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УЧРЕЖДЕНИЕ ОБРАЗОВАНИЯ
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Кафедра высшей математики

DIFFERENTIAL EQUATIONS
MULTIPLE INTEGRALS
INFINITE SEQUENCES AND SERIES

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**

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Настоящая методическая разработка предназначена для иностранных студентов технических специальностей. Данная разработка содержит необходимый материал по разделам «Дифференциальные уравнения», «Кратные интегралы», «Криволинейные интегралы» и «Ряды» изучаемый в общем курсе дисциплины «Математика» изложенный на английском языке. Изложение теоретического материала по всем темам сопровождается рассмотрением большого количества примеров и задач, некоторые понятия и примеры проиллюстрированы.

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I DIFFERENTIAL EQUATIONS

Perhaps the most important of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.

1.1 Modeling with Differential Equations

In describing the process of modeling, we talked about formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on evidence from experiments. The mathematical model often takes the form of a differential equation, that is, an equation that contains an unknown function and some of its derivatives. This is not surprising because in a realworld problem we often notice that changes occur and we want to predict future behavior on the basis of how current values change. Let’s begin by examining several examples of how differential equations arise when we model physical phenomena.

Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let’s identify and name the variables in this model: $t = \text{time}$ (the independent variable); $P =$ the number of individuals in the population (the dependent variable).

The rate of growth of the population is the derivative $\frac{dP}{dt}$. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP \tag{1}$$

where k is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function P and its derivative $\frac{dP}{dt}$.

Having formulated a model, let’s look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all t. So, if $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all t.

This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that $\frac{dP}{dt}$ becomes larger. In other words, the growth rate increases as the population increases.

Equation 1 asks us to find a function whose derivative is a constant multiple of itself. We know that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = (Ce^{kt})' = Cke^{kt} = k(Ce^{kt}) = kP(t).$$

Thus any exponential function of thee form $P(t) = Ce^{kt}$ is a solution of Equation 1.

Allowing C to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 1. But populations have only positive values and so we are interested only in the solutions with $C > 0$. And we are probably concerned only with values of t greater than the initial time $t = 0$. Figure 2 shows the physically meaningful solutions. Putting $t = 0$, we get $P(0) = Ce^{0k} = C$, so the constant C turns out to be the initial population $P(0)$.

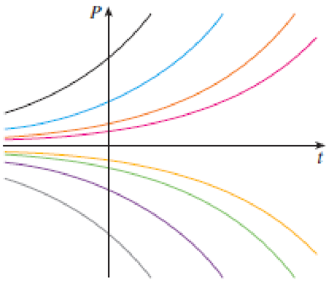


Figure 1

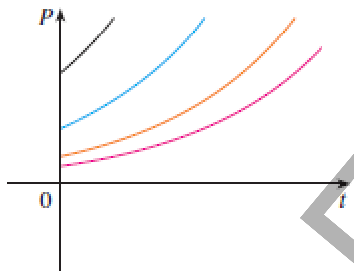


Figure 2

General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus, Equations 1 are first-order equation. In this equation the independent variable is called *t* and represents time, but in general the independent variable doesn't have to represent time. For example, when we consider the differential equation

$$y' = xy$$

it is understood that *y* is an unknown function of x.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple differential equations, namely, those of the form

$$y' = f(x)$$

But, in general, solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations.

When applying differential equations, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(x_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (x_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

Direction Fields

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields).

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y, \ y(0) = 1.$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the *x* - and *y* - coordinates of the point (see Figure 3). In particular, because the curve passes through the point $(0,1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0,1)$ looks like a short line segment through $(0,1)$ with slope 1 (see Figure 4.).

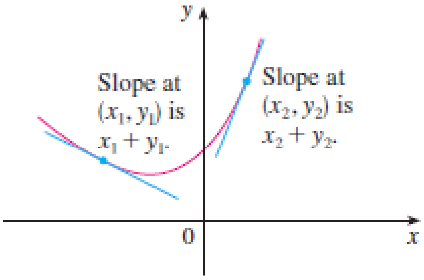


Figure 3

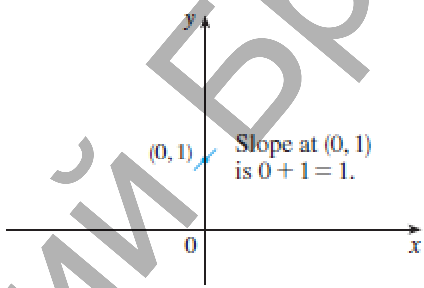


Figure 4

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 5. For instance, the line segment at the point $(1,2)$ has slope $1 + 2 = 3$. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

Now we can sketch the solution curve through the point $(0,1)$ by following the direction field as in Figure 6. Notice that we have drawn the curve so that it is parallel to nearby line segments.

