz$, $\left(|z|=1,\ \mathrm{Im}\,z\ge 0,\ \mathrm{Re}\,z\ge 0\right)$

In Exercises 15 to 23 determine

15. $\int\sqrt{\sin z}\cos z\,dz$

16. $\int\frac{dz}{\sin^2(1-3z)}$

17. $\int\frac{dz}{2z^2+6z+4}$

18. $\int\frac{z^3dz}{\sqrt{5+z^4}}$

19. $\int\frac{dz}{\cos^2(3z+2)}$

20. $\int z^3\,\mathrm{e}^{-z^2}\,dz$

21. $\int(5z^2+1)\mathrm{e}^{-2z}\,dz$

22. $\int\frac{dz}{1+\sqrt{z+3}}$

23. $\int\sin^7 z\cos^5 z\,dz$

Individual Tasks 2.4

1 -5. Evaluate the following integrals.

I.		**II.**	
1.	$\int\limits_\gamma (3-2z^2+\sin z)\,dz,\ \gamma: \lvert z\rvert=2$	*1.*	$\int\limits_\gamma \frac{\bar{z}}{z}dz,\ \gamma: 1\le \lvert z\rvert \le 2,\ \operatorname{Im} z$
2.	$\int\limits_\gamma z\cdot\bar{z}\,dz,\quad \gamma: (\lvert z\rvert=1,\ \operatorname{Im} z\le 0)$	*2.*	$\int\limits_\gamma (\sin iz+z)\,dz,\ \gamma: \lvert z\rvert=1,$
3.	$\int\limits_{3i}^{6i} (z+3)e^{z/3}\,dz$	*3.*	$\int\limits_{i}^{2i} (2z+3)e^{z}\,dz$
4.	$\int \frac{z^2dz}{\sqrt{9+z^3}}$	*4.*	$\int \frac{dz}{2-\sqrt{z-1}}$
5.	$\int \frac{dz}{z^2+6z+10}$	*5.*	$\int \frac{tg(3z+2)dz}{\cos^2(3z+2)}$

2.5 Cauchy's Integral Formulas

Theorem Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 (where C_1 lies inside C as in Figure 11) and on these curves. Then

$$\oint\limits_C f(z)dz = \oint\limits_{C_1} f(z)dz,$$

where C and C_1 are both traversed in the positive sense relative to their interiors (counterclockwise in Figure 18).

The result shows that if we wish to integrate $f(z)$ along curve C, we can equivalently replace C by any curve C_1 so long as $f(z)$ is analytic in the region between C and C_1 as in Figure 18.

Theorem Let $f(z)$ be analytic in a region bounded by the non-overlapping simple closed curves $C, C_1, C_2, ..., C_n$ where $C_1, ..., C_n$ are inside C (as in Figure 19) and on these curves. Then

$$\oint\limits_C f(z)dz = \oint\limits_{C_1} f(z)dz + \oint\limits_{C_2} f(z)dz + \oint\limits_{C_3} f(z)dz + ... + \oint\limits_{C_n} f(z)dz.$$

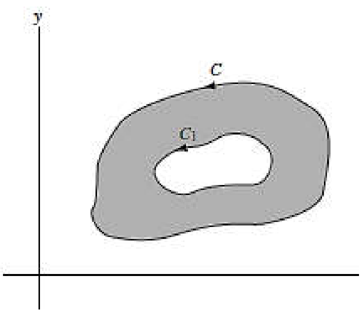

Figure 18

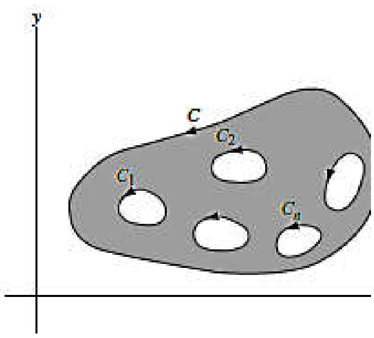

Figure 19

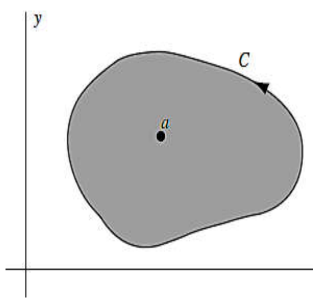

Figure 20

Cauchy's Integral Formulas

Theorem Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C (Figure 20). Then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = f(a) \tag{1}$$

where C is traversed in the positive (counterclockwise) sense.

Also, the n-th derivative of $f(z)$ at $z=a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz, \quad n \ge 1 \tag{2}$$

Formula 2 can be rewritten as the following

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a), \quad n \ge 1 \tag{2a}$$

The result (1) can be considered a special case of (2) with $n=0$ if we define $0!=1$.

The results (1) and (2) are called *Cauchy's integral formulas* and are quite remarkable because they show that if a function $f(z)$ is known on the simple closed curve C, then the values of the function and all its derivatives can be found at all points inside C.

Example 1 Evaluate

(a) $\displaystyle\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz$ (b) $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4}dz$

where C is the circle $|z|=3$.

Solution *(a)* Since $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$, we have

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}dz - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}dz.$$

By Cauchy's integral formula with $a=2$ and $a=1$, respectively, we have

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2 dz}{(z-2)} = 2\pi i\left(\sin\pi 2^2 + \cos\pi 2^2\right) = 2\pi i,$$

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2 dz}{(z-1)} = 2\pi i\left(\sin\pi 1^2 + \cos\pi 1^2\right) = -2\pi i,$$

since $z=1$ and $z=2$ are inside C and $\sin\pi z^2 + \cos\pi z^2$ is analytic inside C. Then, the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z)=e^{2z}$ and $a=-1$ in the Cauchy integral formula $f^{(n)}(a) = \dfrac{n!}{2\pi i}\displaystyle\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$.

If $n=3$, then $f'(z)=2\mathrm{e}^{2z}$, $f''(z)=4\mathrm{e}^{2z}$, $f'''(z)=8\mathrm{e}^{2z}$ and $f'''(-1)=8\mathrm{e}^{-2}$. Hence the Cauchy integral formula becomes

$$8\mathrm{e}^{-2}=\frac{3\,!}{2\pi i}\oint_C \frac{\mathrm{e}^{2z}}{\left(z+1\right)^4}dz$$

from which we see that the required integral has the value $\dfrac{8\pi i\,\mathrm{e}^{-2}}{3}$.

Exercise Set 2.5

In Exercises 1 to 16, evaluate

1. $\displaystyle\oint_{|z-1|=3}\frac{z}{z^2-2z+3}dz$

2. $\displaystyle\oint_{|z-3|=6}\frac{z\,dz}{(z-2)^3(z+4)}$

3. $\displaystyle\oint_{|z|=4}\frac{dz}{(z^2+9)(z+9)}$

4. $\displaystyle\oint_{\gamma}\frac{z-1}{(z^2-2z+3)^2}dz,\ \ \gamma:\left|z-1-i\right|=\frac{3}{2}$

5. $\displaystyle\oint_{|z+i|=1}\frac{i\,dz}{(z^2+1)^2}$

6. $\displaystyle\oint_{|z|=1}\frac{3z^4-2z^3+5}{z^4}dz$

7. $\displaystyle\oint_{\gamma}\frac{2\,dz}{z^2(z-1)},\ \ \gamma:\left|z-1-i\right|=\frac{5}{4}$

8. $\displaystyle\frac{1}{2\pi i}\oint_{|z-2|=3}\frac{\cos^2 z+1}{z(z-\pi)}dz$

9. $\displaystyle\oint_{|z-3|=1}\frac{\sin 3z+2}{z^2(z-\pi)}dz$

10. $\displaystyle\oint_{|z-1|=1}\frac{\cos\frac{\pi}{4}z}{(z^2-1)^2}dz$

11. $\displaystyle\oint_{|z|=1}\frac{\mathrm{e}^{iz}-1}{z^3}dz$

12. $\displaystyle\oint_{|z|=1}\frac{2+\sin z}{z(z+2i)}dz$

13. $\displaystyle\frac{1}{2\pi i}\oint_{|z+1|=3}\frac{3z^2+2z+4}{\left(z^2+4\right)\cdot\sin\frac{z}{2}}dz$

14. $\displaystyle\oint_{|z|=2}\frac{\sin z\cdot\sin(z-1)}{z^2-z}dz$

15. $\displaystyle\oint_{|z|=1/3}\frac{1-\sin z}{z^2}dz$

16. $\displaystyle\oint_{|z|=3}\frac{z\,\mathrm{sh}\,z}{(z^2-1)^2}dz$

Individual Tasks 2.5

1-4. Evaluate the following integrals.

I.	**II.**
1. $\oint\limits_{\lvert z\rvert=2} \frac{2z+1}{(z-3)(z+4)}dz$	*1.* $\oint\limits_{\lvert z\rvert=4} \frac{z-3}{(z+5)(z-6)}dz$
2. $\oint\limits_{\lvert z\rvert=2} \frac{2z-3}{z^2-2z-3}dz$	*2.* $\oint\limits_{\lvert z\rvert=3} \frac{3z+1}{z^2-2z-8}dz$
3. $\oint\limits_{\lvert z-1\rvert=1} \frac{\cos(\pi z/4)}{z^2-1}dz$	*3.* $\oint\limits_{\lvert z+1{,}5\rvert=1} \frac{\cos^2 z+3}{2z^2+\pi z}dz$
4. $\oint\limits_{C} \frac{\cos z}{(z-1+3i)(z-7+3i)}dz,$ $C:\lvert z-1+3i\rvert$	*4.* $\oint\limits_{C} \frac{\sin z dz}{(z-1-2i)(z-1-10i)}$ $C:\lvert z-1-2i\rvert=4$

2.6 Series of Functions. Power Series. Taylor's Theorem

Definition The sequence $\{S_n(z)\}$ symbolized by

$$u_1(z)+u_2(z)+u_3(z)+...+u_n(z)+...=\sum_{n=1}^{\infty}u_n(z) \qquad (1)$$

and is called *an infinite series*.

If $\lim\limits_{n\to\infty}S_n(z)=S(z)$, the series is called *convergent* and $S(z)$ is its sum; otherwise, the series is called *divergent*. We sometimes write $\sum\limits_{n=1}^{\infty}u_n(z)$ as $\sum u_n(z)$ or $\sum u_n$ for brevity.

Definition If a series converges for all values of z (points) in a region $\Re$, we call $\Re$ the *region of convergence* of the series.

Definition A series $\sum\limits_{n=1}^{\infty}u_n(z)$ is called *absolutely convergent* if the series of absolute values, i.e., $\sum\limits_{n=1}^{\infty}\lvert u_n(z)\rvert$, converges.

Definition If $\sum\limits_{n=1}^{\infty}u_n(z)$ converges but $\sum\limits_{n=1}^{\infty}\lvert u_n(z)\rvert$ does not converge, we call $\sum\limits_{n=1}^{\infty}u_n(z)$ *conditionally convergent*.

Definition A series having the form

$$\sum_{n=0}^{\infty}a_n(z-a)^n=a_0+a_1(z-a)+a_2(z-a)^2+...+a_n(z-a)^n+... \qquad (2)$$

is called *a power series in* $z-a$.

Geometrically, if Γ is a circle of radius R with the center at $z = a$, then the series (2) converges at all points inside Γ and diverges at all points outside Γ, while it may or may not converge on the circle Γ. We can consider the special cases $R = 0$ and $R = \infty$, respectively, to be the cases where (2) converges only at $z = a$ or converges for all (finite) values of z. Because of this geometrical interpretation, R is often called *the radius of convergence* of (2) and the corresponding circle is called *the circle of convergence.*

Example 1 Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{4^n \cdot (n+1)^3}$.

Solution

If $u_n = \frac{(z+2)^{n-1}}{4^n \cdot (n+1)^3}$, then $u_{n+1} = \frac{(z+2)^n}{4^{n+1} \cdot (n+2)^3}$. Hence, excluding $z = -2$ for which the given series converges, we have

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(z+2)}{4} \cdot \frac{(n+1)^3}{(n+2)^3} \right| = \frac{|z+2|}{4}$$

Then the series converges (absolutely) for $\frac{|z+2|}{4} < 1$, i.e., $|z+2| < 4$. The point $z = -2$ is included in $|z+2| < 4$.

If $\frac{|z+2|}{4} = 1$, i.e., $|z+2| = 4$, the ratio test fails. However, it is seen that in this case

$$\left| \frac{(z+2)^{n-1}}{4^n \cdot (n+1)^3} \right| = \frac{1}{4(n+1)^3} \le \frac{1}{n^3}$$

and since $\sum \frac{1}{n^3}$ converges [α series with $\alpha = 3$], the given series converges (absolutely).

It follows that the given series converges (absolutely) for $|z+2| \le 4$. Geometrically, this is the set of all points inside and on the circle of radius 4 with the center at $z = -2$, called the circle of convergence (shown shaded in Figure 21). The radius of convergence is equal to 4.

Let $f(z)$ be analytic inside and on a simple closed curve C. Then

$$f(z) = f(a) + f'(a) \cdot (z-a) + \frac{f''(a)}{2!}(z-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(z-a)^n + ... \qquad (3)$$

is called Taylor's theorem and the series (3) is called *a Taylor series* or expansion for $f(z)$.

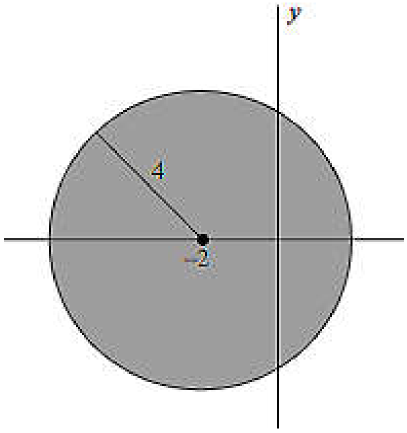

Figure 21

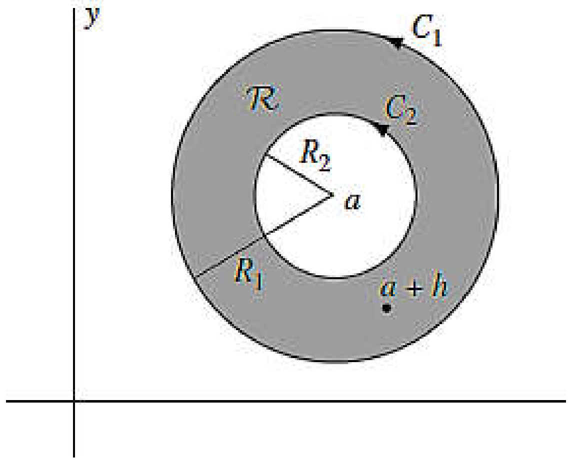

Figure 22

The region of convergence of the series (3) is given by $|z-a|<R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z-a|=R$, the series may or may not converge. For $|z-a|>R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e., the series converges for all z. If $a=0$ in (3), the resulting series is often called a *Maclaurin series.*

The following list shows some special series together with their regions of convergence. In the case of multiple-valued functions, the principal branch is used.