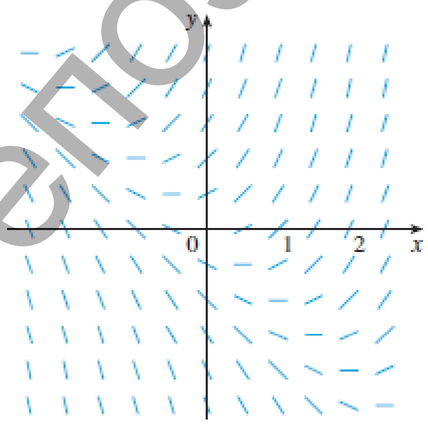


Figure 5

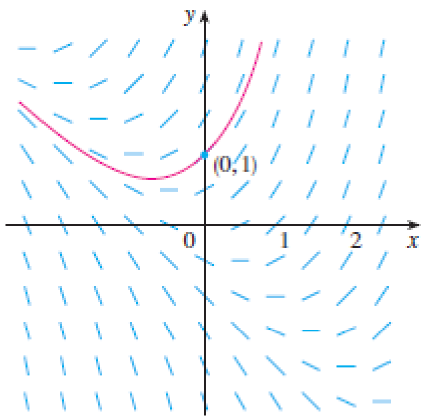


Figure 6

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x,y)$ is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x,y) on the curve is $F(x,y)$. If we draw short line segments with slope $F(x,y)$ at several points (x,y), the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

1.2 Separable Equations

We have looked at first-order differential equations from a geometric point of view (direction fields). What about the symbolic point of view? It would be nice to have an explicit formula for a solution of a differential equation. Unfortunately, that is not always possible. But in this section we examine a certain type of differential equation that *can* be solved explicitly.

A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of *x* times a function of *y*. In other words, it can be written in the form

$$\frac{dy}{dx} = f(x)\varphi(y) \tag{1}$$

the name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if $\varphi(y) \neq 0$, we could solve this equation we rewrite it in the differential form

$$dy = f(x)\varphi(y)dx,$$

$$\frac{dy}{\varphi(y)} = \frac{dx}{f(x)}.$$

Then we integrate both sides of the equation:

$$\int \frac{dy}{\varphi(y)} = \int \frac{dx}{f(x)} \tag{2}$$

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

Note 1. If

$$f_1(x) \times f_2(y)dx + \varphi_1(x) \times \varphi_2(y)dy = 0,$$

then

$$\int \frac{f_1(x)}{\varphi_1(x)}dx + \int \frac{\varphi_2(y)}{f_2(y)}dy = C, \qquad \varphi_1(x) \neq 0, \qquad f_2(y) \neq 0.$$

Note 2. The differential equation of form

$$M(x,y)dx + N(x,y)dy = 0 \tag{3}$$

is called **homogeneous differential first order equation.** If function $M(x,y); N(x,y)$ - the uniform functions of one and the same measurement. Equation (3) it is possible to lead to the form

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right) \tag{4}$$

With the aid of the substitution

$$y = x u(x)$$

of equation (3) or (4) are converted to the separable equation.

Example 1. Solve the differential equation

$$\sin^2 3x \times dy + 3y\,dx = 0.$$

Solution. We write the equation in terms of differentials and integrate both sides:

$$\frac{dy}{y} = -\frac{3dx}{\sin^2 3x} \quad \text{or} \quad \frac{dy}{y} = -\frac{d(3x)}{\sin^2 3x};$$

$$\int \frac{dy}{y} = -\int \frac{d(3x)}{\sin^2 3x} + C;$$

$$\ln|y| = \operatorname{ctg} 3x + C$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$) Solving for y, we get

$$y = C e^{\operatorname{ctg} 3x}.$$

Example 2. Solve the differential equation

$$(y^2 + xy^2) \cdot y' + x^2 - yx^2 = 0.$$

Solution. We write the equation in terms of differentials and integrate both sides:

$$y^2(1+x)dy = x^2(y-1)dx.$$

If $x \neq 0$, $y \neq 0$, $x \neq -1$, $y \neq 1$, then

$$\frac{y^2}{y-1}dy = \frac{x^2}{x+1}dx;$$

$$\int\left(y+1+\frac{1}{y-1}\right)dy = \int\left(x-1+\frac{1}{x+1}\right)dx;$$

$$\frac{y^2}{2} + y + \ln|y-1| = \frac{x^2}{2} - x + \ln|x+1| + C$$

where C is an arbitrary constant.

Example 3. Find the solution of the initial-value problem

$$\left(x^2 - 3y^2\right)dx + 2xy\,dy = 0, \; y(2) = 1.$$

Solution. Let us write down equation in the form

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy};$$

$$\frac{dy}{dx} = \frac{1}{2}\left(3\frac{y}{x} - \frac{x}{y}\right).$$

Let us introduce the replacement of the unknown function of $y = xu(x)$, then

$$y' = u(x) + x \cdot u'(x).$$

Substituting these expressions into the initial equation, we will obtain the separable equation.

$$x\cdot u'(x)+u=\frac{1}{2}\left(3u-\frac{1}{u}\right);\quad xu'=\frac{1}{2}\left(u-\frac{1}{u}\right);\quad xu'=\frac{u^2-1}{2u};\quad x\,du=\frac{u^2-1}{2u}dx;$$

$$\frac{2u\,du}{u^2-1}=\frac{dx}{x};\quad \int\frac{2u\,du}{u^2-1}=\int\frac{dx}{x};\quad \ln|u^2-1|=\ln|x|+\ln|C|;\quad u^2-1=Cx;$$

$$\left(\frac{y}{x}\right)^2-1=Cx.$$

Since $y(2)=1$, we have $1-4=8C$; $C=-\frac{3}{8}$. Therefore the solution to the initial-value problem is

$$\frac{y^2}{x^2}=1-\frac{3}{8}x,\quad y=\pm x\sqrt{1-\frac{3}{8}x}.$$

1.3 Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$y'+p(x)y=q(x)\quad (\text{or}\quad A(x)y'+B(x)y+C(x)=0) \tag{1}$$

where $p(x)$ and $q(x)$ are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function called an *integrating factor*

$$I(x)=e^{\int p(x)dx}.$$

Thus a formula for the general solution to Equation 1 is provided by solution

$$y(x)=\frac{1}{I(x)}\left(\int I(x)q(x)dx+C\right),$$

where $I(x)=e^{\int p(x)dx}$. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

Note 1. It turns out that every first-order linear differential equation can be solved with the aid of the substitution $y=u(x)v(x)$, where $u(x),v(x)$ -the unknown functions. This equation we reduce to the form

$$u'v+uv'+p(x)uv=q(x),\qquad u'v+u(v'+p(x)v)=q(x).$$

Since one of the unknown functions can be selected arbitrarily, then as $v(x)$ is taken any particular solution of the equation

$$v'+p(x)v=0,$$

function $u(x)$ will be determined from the equation

$$u'(x)v(x)=q(x).$$

Thus, the solution of linear equation is reduced to the sequential solution of two equations with the divided variables relative to each of the auxiliary functions.

Example 1. Solve the differential equation

$$y'+\frac{3y}{x}=x^2.$$

Solution. We use the substitution $y = uv$, $y' = u'v + uv'$. We obtain the following equation

$$u'v + uv' + \frac{3uv}{x} = x^2;$$

$$u'v + u\left(v' + \frac{3v}{x}\right) = x^2.$$

We solve the consecutively two equations

$$v' + \frac{3v}{x} = 0 \text{ and } u'v = x^2.$$

$$\frac{dv}{dx} = -\frac{3v}{x}; \ \frac{dv}{v} = -\frac{3dx}{x}; \ \int\frac{dv}{v} = -\int\frac{3dx}{x}; \quad \ln|v| = -3\ln|x|; \quad v = \frac{1}{x^3}.$$

$$u' \cdot \frac{1}{x^3} = x^2; \ \frac{du}{dx} = x^5; \ du = x^5 dx; \quad u = \int x^5 dx = \frac{x^6}{6} + C.$$

Multiplying $u(x)$ on $v(x)$, we obtain the general solution of this equation

$$y = \frac{1}{x^3}\left(\frac{x^6}{6} + C\right) \text{ or } y = \frac{x^3}{6} + \frac{C}{x^3}, \ C - \text{const}.$$