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$	$\lvert z\rvert < \infty$
$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} + \ldots$	$\lvert z\rvert < \infty$
$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + (-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!} + \ldots$	$\lvert z\rvert < \infty$
$\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^{n-1} + \ldots$	$\lvert z\rvert < 1$
$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + (-1)^{n-1}\frac{z^n}{n} + \ldots$	$\lvert z\rvert < 1$
$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}z^n + \ldots$	$\lvert z\rvert < 1$

Laurent Series. Classification of Singularities

Let C_1 and C_2 be concentric circles of radii R_1 and R_2, respectively, and the center at a (Figure 22). Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 in the ring-shaped region $\Re$ (also called the annulus or annular region) between C_1 and C_2, is shown shaded in Figure 22. Let $a+h$ be any point in $\Re$. Then we have

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+...+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+..., \qquad (4)$$

where

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(t)}{(t-a)^{n+1}}dt,\ n\in Z \qquad (5)$$

This is called Laurent's theorem and (1) or (4) with coefficients (5) is called *a Laurent series* or expansion.

The part $a_0+a_1(z-a)+a_2(z-a)^2+...$ is called *the analytic part of the Laurent series*, while the remainder of the series, which consists of inverse powers of $z-a$, is called *the principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

It is possible to classify the singularities of a function $f(z)$ by examination of its Laurent series. For this purpose, we assume that in Figure 21, $R_2=0$, so that $f(z)$ is analytic inside and on C_1 except at $z=a$, which is an isolated singularity. In the following, all singularities are assumed isolated unless otherwise indicated.

1. Poles. If $f(z)$ has the form (4) in which the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+...+\frac{a_{-n}}{(z-a)^n},$$

where $a_{-n}\neq 0$, then $z=a$ is called *a pole of order* n. If $n=1$, then it is called *a simple pole*.

If $f(z)$ has a pole at $z=a$, then $\lim\limits_{z\to a}f(z)=\infty$.

2. Removable singularities. If a single-valued function $f(z)$ is not defined at $z=a$ but $\lim\limits_{z\to a}f(z)$ exists, then $z=a$ is called *a removable singularity*. In such case, we define $f(z)$ at $z=a$ as equal to $\lim\limits_{z\to a}f(z)$, and $f(z)$ will then be analytic at a.

Example 2 If $f(z)=\dfrac{\sin z}{z}$, then $z=0$ is a removable singularity since $f(0)$ is not defined but $\lim\limits_{z\to 0}\dfrac{\sin z}{z}=1$. We define $f(0)=\lim\limits_{z\to 0}\dfrac{\sin z}{z}=1$. Note that in this case

$$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-...+(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+...\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-.$$

3. Essential singularities. If $f(z)$ is single-valued, then any singularity that is not a pole or removable singularity is called *an essential singularity*. If $z=a$ is an essential singularity of $f(z)$, then *the principal part of the Laurent expansion has infinitely many terms.*

Example 3 Since $e^{\frac{1}{z}}=1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+...$, $z=0$ is an essential singularity.

4. Branch points. A point $z = z_0$ is called *a branch point* of a multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged, when z describes a closed path about z_0. A branch point is a non-isolated singularity. Since each of the branches of a multiple-valued function is analytic, all of the theorems for analytic functions, in a particular Taylor's theorem, can be applied.

Example 4 The branch of $f(z) = z^{\frac{1}{2}}$, which has the value 1 for $z = 1$, has a Taylor series of the form $a_0 + a_1(z-1) + a_2(z-1)^2 + \ldots$ and the radius of convergence $R = 1$ (the distance from $z = 1$ to the nearest singularity, namely the branch point $z = 0$).

Example 5 Find Laurent series about the indicated singularity for each of the following functions:

(a) $\dfrac{e^{2z}}{(z-1)^3}, z = 1;$ *(b)* $\dfrac{z - \sin z}{z^3}, z = 0;$ *(c)* $\dfrac{1}{z^2(z-3)^2}, z = 3$

Name the singularity in each case and give the region of convergence of each series.

Solution

(a) Let $z - 1 = u$ .Then $z = u + 1$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2u+2}}{u^3} = \frac{e^2}{u^3}e^{2u} = \frac{e^2}{u^3}\left(1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \ldots\right) =$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \ldots.$$

$z = 1$ is a pole of order 3, or a *triple pole*.

(b)

$$\frac{z - \sin z}{z^3} = \frac{z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \ldots\right)}{z^3} = \frac{1}{z^3}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots\right) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^5}{7!} + \ldots.$$

$z = 0$ is a *removable singularity*. The series converges for all values of z .

(c) Let $z - 3 = u$. Then, by the binomial theorem,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(u+3)^2} = \frac{1}{9u^2\left(1 + \dfrac{u}{3}\right)^2} =$$

$$= \frac{1}{9u^2}\left(1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \ldots\right) =$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \ldots = \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \ldots.$$

$z = 3$ is a pole of order 2 or a *double pole*.

Example 6 Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for:

(a) $1<|z|<3$ (b) $|z|>3$ (c) $0<|z+1|<2$ (d) $|z|<1$.

Solution

(a) Resolving into partial fractions, we have

$$\frac{1}{(z+1)(z+3)}=\frac{1}{2}\cdot\frac{1}{z+1}-\frac{1}{2}\cdot\frac{1}{z+3}.$$

If $|z|>1$, then

$$\frac{1}{2(z+1)}=\frac{1}{2z\left(1+\frac{1}{z}\right)}=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+...\right)=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-....$$

If $|z|<3$, then

$$\frac{1}{2(z+3)}=\frac{1}{6\left(1+\frac{z}{3}\right)}=\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+...\right)=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+....$$

Then, the required Laurent expansion valid for both $|z|>1$ and $|z|<3$, i.e., $1<|z|<3$, is

$$...-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}-...$$

(b) If $|z|>1$, we have the result such as in the part (a),

$$\frac{1}{2(z+1)}=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-....$$

If $|z|>3$, then

$$\frac{1}{2(z+3)}=\frac{1}{2z\left(1+\frac{3}{z}\right)}=\frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\frac{81}{z^4}+...\right)=\frac{1}{2z}-\frac{3}{2z^2}+\frac{9}{2z^3}-....$$

Then, the required Laurent expansion valid for both $|z|>1$ and $|z|>3$, i.e., $|z|>3$, is

$$\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+....$$

(c) Let $z+1=u$. Then

$$\frac{1}{(z+1)(z+3)}=\frac{1}{u(u+2)}=\frac{1}{2u\left(1+\frac{u}{2}\right)}=\frac{1}{2u}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+...\right)=$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\frac{1}{16}(z+1)^2+...,\text{ valid for } 0<|z+1|<2.$$

(d) If $|z|<1$,

$$\frac{1}{2(z+1)}=\frac{1}{2(1+z)}=\frac{1}{2}\left(1-z+z^2-z^3+z^4+...\right)=\frac{1}{2}-\frac{1}{2}z+\frac{1}{2}z^2-\frac{1}{2}z^3+\dots.$$

If $|z|<3$, we have the result such as in the part (a),

$$\frac{1}{2(z+3)}=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\dots.$$

Then the required Laurent expansion, valid for both $|z|<1$ and $|z|<3$, i.e., $|z|<1$, is obtained by subtraction

$$\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\dots.$$

This is a Taylor series.