Note 2. A first-order **linear** differential equation is one that can be put into the form

$$x' + p(y)x = q(y).$$

This linear differential equation can be solved with the aid of the substitution

$$x(y) = u(y)v(y).$$

Note 3. The **Bernoulli equations** take the form

$$y' + p(x)y = q(x)y^n \text{ or } x' + p(y)x = q(y)x^n, \quad n \in R.$$

These equations can be reduced to the appropriate linear equations, but them they are usually solved with the aid of the substitution

$$y = u(x)v(x) \text{ or } x = u(y)v(y).$$

Example 2. Find the solution of the initial-value problem

$$2ydx + \left(y^2 - 6x\right)dy = 0, \ y(6) = 2.$$

Solution. It is easy to see that this equation is not linear relative to y. Let us write it down in the form

$$2y\frac{dx}{dy} + y^2 - 6x = 0; \quad \frac{dx}{dy} - \frac{3}{y}x = -\frac{y}{2}; \quad x = uv; \quad u'v + uv' - \frac{3}{y}uv = -\frac{y}{2};$$

$$u'v + u\left(v' - \frac{3}{y}v\right) = -\frac{y}{2}; \quad \frac{dv}{dy} = \frac{3v}{y}; \quad \frac{dv}{v} = \frac{3dy}{y}; \quad \ln|v| = 3\ln|y|; \quad v = y^3.$$

Then from the equation $u'v = -\frac{y}{2}$ we determine the function $u(y)$

$$u'y^3 = -\frac{y}{2}; \quad u' = -\frac{1}{2y^2}; \ du = -\frac{dy}{2y^2}; \quad u = \frac{1}{2y} + C.$$

Let us extract the general solution of the initial equation

$x = uv;\quad x = \left(\frac{1}{2y} + C\right)y^3;\qquad x = Cy^3 + \frac{y^2}{2}.$

Since $y(6) = 2$, we have $6 = 8C + 2$, $C = \frac{1}{2}$. Therefore the solution to the initial-value problem is $x = 0{,}5\left(y^3 + y^2\right)$.

Exercise Set 1

In Exercise 1 to 10, solve the differential equation.

1. $\mathrm{tg}x\,dy - y\,dx = 0$.
2. $\sqrt{16 + x^2}\,dy - y\,dx = 0$.
3. $x^2\,dy + y\,dx = 0$.
4. $\left(e^x + 2\right)dy - y\,e^x\,dx = 0$.
5. $\left(4 + x^2\right)dy - y\,dx = 0$.
6. $\mathrm{ctg}x\,dy + y\,dx = 0$.
7. $\sqrt{x^2 - 25}\,dy - y\,dx = 0$.
8. $xy' = \sqrt{y^2 - x^2}$.
9. $\left(x^2 + y^2\right)dx = 2xydy$.
10. $(x^2 - 3y^2)dx + 2xydy = 0$.

In Exercise 11 to 15, solve the initial-value problem.

11. $y' - \frac{2}{x}y = 3e^{3x-6}x^2$, $\quad y(2) = 8$.
12. $y' - \frac{5}{x}y = 3e^{3x+6}x^5$, $\quad y(-2) = 32$.
13. $y' - \frac{5}{x}y = 6x^5\sin(6x + 6)$, $\quad y(-1) = 2$.
14. $y' - \frac{3}{x}y = 2e^{2x-6}x^3$, $\quad y(3) = 54$.
15. $y' - \frac{4}{x}y = 2e^{2x+4}x^4$, $\quad y(-2) = 32$.

1.4 Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$P(x)y'' + Q(x)y' + R(x)y = G(x) \tag{1}$$

where $P(x), Q(x), R(x), G(x)$ are continuous functions.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1. Such equations are called **homogeneous** linear equations. Thus the form of a second-order linear homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous** and is discussed in Section 1.5.

Theorem 1. If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and C_1 and C_2 are any constants, then the function $y = C_1y_1 + C_2y_2$ is also a solution of Equation 2.

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions $y_1(x)$ and $y_2(x)$. This means that neither $y_1(x)$ nor $y_2(x)$ is a constant multiple of the other.

Theorem 2. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of Equation 2, and $P(x) \neq 0$, then the general solution is given by $y = C_1 y_1 + C_2 y_2$, where C_1 and C_2 are arbitrary constants.

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions $P(x), Q(x), R(x)$ are constant functions, that is, if the differential equation has the form

$$y'' + py' + qy = 0 \qquad (3)$$

where p, q are constants.

It's not hard to think of some likely candidates for particular solutions of Equation 3 if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{kx}$ (where k is a constant) has the property that its derivative is a constant multiple of itself: $y' = ke^{kx}$. Furthermore, $y'' = k^2 e^{kx}$. If we substitute these expressions into Equation 3, we see that $y = e^{kx}$ is a solution if

$$e^{kx}(k^2 + pk + q) = 0.$$

Thus $y = e^{kx}$ is a solution of Equation 3 if k is a root of the equation

$$k^2 + pk + q = 0 \qquad (4)$$

Equation 4 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation (3).

The general solution of initial equation takes the form:

1. $y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$, if $k_1 \neq k_2$, $k_1, k_2 \in R$;
2. $y = e^{k_1 x}(C_1 + C_2 x)$, if $k_1 = k_2$;
3. $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$, if $k_{1,2} = \alpha \pm i\beta$.

Example 1. Solve the equations

a) $y'' - 5y' + 6y = 0$; b) $y'' + 8y' + 16y = 0$; c) $y'' - 6y' + 13y = 0$.

Solution. For each case we compile characteristic equation, we find its roots, we extract the appropriate linearly independent solutions of differential equation and their general solution:

a) $k^2 - 5k + 6 = 0 \Rightarrow k_1 = 2,\ k_2 = 3 \Rightarrow y_1 = e^{2x},\ y_2 = e^{3x} \Rightarrow y = C_1 e^{2x} + C_2 e^{3x}$;

b) $k^2 + 8k + 16 = 0 \Rightarrow k_1 = -4,\ k_2 = -4 \Rightarrow y_1 = e^{-4x},\ y_2 = xe^{-4x} \Rightarrow y = e^{-4x}(C_1 + C_2 x)$;

c) $k^2 - 6k + 13 = 0 \Rightarrow k_{1,2} = 3 \pm 2i \Rightarrow \alpha = 3, \beta = 2 \Rightarrow y_1 = e^{3x}\cos 2x,\ y_2 = e^{3x}\sin 2x \Rightarrow$

$y = e^{3x}\left(C_1 \cos 2x + C_2 \sin 2x\right)$.

Example 2. Solve the initial-value problem

$$y'' - 5y' + 6y = 0; \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution. From Example 1 we know that the general solution of the differential equation is

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

Differentiating this solution, we get

$$y' = 2C_1e^{2x} + 3C_2e^{3x}.$$

To satisfy the initial conditions we require that

$$y(0) = C_1 + C_2 = 1 \qquad \text{(a)}$$

$$y'(0) = 2C_1 + 3C_2 = 0 \qquad \text{(b)}$$

From (b), we have $C_2 = -\frac{2}{3}C_1$ and so (a) gives

$$C_1 - \frac{2}{3}C_1 = 1;\ \frac{1}{3}C_1 = 1;\ C_1 = 3;\ C_2 = -\frac{2}{3}C_1 = -\frac{2}{3}\cdot 3 = -2.$$

Thus the required solution of the initial-value problem is

$$y = 3e^{2x} - 2e^{3x}.$$

1.5 Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$y'' + py' + qy = f(x) \qquad (1)$$

where p, q are constants and $f(x)$ are continuous function. The related homogeneous equation

$$y'' + py' + qy = 0 \qquad (2)$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

Theorem. The general solution of the nonhomogeneous differential equation (1) can be written as

$$y = \overline{y} + y^*,$$

where $\overline{y}$ is a particular solution of Equation 1 and y^* is the general solution of the complementary Equation 2.

Therefore Theorem says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution $\overline{y}$. There are two methods for finding a particular solution: The method of undetermined coefficients is straightforward but works only for a restricted class of functions $f(x)$. The method of variation of parameters works for every function $f(x)$ but is usually more difficult to apply in practice.

The Method of Undetermined Coefficients

1. If $f(x) = P_n(x)\cdot e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree n, then try

$$y^* = x^r Q_n(x)\cdot e^{\alpha x},$$

where $Q_n(x)$ is an n th-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $f(x) = e^{ax}(P_n(x)\cos bx + Q_m(x)\sin bx)$, where $P_n(x)$ is an n th-degree polynomial ($Q_n(x)$ is an m th-degree polynomial), then try

$$y^* = x^r e^{ax}(S_N(x)\cos bx + T_N(x)\sin bx),$$

where $S_N(x), T_N(x)$ and are N th-degree polynomials ($N = \max\{n,m\}$).

Modification: r number is equal to the multiplicity of the number with respect to the roots of the characteristic equation.

Example 1. Solve the equation $y'' + 3y' - 4y = -4x^2 - 6x + 19$.