Exercise Set 2.6

In Exercises 1 to 3, investigate the convergence of:

1. $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}+i\,\frac{n}{3^n}\right)$ *2.* $\sum_{n=1}^{\infty}\left(\frac{\cos 3n}{n^3}+\frac{\sin 4n}{n^4}\cdot i\right)$ *3.* $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+i\,\frac{2n-1}{3n+1}\right)$

In Exercises 4 to 7, find the region of the convergence of:

4. $\sum_{n=1}^{\infty}\frac{n+1}{2^n}\left(z-2+i\right)^n$ *5.* $\sum_{n=1}^{\infty}\frac{(n+1)!\,(4+3i)^n}{(2n+1)!}z^n$

6. $\sum_{n=0}^{\infty}\frac{3n+2}{\left(1+i\sqrt{3}\right)^n}\left(z+2-i\right)^n$ *7.* $\sum_{n=0}^{\infty}\frac{(2n+1)}{(4+i)^n}\left(z-3+i\right)^n$

In Exercises 8 to 17, expand $f(z)$ in a Laurent series valid for the given K:

8. $f(z)=\dfrac{1}{(z-2)(z-3)}$, $K: 2<|z|<3$;

9. $f(z)=\dfrac{2z-3}{z^2-3z+2}$, $K: 0<|z-2|<1$;

10. $f(z)=\dfrac{2}{(z-1)(z-3)}$, $K: 3<|z-1|<+\infty$;

11. $f(z)=\dfrac{z+2}{z^2+2z-8}$, $K:\ 2<|z+2|<4$;

12. $f(z)=\dfrac{2z+3}{z^2+3z+2}$, $K:\ 1<|z|<2$;

13. $f(z)=\dfrac{2z-3}{z^2-3z+2}$, $K:\ |z-1|<2$

14. $f(z)=\dfrac{2}{z^2-1}$, $K:\ 1<|z+2|<3$;

15. $f(z)=\dfrac{1}{z^2+1}$, $K:\ 0<|z-i|<2$;

16. $f(z)=\dfrac{4z-8}{(z+1)(z-3)}, \quad K:\ 3<|z|<\infty;$

17. $f(z)=(z^2+z)^{-1}, \quad K:\ 0<|z|<1.$

In Exercises 18 to 25, expand each of the following functions in a Laurent series at a given point z_0:

18. $f(z)=\ln\dfrac{z-3}{z},\ \ z_0=\infty$

19. $f(z)=\sin\dfrac{z}{z-1},\ \ z_0=1$

20. $f(z)=\cos\dfrac{i}{z^2}+\dfrac{z}{z-1},\ z_0=0$

21. $f(z)=e^{\frac{z}{z-3}},\ z_0=3$

22. $f(z)=\dfrac{1}{z}\sin^2\dfrac{2}{z},\ z_0=0$

23. $f(z)=\ln\dfrac{z-1}{z-2},\ \ z_0=\infty$

24. $f(z)=z\,e^{\frac{z}{z-4}},\ \ z_0=4$

25. $f(z)=\cos\dfrac{3z}{z-i},\ \ z_0=i$

In Exercises 26 to 35, determine and classify all the singularities of the functions

26. $f(z)=\dfrac{1+\cos z}{z-\pi},\ z_0=\pi$

27. $f(z)=\dfrac{e^{z+i}}{z+i},\ z_0=-i$

28. $f(z)=\dfrac{\sin 4z-4z}{e^z-1-z},\ z_0=0$

29. $f(z)=\dfrac{\sin z}{z^3(1-\cos z)},\ z_0=2\pi$

30. $f(z)=\cos\dfrac{1}{\pi+z},\ z_0=-\pi$

31. $f(z)=\dfrac{z^2-1}{z^6+2z^5+z^4},\ z_0=-1$

32. $f(z)=\dfrac{z^2-1}{z^6+2z^5+z^4},\ z_0=0$

33. $f(z)=\dfrac{e^z-1}{\sin\pi z},\ z_0=0$

34. $f(z)=\dfrac{z-\pi}{\sin^2 z},\ z_0=0$

35. $f(z)=z^2\sin\dfrac{1}{z},\ z_0=0$

Individual Tasks 2.6

1-2. Expand $f(z)$ in a Laurent series valid for a given K.
3. Expand $f(z)$ in a Laurent series at a given point z_0.
4. Determine and classify all the singularities of the functions.

I.	**II.**
1. $f(z)=\dfrac{1}{z(1-z)},\ K:0<\lvert z-1\rvert<1$	*1.* $f(z)=\dfrac{1}{(z-1)(z-2)},K:0<\lvert z-2\rvert<1$
2. $f(z)=\dfrac{z+2}{z^2-4z+3},\ K:\ \lvert z-1\rvert>2$	2. $f(z)=\dfrac{2z-3}{z^2-3z+2},\ K:1<\lvert z\rvert<2$
3. $f(z)=z\cos\dfrac{1}{z-2},\quad z_0=2$	3. $f(z)=\sin\dfrac{z}{z-3},\quad z_0=3$
4. $f(z)=\dfrac{e^{z-2i}}{z-2i},\ z_0=2i$	4. $f(z)=\dfrac{1+\cos z}{z+\pi},\ z_0=-\pi$

2.7 Residues

Let $f(z)$ be single-valued and analytic inside and on a circle C, except at the point $z=a$ chosen as the center of C. Then $f(z)$ has a Laurent series about $z=a$ given by

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n=a_0+a_1(z-a)+a_2(z-a)^2+...+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+... \qquad (1)$$

where

$$a_n=\frac{1}{2\pi i}\oint_C\frac{f(z)}{\left(z-a\right)^{n+1}}dz,\quad n\in Z \qquad (2)$$

In the special case $n=-1$, we have the following formula

$$\oint_C f(z)dz=2\pi i a_{-1} \qquad (3)$$

Formally, we can obtain (3) from (1) by integrating term by term and using the results

$$\oint_C\frac{dz}{(z-a)^n}=\begin{cases}2\pi i, & n=1\\ 0, & n\neq 1\end{cases} \qquad (4)$$

Because of the fact that (3) involves only the coefficient a_{-1} in (1), we call a_{-1} *the residue of* $f(z)$ at $z=a$.

$$\operatorname*{Res}_{z=a} f(z)=a_{-1} \qquad (5)$$

To obtain the residue of a function $f(z)$ at $z=a$, it may appear from (1) that the Laurent expansion of $f(z)$ about $z=a$ must be obtained. However, in the case where $z=a$ is a pole of order k, there is a simple formula for a_{-1} given by

$$\operatorname*{Res}_{z=a} f(z)=\frac{1}{(k-1)!}\lim_{z\to a}\left((z-a)^k f(z)\right)^{(k-1)},\quad k\geq 1 \qquad (6)$$

If $k=1$ (*a simple pole*), then the result is especially simple and is given by

$$\operatorname*{Res}_{z=a} f(z)=a_{-1}=\lim_{z\to a}(z-a)f(z) \qquad (7)$$

which is a special case of (6) with $k=1$ if we define $0!=1$.

Example 1 Find the residues of $f(z)=\dfrac{z}{(z-1)(z+1)^2}$.

Solution

If $f(z)=\dfrac{z}{(z-1)(z+1)^2}$, then $z=1$ and $z=-1$ are the poles of orders one and two, respectively. We have, using (7) and (6) with $k=2$,

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+1)^2} = \lim_{z\to 1}\frac{z}{(z+1)^2} = \frac{1}{4},$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(2-1)!}\lim_{z\to -1}\left((z+1)^2\frac{z}{(z-1)(z+1)^2}\right)' = \lim_{z\to -1}\left(\frac{z}{(z-1)}\right)' =$$
$$= \lim_{z\to -1}\frac{z-1-z}{(z-1)^2} = -\frac{1}{4}.$$

If $z=a$ is an essential singularity, then the residue can sometimes be found by using known series expansions.

Example 2 Let $f(z)=e^{-\frac{1}{z}}$. Then, $z=0$ is an essential singularity and from the known expansion for e^u with $u=-\frac{1}{z}$, we find

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \ldots$$

from which we see that the residue at $z=0$ is the coefficient of $1/z$ and equals -1.

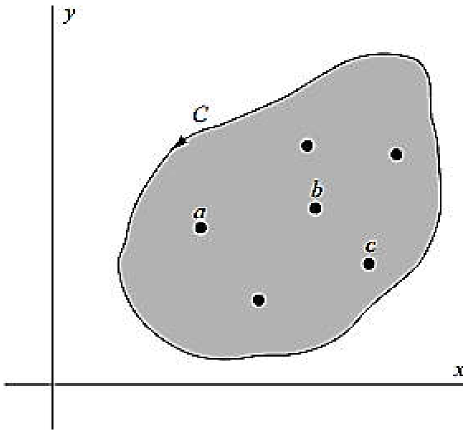

Figure 23

Theorem (residue theorem) Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a,b,c,\ldots$ inside C, which have the residues given by $a_{-1},b_{-1},c_{-1},\ldots$ (see Figure 23). Then, the residue theorem states that

$$\oint_C f(z)\,dz = 2\pi i\left(a_{-1}+b_{-1}+c_{-1}+\ldots\right) \tag{8}$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C. Note that (8) is a generalization of (3). Cauchy's theorem and integral formulas are special cases of this theorem.

Example 3 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ around the circle C with equation $|z|=3$.

Solution

The integrand $\dfrac{e^{zt}}{z^2(z^2+2z+2)}$ has a double pole at $z=0$ and two simple poles at $z=-1\pm i$ (roots of $z^2+2z+2=0$). All these poles are inside C.

Residue at $z=0$ is

$$\operatorname*{Res}_{z=0} f(z)=\frac{1}{(2-1)!}\lim_{z\to 0}\left(z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right)'=$$

$$=\lim_{z\to 0}\frac{(z^2+2z+2)te^{zt}-e^{zt}(2z+2)}{(z^2+2z+2)^2}=\frac{t-1}{2}.$$

Residue at $z=-1+i$ is

$$\operatorname*{Res}_{z=-1+i} f(z)=\lim_{z\to -1+i}\left((z-(-1+i))\frac{e^{zt}}{z^2(z^2+2z+2)}\right)=$$

$$=\lim_{z\to -1+i}\frac{e^{zt}}{z^2}\lim_{z\to -1+i}\frac{z+1-i}{z^2+2z+2}=\frac{e^{(-1+i)t}}{4}.$$

Residue at $z=-1-i$ is

$$\operatorname*{Res}_{z=-1-i} f(z)=\lim_{z\to -1-i}\left((z-(-1-i))\frac{e^{zt}}{z^2(z^2+2z+2)}\right)=$$

$$=\lim_{z\to -1-i}\frac{e^{zt}}{z^2}\cdot\lim_{z\to -1-i}\frac{z+1+i}{z^2+2z+2}=\frac{e^{(-1-i)t}}{4}.$$

Then, by the residue theorem

$$\frac{1}{2\pi i}\oint_C\frac{e^{zt}}{z^2(z^2+2z+2)}dz=\frac{1}{2\pi i}2\pi i\left(\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right)=\frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t.$$

The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour C, the choice of which may require great ingenuity. The following types are most common in practice.

1. $\displaystyle\int_{-\infty}^{+\infty}F(x)dx$, where $F(x)$ is a rational function.

Consider $\displaystyle\oint_C F(z)dz$ along a contour C consisting of a line along the x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as a diameter (Figure 24). Let $R\to\infty$. If $F(x)$ is an even function, this can be used to evaluate $\displaystyle\int_0^{+\infty}F(x)dx$.

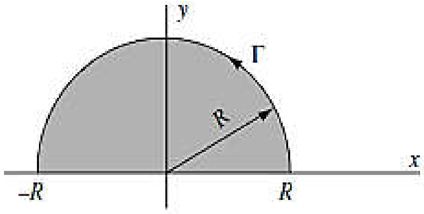

Figure 24

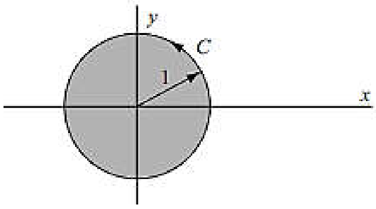

Figure 25

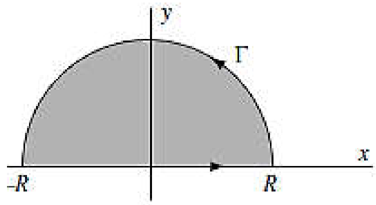

Figure 26

2. $\int\limits_0^{2\pi} G(\sin\varphi,\cos\varphi)d\varphi$, where $G(\sin\varphi,\cos\varphi)$ is a rational function of $\sin\varphi$ and $\cos\varphi$.

Let $z = \mathrm{e}^{i\varphi}$. Then $\sin\varphi = \dfrac{z - z^{-1}}{2i}$, $\cos\varphi = \dfrac{z + z^{-1}}{2}$ and $dz = i\,\mathrm{e}^{i\varphi}\,d\varphi$ or $d\varphi = \dfrac{dz}{iz}$. The given integral is equivalent to $\oint\limits_C F(z)\,dz$ where C is the unit circle with the center at the origin (Figure 25).

3. $\int\limits_{-\infty}^{+\infty} F(x)\begin{Bmatrix}\cos mx\\ \sin mx\end{Bmatrix}dx$, where $F(x)$ is a rational function.

Here, we consider $\oint\limits_C F(z)\mathrm{e}^{imz}\,dz$, where C is the same contour as that in Type 1.