Solution. The auxiliary equation of $y'' + 3y' - 4y = 0$ is $k^2 + 3k - 4 = 0$ with roots

$$k_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot(-4)}}{2\cdot 1} = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2},$$

$$k_1 = \frac{-3-5}{2} = \frac{-8}{2} = -4, \; k_2 = \frac{-3+5}{2} = \frac{2}{2} = 1.$$

So the solution of the complementary equation is $\overline{y} = C_1 e^{-4x} + C_2 e^{x}$.

Since $f(x) = -4x^2 - 6x + 19$ is a polynomial of degree 2, we seek a particular solution of the form $y^* = Ax^2 + Bx + C$. Then $(y^*)' = 2Ax + B$ and $(y^*)'' = 2A$ so, substituting into the given differential equation, we have

$$2A + 3\left(2Ax + B\right) - 4\left(Ax^2 + Bx + C\right) = -4x^2 - 6x + 19$$

$$-4Ax^2 + \left(6A - 4B\right)x + \left(2A + 3B - 4C\right) = -4x^2 - 6x + 19.$$

Polynomials are equal when their coefficients are equal. Thus

$$\begin{cases} -4A = -4, \\ 6A - 4B = -6, \\ 2A + 3B - 4C = 19, \end{cases} \quad \text{or} \quad \begin{cases} A = 1, \\ B = 3, \\ C = -2. \end{cases}$$

A particular solution is therefore $y^* = Ax^2 + Bx + C = x^2 + 3x - 2$.

The general solution is $y = \overline{y} + y^* = C_1 e^{-4x} + C_2 e^{x} + x^2 + 3x - 2$.

Example 2. Solve the equation $y'' - y' - 2y = 4xe^{x}$.

Solution. The auxiliary equation of $y'' - y' - 2y = 0$ is $k^2 - k - 2 = 0$ with roots $k_1 = -1, \; k_2 = 2$. So the solution of the complementary equation is $\overline{y} = C_1 e^{-x} + C_2 e^{2x}$.

For a particular solution we try $y^* = (Ax + B)e^{x}$.

Then $(y^*)' = Ae^{x} + (Ax + B)e^{x} = (Ax + A + B)e^{x}$ and $(y^*)'' = (Ax + 2A + B)e^{x}$ so, substituting into the given differential equation, we have

$$2Ae^{x} + (Ax + B)e^{x} - Ae^{x} - (Ax + B)e^{x} - 2(Ax + B)e^{x} = 4xe^{x},$$

$$A - 2Ax - 2B = 4x.$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 4, \;\; A - 2B = 0; \quad A = -2, \;\; B = -1.$$

A particular solution is therefore

$$y^* = -(2x + 1)e^{x}.$$

The general solution is

$$y = C_1 e^{-x} + C_2 e^{2x} - (2x + 1)e^{x}.$$

Example 3. Solve the equation $y'' + y = x\sin x$.

Solution. The auxiliary equation of $y'' + y = 0$ is $k^2 + 1 = 0$ with roots

$$k = \pm i = 0 \pm 1 \cdot i \ (\alpha = 0, \beta = 1).$$

So the solution of the complementary equation is

$$\overline{y} = C_1 \cos x + C_2 \sin x.$$

For a particular solution we try

$$y^* = x((Ax + B)\cos x + (Cx + D)\sin x).$$

We find derivatives $(y^*)'$, $(y^*)''$ and substitute them in the assigned equation:

$$y^* = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x;$$

$$(y^*)' = (2Ax + B)\cos x - (Ax^2 + Bx)\sin x + (2Cx + D)\sin x + (Cx^2 + Dx)\cos x;$$

$$(y^*)'' = 2A\cos x - 2(2Ax + B)\sin x - (Ax^2 + Bx)\cos x + 2C\sin x + 2(2Cx + D)\cos x - (Cx^2 + Dx)\sin x.$$

$$2A\cos x - 2(2Ax + B)\sin x + 2C\sin x + 2(2Cx + D)\cos x = x\sin x.$$

This expressions are equal when their coefficients before $\sin x$, $\cos x$, $x\sin x$, $x\cos x$ are equal

$$\begin{cases} 2A + 2D = 0, \\ 4C = 0, \\ -2B + 2C = 0, \\ -4A = 1 \end{cases} \quad \text{or} \quad \begin{cases} A = -\dfrac{1}{4}, \\ D = \dfrac{1}{4}, \\ B = C = 0. \end{cases}$$

A particular solution is therefore $y^* = -\dfrac{x^2}{4}\cos x + \dfrac{x}{4}\sin x$.