Example 4 Evaluate $\int\limits_0^{+\infty}\dfrac{dx}{1+x^6}$.

Solution

Consider $\oint\limits_C \dfrac{dz}{1+z^6}$, where C is the closed contour of Figure 26, consisting of the line from $-R$ to $+R$ and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6 + 1 = 0$, when $z_1 = \mathrm{e}^{\frac{\pi i}{6}}, z_2 = \mathrm{e}^{\frac{3\pi i}{6}}, z_3 = \mathrm{e}^{\frac{5\pi i}{6}}, z_4 = \mathrm{e}^{\frac{7\pi i}{6}}, z_5 = \mathrm{e}^{\frac{9\pi i}{6}}, z_6 = \mathrm{e}^{\frac{11\pi i}{6}}$, these are simple poles of $\dfrac{1}{z^6+1}$. Only the poles $z_1 = \mathrm{e}^{\frac{\pi i}{6}}$, $z_2 = \mathrm{e}^{\frac{3\pi i}{6}}$, $z_3 = \mathrm{e}^{\frac{5\pi i}{6}}$ lie within C. Then, using L'Hospital's rule,

$$\operatorname*{Res}_{z=\mathrm{e}^{\frac{\pi i}{6}}} f(z) = \lim_{z\to \mathrm{e}^{\frac{\pi i}{6}}}\left(z - \mathrm{e}^{\frac{\pi i}{6}}\right)f(z) = \lim_{z\to \mathrm{e}^{\frac{\pi i}{6}}}\left(z - \mathrm{e}^{\frac{\pi i}{6}}\right)\frac{1}{z^6+1} = \lim_{z\to \mathrm{e}^{\frac{\pi i}{6}}}\frac{1}{6z^5} = \frac{1}{6}\mathrm{e}^{-\frac{5\pi i}{6}},$$

$$\operatorname*{Res}_{z=\mathrm{e}^{\frac{3\pi i}{6}}} f(z) = \lim_{z\to \mathrm{e}^{\frac{3\pi i}{6}}}\left(z - \mathrm{e}^{\frac{3\pi i}{6}}\right)f(z) = \lim_{z\to \mathrm{e}^{\frac{3\pi i}{6}}}\left(z - \mathrm{e}^{\frac{3\pi i}{6}}\right)\frac{1}{z^6+1} = \lim_{z\to \mathrm{e}^{\frac{3\pi i}{6}}}\frac{1}{6z^5} = \frac{1}{6}\mathrm{e}^{-\frac{5\pi i}{2}},$$

$$\operatorname*{Res}_{z=\mathrm{e}^{\frac{5\pi i}{6}}} f(z) = \lim_{z\to \mathrm{e}^{\frac{5\pi i}{6}}}\left(z - \mathrm{e}^{\frac{5\pi i}{6}}\right)f(z) = \lim_{z\to \mathrm{e}^{\frac{5\pi i}{6}}}\left(z - \mathrm{e}^{\frac{5\pi i}{6}}\right)\frac{1}{z^6+1} = \lim_{z\to \mathrm{e}^{\frac{5\pi i}{6}}}\frac{1}{6z^5} = \frac{1}{6}\mathrm{e}^{-\frac{25\pi i}{6}}$$

Thus,

$$\oint_C \frac{dz}{1+z^6} = 2\pi i\left(\frac{1}{6}e^{-\frac{5\pi i}{6}} + \frac{1}{6}e^{-\frac{5\pi i}{2}} + \frac{1}{6}e^{-\frac{25\pi i}{6}}\right) = \frac{2\pi}{3}$$

that is,

$$\int_{-R}^{R} \frac{dx}{1+x^6} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

Taking the limit of both sides as $R \to \infty$

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{1+x^6} = \int_{-\infty}^{+\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

Since

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^6} = 2\int_{0}^{+\infty} \frac{dx}{1+x^6}$$

the required integral has the value $\frac{\pi}{3}$.

Exercise Set 2.7

In Exercises 1 to 18, evaluate

1. $\oint\limits_{|z|=2} \frac{e^z}{z^2(z+1)}dz$

2. $\oint\limits_{|z-i|=1} \frac{e^z}{z^4+2z^2+1}dz$

3. $\oint\limits_{|z|=3} \frac{\sin z\, dz}{z^2(z^2-4)}$

4. $\oint\limits_{|z|=2} \frac{e^z\, dz}{z^2(z^2-9)}$

5. $\oint\limits_{|z-i|=3} \frac{e^{z^2}-1}{z^3-i z^2}dz$

6. $\oint\limits_{|z-1|=2} \frac{\cos\frac{z}{2}}{z^2-4}dz$

7. $\int_0^{2\pi} \frac{dt}{13-5\cos t}$

8. $\int_0^{2\pi} \frac{dt}{(3+\sin t)^3}$

9. $\int_0^{2\pi} \frac{dt}{(5-4\cos t)^2}$

10. $\int_{-\infty}^{\infty} \frac{x^2+2}{x^4+x^2+12}dx$

11. $\int_{-\infty}^{\infty} \frac{x-1}{(x^2+4)^2}dx$

12. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 - 8x + 20} dx$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^{2ix}}{x^4 + 10x^2 + 9} dx$

15. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^2 (x^2 + 16)}$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2 (x^2 + 4)}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{2ix}}{x^2 - 10x + 26} dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^{2ix}}{x^4 + 13x^2 + 36} dx$

Individual Tasks 2.7

1-4. Evaluate the given integrals.

I.	II.
1. $\displaystyle\frac{1}{2\pi i} \oint_{\lvert z\rvert=2} \frac{z^6 + 2z^5 - i}{(z+i)^4 z^2} dz$	1. $\displaystyle\oint_{\lvert z\rvert=3} \frac{2z^2 - 3z + 6}{z(1+z^2)} dz$
2. $\displaystyle\oint_{\lvert z-1\rvert=\frac{1}{4}} \frac{z^2 + z - 4}{z^2(z-1)} dz$	2. $\displaystyle\frac{1}{2\pi i} \oint_{\lvert z\rvert=5} \frac{e^z dz}{z^2(z^2+16)}$
3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$	3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^3}$
4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2(x^2+25)}$	4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)}$

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APENDIX

Individual Tests 1.1

Variant 1	Variant 2
1. Find the sum of the series: a) $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+2)}$, b) $\sum\limits_{n=1}^{\infty}\frac{5^n+2^n}{10^n}$. 2. Investigate the series for convergence: a) $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^3}$, b) $\sum\limits_{n=1}^{\infty}\frac{n}{\sqrt{n(n-1)}}$.	1. Find the sum of the series: a) $\sum\limits_{n=1}^{\infty}\frac{1}{(n+2)(n+3)}$, b) $\sum\limits_{n=1}^{\infty}\frac{3^n+4^n}{12^n}$. 2. Investigate the series for convergence: a) $\sum\limits_{n=1}^{\infty}\frac{\sin 3^n}{3^n}$, b) $\sum\limits_{n=1}^{\infty}\frac{\sqrt{n}-1}{n}$.