The general solution is $y = C_1\cos x + C_2\sin x - \dfrac{x^2}{4}\cos x + \dfrac{x}{4}\sin x$.

The Method of Variation of Parameters

Suppose we have already solved the homogeneous equation

$$y'' + a_1 y' + a_2 y = 0.$$

and written the solution as $\overline{y} = C_1 y_1(x) + C_2 y_2(x)$, where $C_1, C_2 - \forall\,\text{const}$, $y_1(x), y_2(x)$ are linearly independent solutions. We look for a particular solution of the nonhomogeneous equation of the form

$$y^* = C_1(x) y_1(x) + C_2(x) y_2(x).$$

(This method is called **variation of parameters** because we have varied the parameters C_1, C_2 to make them functions.)

Functions $C_1(x), C_2(x)$ are determined from the system of equations:

$$\begin{cases} C_1'(x) y_1(x) + C_2'(x) y_2(x) = 0, \\ C_1'(x) y_1'(x) + C_2'(x) y_2'(x) = f(x). \end{cases}$$

Example 4. Solve the equation $y'' + 4y = \dfrac{1}{\sin 2x}$.

Solution. The auxiliary equation of $y'' + 4y = 0$ is $k^2 + 4 = 0$ with roots $k = \pm 2i$.

So the solution of the complementary equation is

$$\overline{y} = C_1 \cos 2x + C_2 \sin 2x .$$

For a particular solution we try

$$y^* = C_1(x)\cos 2x + C_2(x)\sin 2x.$$

$$y_1(x) = \cos 2x, \quad y_1'(x) = -2\sin 2x,$$

$$y_2(x) = \sin 2x, \quad y_2'(x) = 2\cos 2x.$$

Functions $C_1(x), C_2(x)$ are determined from the system of equations:

$$\begin{cases} C_1'(x)\cos 2x + C_2'(x)\sin 2x = 0, \\ -2C_1'(x)\sin 2x + 2C_2'(x)\cos 2x = \dfrac{1}{\sin 2x}. \end{cases}$$

We solve this system according to Cramars' Rules.

$$\Delta = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2 .$$

Then

$$C_1'(x) = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ \dfrac{1}{\sin 2x} & 2\cos 2x \end{vmatrix} = -\frac{1}{2},$$

$$C_2'(x) = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \dfrac{1}{\sin 2x} \end{vmatrix} = \frac{1}{2}\operatorname{ctg} 2x.$$

Integrating last two equalities, we have:

$$C_1(x) = -\frac{1}{2}x,$$

$$C_2(x) = \frac{1}{4}\ln|\sin 2x| .$$

The general solution is

$$y = \overline{y} + y^* = C_1\cos 2x + C_2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \ln|\sin 2x| .$$

1.6 Systems of the differential equations

Method of the exception. Solution of the normal system of two differential first order equations, i.e., the system of the form

$$\begin{cases} \dfrac{dy}{dx} = f(x,y,z), \\ \dfrac{dz}{dx} = g(x,y,z), \end{cases}$$

by that permitted relative to derivatives of two unknown functions $y(x)$ and $z(x)$ (or $x(t), y(t)$), it is reduced to the solution of one differential equation of the second order relative to one of the functions. Let us examine the process of information based on example.

Example 1. Find the general solution $\begin{cases} x = x(t), \\ y = y(t). \end{cases}$ of the system of the differential equations

$$\begin{cases} x' = x - y, \\ y' = -4x + y. \end{cases}$$

Solution. We differentiate the first equation of the system $x'' = x' - y'$. Let us replace in the last equation y' with its expression from the second equation of the system $x'' = x' - (-4x + y)$, $x'' = x' + 4x - y$. Let us replace in the last equation y with its expression from the second equation of the system $x'' = x' + 4x - (x - x')$, $x'' - 2x' - 3x = 0$.

The auxiliary equation of $x'' - 2x' - 3x = 0$ is $k^2 - 2k - 3 = 0$ with roots $k_1 = -1,\ k_2 = 3$.

So the solution of the complementary equation is

$$x(t) = C_1 e^{-t} + C_2 e^{3t}.$$

Differentiating this equation for variable *t*, we will obtain: $x' = -C_1 e^{-t} + 3C_2 e^{3t}$.

Then we find y from the equation $y = x - x'$:

$$y(t) = C_1 e^{-t} + C_2 e^{3t} - \left(-C_1 e^{-t} + 3C_2 e^{3t}\right) = 2C_1 e^{-t} - 2C_2 e^{3t}.$$

Answer: $\begin{cases} x(t) = C_1 e^{-t} + C_2 e^{3t}, \\ y(t) = 2C_1 e^{-t} - 2C_2 e^{3t}. \end{cases}$

Euler's method of the solution of the linear uniform systems of differential equations with the constant coefficients.