Individual Tests 1.2

Variant 1	Variant 2
Investigate the series for convergence: 1. $\sum\limits_{n=1}^{\infty}\frac{n(n+1)}{3^n}$, 2. $\sum\limits_{n=1}^{\infty}\left(\frac{3n}{n+1}\right)^n$. 3. $\sum\limits_{n=1}^{\infty}\frac{2}{3+n^2}$, 4. $\sum\limits_{n=1}^{\infty}\left(\frac{3n}{3n+1}\right)^n$, 5. $\sum\limits_{n=1}^{\infty}ne^{-n^2}$	Investigate the series for convergence: 1. $\sum\limits_{n=1}^{\infty}\frac{n^3}{(n+1)!}$, 2. $\sum\limits_{n=1}^{\infty}\left(\frac{5n}{n+3}\right)^n$. 3. $\sum\limits_{n=1}^{\infty}\frac{n}{2^{n^2}}$, 4. $\sum\limits_{n=1}^{\infty}\frac{5n}{3n+1}$, 5. $\sum\limits_{n=1}^{\infty}\frac{2}{1+\sqrt{n}}$

Individual Tests 1.3

Variant 1	Variant 2
Find the domain of convergence of the series: 1. $\sum_{n=1}^{\infty}\frac{(2n+3)(x+1)^n}{4^{n-1}}$ 2. $\sum_{n=1}^{\infty}\frac{(x^2+x-6)^n}{2^n}$ Determine whether the series is absolutely convergent, conditionally convergent, or divergent: 3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n(n+1)}$ 4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(3n+2)}{n\sqrt{n}+1}$	Find the domain of convergence of the series: 1. $\sum_{n=1}^{\infty}\frac{1}{n^n(x+2)^n}$ 2. $\sum_{n=1}^{\infty}\frac{(x+4)^n}{6^n}\cdot\frac{n+1}{n+3}$ Determine whether the series is absolutely convergent, conditionally convergent, or divergent: 3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(3n+2)}{4^n}$ 4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{2^n}$

Individual Tests 1.4

Variant 1	Variant 2
Investigate the series for convergence: 1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n+1}}$, 2. $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n\cdot(n+1)!}$, 3. $\sum_{n=1}^{\infty}n\sin\frac{\pi}{3^n}$, 4. $\sum_{n=1}^{\infty}\frac{1}{\ln^n(n+3)}$. 5. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[7]{(3+7n)^{10}}}$, 6. $\sum_{n=1}^{\infty}\frac{1}{\ln(n+3)}$, 7. $\sum_{n=1}^{\infty}\frac{1}{5^n}\left(\frac{n}{n+3}\right)^{n^2}$, 8. $\sum_{n=1}^{\infty}\frac{2n-1}{3^n(n+1)}x^n$	Investigate the series for convergence: 1. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{2n^3+1}}$, 2. $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n!}{5^n}$, 3. $\sum_{n=1}^{\infty}\frac{n!}{5^n(n+3)!}$, 4. $\sum_{n=1}^{\infty}\left(\frac{2n-1}{2n}\right)^{n^2}$. 5. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{(3n-1)^4}}$, 6. $\sum_{n=1}^{\infty}\frac{2n-1}{3n^2+5n}$, 7. $\sum_{n=1}^{\infty}\frac{2}{3^n+n}$. 8. $\sum_{n=1}^{\infty}\frac{n+1}{4^n}x^n$

Individual Tests 1.5

Variant 1	Variant 2
Find a power series representation for the function and determine the interval of convergence: 1. $f(x)=\dfrac{1}{7+2x}$, 2. $f(x)=\dfrac{3x-5}{x^2-3x+2}$, 3. $f(x)=\cos^2 5x^3$, 4. $f(x)=\mathrm{e}^{5x^3}$.	Find a power series representation for the function and determine the interval of convergence: 1. $f(x)=\dfrac{1}{3+4x}$, 2. $f(x)=\dfrac{4x-5}{x^2+6x+5}$, 3. $f(x)=\sin^2 3x^2$, 4. $f(x)=x\mathrm{e}^{-x^2}$.

Individual Tests 1.6

Variant 1	Variant 2
Use a power series to approximate the definite integral to three decimal places: 1. $\int\limits_0^1 \sin\sqrt{x}\,dx$, 2. $\int\limits_0^{0,1} \dfrac{dx}{1+x^4}$, 3. $\int\limits_1^3 \dfrac{\ln\left(1+x^3\right)}{x^3}dx$. Use power series to solve the initial-value problem. $y'=4y-y^2,\ y(0)=1.$	Use a power series to approximate the definite integral to three decimal places: 1. $\int\limits_0^1 \cos\sqrt[3]{x}\,dx$, 2. $\int\limits_0^{0,2} \dfrac{dx}{1+x^5}$, 3. $\int\limits_0^{0.3} \sqrt{1+x^5}\,dx$. Use power series to solve the initial-value problem. $y'=2\sin x+y^3,\ y(0)=1.$

Individual Tests 1.7

Variant 1	Variant 2
Determine the Fourier coefficients of the given functions on the given intervals: 1. $f(x)=\begin{cases}1, & if\ -\pi\le x\le 0,\\ x, & if\ \ 0<x<\pi.\end{cases}$, 2. $f(x)=1-x,\ if\ x\in(-1;1)$.	Determine the Fourier coefficients of the given functions on the given intervals: 1. $f(x)=\begin{cases}-x, & if\ -\pi\le x\le 0,\\ \pi, & if\ \ 0<x<\pi.\end{cases}$, 2. $f(x)=x+2,\ if\ x\in(-3;3)$.

Individual Tests 1.8

Variant 1	Variant 2
Determine the Fourier sine series of the given functions on the given intervals: 1. $f\left(x\right)=x-3\pi, x\in(3\pi;4\pi)$. 2. $f\left(x\right)=\begin{cases}3, if \quad 0<x<\pi/2; \\ 0, if \quad \pi/2<x<\pi.\end{cases}$	Determine the Fourier cosine series of the given functions on the given intervals: 1. $f(x)=\begin{cases}-x, \ if \ -\pi\le x\le 0, \\ x, \ if \ 0<x<\pi.\end{cases}$. 2. $f\left(x\right)=\sin x, \ x\in(0;\pi)$

Individual Tests 2.1

Variant 1	Variant 2
Perform each of the indicated operations: $\dfrac{(1-3i)(i+2)}{1-i}$ Describe and graph the locus represented by the following expression: $\lvert z-i+3\rvert=2$ Describe graphically the region represented by the following expression: $-\pi/6\le\arg(z+i)<\pi/4$ Solve the following equations, obtaining all roots: $z^4+81=0$	Perform each of the indicated operations: $\dfrac{(2-3i)(2i-3)}{2+i}$ Describe and graph the locus represented by the following expression: $\lvert z-2-3i\rvert=2$ Describe graphically the region represented by the following expression: $1<\lvert z-2+i\rvert\le 3$ Solve the following equations, obtaining all roots: $z^3-64=0$

Individual Tests 2.2

Variant 1	Variant 2
Separate each of the following expressions into real and imaginary parts: 1. $w=(2-i)z^2-iz^2+4i+1$, 2. $w=z^2\cdot\mathrm{Re}(z-2iz)$. Find the value of the given numbers. $\mathrm{Ln}\left(2\sqrt{3}+2i\right)$.	Separate each of the following expressions into real and imaginary parts: 1. $w=(3-i)z-iz^2-7-2i$, 2. $w=z^2\cdot\mathrm{Im}(z-2iz)$. Find the value of the given numbers. $\left(\sqrt{3}-i\right)^{2i}$.

Individual Tests 2.3

Variant 1	Variant 2
Differentiate the following functions: 1. $w=\cos^4(3z+7i)$, 2. $(2z-4i)^{i+z}$. Find out which of the following functions are analytic at least at one point: $w=(2-i)z^2-iz+4i+1$ Prove that the given function is harmonic. Find $u(x,y)$ such that $f(z)=u+iv$ is analytic: $v=3y^2x-x^3$	Differentiate the following functions: 1. $w=(z^3-3z^2)\sin 4z$, 2. $(3z+5i)^{2z-i}$. Find out which of the following functions are analytic at least at one point: $w=(3-i)z-iz^2-7-2i$ Prove that the given function is harmonic. Find $u(x,y)$ such that $f(z)=u+iv$ is analytic: $v=x^2-y^2-2y+1$

Individual Tests 2.4

Variant 1	Variant 2
Evaluate the following integrals: 1. $\int_\gamma (3z^2+\cos z)\,dz$, $\gamma: \vert z\vert=1,\ \mathrm{Re}\,z\ge 0$. 2. $\int_0^i z\sin 2z\,dz$, 3. $\int \frac{dz}{1+\sqrt{z-2}}$, 4. $\int \frac{z^7 dz}{\sqrt{7+z^8}}$.	Evaluate the following integrals: 1. $\int_\gamma (z^2-2\sin z)\,dz$, $\gamma: \vert z\vert=3,\ \mathrm{Im}\,z\le 0$ 2. $\int_0^i z\cos 3z\,dz$, 3. $\int \frac{dz}{3+\sqrt{z-1}}$, 4. $\int \frac{\ln^3(3z-1)\,dz}{3z-1}$.

Individual Tests 2.5

Variant 1	Variant 2
Evaluate the following integrals: 1. $\oint_{\vert z\vert=2} \frac{z+1}{(z-1)(z+4)}dz$, 2. $\oint_{\vert z-i\vert=6} \frac{5z+3}{z^2+2z-3}dz$, 3. $\oint_{\vert z\vert=1} \frac{3z^3-7z^2+4z-1}{z^4}dz$, 4. $\oint_{\vert z+i\vert=2} \frac{i\,dz}{(z^2+4)^2}$.	Evaluate the following integrals: 1. $\oint_{\vert z\vert=3} \frac{3z+5}{(z-2)(z-5)}dz$, 2. $\oint_{\vert z+1\vert=5} \frac{3z-2}{z^2-2z-3}dz$, 3. $\oint_{\vert z\vert=2} \frac{z^3-z^2+5z-2}{z^3}dz$, 4. $\oint_{\vert z-i\vert=3} \frac{i\,dz}{(z^2+9)^2}$.

Individual Tests 2.6

Variant 1	Variant 2
Determine and classify all the singularities of the functions: 1. $f(z)=\dfrac{z^2-4}{z^5+4z^4+4z^3}$, 2. $f(z)=\dfrac{e^z-1-z}{\sin 2z-2z}$. Expand $f(z)$ in a Laurent series valid for given K. $f(z)=\dfrac{4z-8}{z^2-2z-3}$, $K:\ 3<\lvert z\rvert<\infty$.	Determine and classify all the singularities of the functions: 1. $f(z)=\dfrac{z^2-9}{z^4+6z^3+9z^2}$, 2. $f(z)=\dfrac{z^3(1-\cos z)}{\sin z}$. Expand $f(z)$ in a Laurent series valid for given K. $f(z)=\dfrac{z+2}{z^2+2z-8}$, $K: 2<\lvert z-2\rvert<$.

Individual Tests 2.7

Variant 1	Variant 2
Evaluate the following integrals: 1. $\oint\limits_{\lvert z\rvert=3}\dfrac{e^z\,dz}{z^2(z^2+16)}$, 2. $\oint\limits_{\lvert z\rvert=2}\dfrac{\sin z\,dz}{z^2(z^2-9)}$, 3. $\int\limits_{-\infty}^{\infty}\dfrac{x+1}{(x^2+1)^2}dx$.	Evaluate the following integrals: 1. $\oint\limits_{\lvert z\rvert=3}\dfrac{e^z}{z^2(z+5)}dz$, 2. $\oint\limits_{\lvert z\rvert=1}\dfrac{\sin z\,dz}{z^2(z^2+2)}$, 3. $\int\limits_{-\infty}^{\infty}\dfrac{x-3}{(x^2+9)^2}dx$.

Attestation test "Series"

1. Determine whether the series $\sum_{n=1}^{\infty}\frac{4n-4}{5n+1}$ is convergent or divergent.

2. Determine whether the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}-1}{\sqrt{n^3+2n^2-n}}$ is convergent or divergent.

3. Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^2(n+1)}$ is convergent or divergent.

4. Determine whether the series $\sum_{n=1}^{\infty}\frac{n!}{(2n+3)!}$ is convergent or divergent.

5. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{4n-4}{5n^2+1}$ is absolutely convergent, conditionally convergent, or divergent.

6. Find interval of convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+1}{2n}\right)^n(x+1)^n$.

7. Find a power series representation for the function and determine the interval of convergence $f(x)=\cos\frac{2x^3}{3}$.

8. Use a power series to approximate the definite integral to three decimal places $\int_{0.01}^{0.1}\frac{\ln(1+x)}{x}dx$.

9. Use power series to solve the initial-value problem $y'=x^2+2y^2$, $y(0)=2$.

10. Determine the Fourier sine series of the given functions on the given intervals $f(x)=1-x,\ x\in(0;\pi)$.

Attestation test "Theory of analytic functions of one complex variable"

1. Describe graphically the region represented by the following $|z-2-i|\geq 1$, $1\leq \operatorname{Re} z\leq 3$.

2. Separate of the following into real and imaginary parts $z=\frac{-3-i}{1-3i}$.

3. Find the derivative of the following

$f(z)=\left(x^3-3xy^2+3x\right)+i\left(3x^2y-y^3+3y-1\right)$ at the point $z_0=i-1$.

4. Find the derivative of the following $\left(z^2+3z\right)\sin 3z$

5. Find a coefficient of expansion and the rotation angle at this point when mapping $f(z)=u(x,y)+i\,v(x,y)$

$$u(x,y)=x^3-3xy^2+3x,\quad v(x,y)=3x^2y-y^3+3y-1,\quad z_0=-1-2i$$

6. Evaluate $\int\limits_C z\cdot \operatorname{Im} z\,dz$, $C-$ the line from $z_1=0$ to $z_2=1+i$.

7. Evaluate $\int \frac{dz}{z^2-25}$.

8. Evaluate $\oint\limits_{|z-i|=1} \frac{z-3}{z^2+1}dz$.

9. Expand $f(z)$ in a Laurent series valid for given K:

$$f(z)=\frac{1}{(z-2)(z-3)},\quad K:\ 2<|z|<3.$$

10. Evaluate $\oint\limits_{|z|=3} \frac{\cos z\,dz}{z^2(z^2-4)}$.

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Series
Theory of Functions
of a Complex Variable

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по дисциплине **«Математика»**

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