Assume that the system of three equations with three unknown functions is assigned:

$$\begin{cases} x'(t) = a_{11}x + a_{12}y + a_{13}z, \\ y'(t) = a_{21}x + a_{22}y + a_{23}z, \\ z'(t) = a_{31}x + a_{32}y + a_{33}z. \end{cases}$$

Find $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$.

We will search for unknown functions in the form

$$\begin{cases} x(t) = \alpha \cdot e^{kt}, \\ y(t) = \beta \cdot e^{kt}, \\ z(t) = \gamma \cdot e^{kt}. \end{cases}$$

Substituting these expressions into the system and converting it, we will obtain the system of linear homogeneous algebraic equations relatively α, β, γ:

$$\begin{cases}(a_{11}-k)\alpha+a_{12}\beta+a_{13}\gamma=0,\\ a_{21}\alpha+(a_{22}-k)\beta+a_{23}\gamma=0,\\ a_{31}\alpha+a_{32}\beta+(a_{33}-k)\gamma=0.\end{cases} \tag{1}$$

System (1) has non-trivial solutions, if its determinant is equal to zero. We will obtain cubic equation for determining the number k:

$$\Delta=\begin{vmatrix}a_{11}-k & a_{12} & a_{13}\\ a_{21} & a_{22}-k & a_{23}\\ a_{31} & a_{32} & a_{33}-k\end{vmatrix}=0. \tag{2}$$

Equation (2) is called the characteristic equation of reference system. We solve it, we find values k, for each value from system (1) find α,β,γ, write the linearly independent solutions for each unknown function, we compose the general solution of the system.

Example 2. Find the general solution of the system of the differential equations

$$\begin{cases}x'=x-y+z,\\ y'=x+y-z,\\ z'=2x-y.\end{cases}$$

Solution. The characteristic equation of this system takes the form

$$\begin{vmatrix}1-k & -1 & 1\\ 1 & 1-k & -1\\ 2 & -1 & -k\end{vmatrix}=0,$$

$$(1-k)^2(-k)-1+2-2(1-k)-(1-k)-k=0,$$

$$(k-1)(k-2)(k+1)=0,$$

$$k_1=1, k_2=2, k_3=-1.$$

The appropriate values α,β,γ for each k let us find from the system of equations

$$\begin{cases}(1-k)\alpha-\beta+\gamma=0,\\ \alpha+(1-k)\beta-\gamma=0,\\ 2\alpha-\beta-k\gamma=0.\end{cases} \tag{3}$$

If $k=1$, then

$$\begin{cases}-\beta+\gamma=0,\\ \alpha-\gamma=0,\\ 2\alpha-\beta-\gamma=0,\end{cases}\Rightarrow\begin{cases}\beta=\gamma,\\ \alpha=\gamma,\\ 0=0,\end{cases}\Rightarrow\begin{cases}\alpha=1,\\ \beta=1,\\ \gamma=1,\end{cases}\Rightarrow\begin{cases}x_1=e^t,\\ y_1=e^t,\\ z_1=e^t.\end{cases}$$

If $k=2$, then

$$\begin{cases}-\alpha-\beta+\gamma=0,\\ \alpha-\beta-\gamma=0,\\ 2\alpha-\beta-2\gamma=0,\end{cases}\Rightarrow\begin{cases}2\beta=0,\\ \alpha=\gamma,\\ 0=0,\end{cases}\Rightarrow\begin{cases}\alpha=1,\\ \beta=0,\\ \gamma=1,\end{cases}\Rightarrow\begin{cases}x_2=e^{2t},\\ y_2=0,\\ z_2=e^{2t}.\end{cases}$$

If $k=-1$, then

$$\begin{cases}2\alpha-\beta+\gamma=0,\\ \alpha+2\beta-\gamma=0,\\ 2\alpha-\beta+\gamma=0,\end{cases}\Rightarrow\begin{cases}2\alpha-\beta+\gamma=0,\\ 3\alpha+\beta=0,\end{cases}\Rightarrow\begin{cases}\beta=-3\alpha,\\ \gamma=-5\alpha,\end{cases}\Rightarrow\begin{cases}\alpha=1,\\ \beta=-3,\\ \gamma=-5,\end{cases}\Rightarrow\begin{cases}x_3=e^{-t},\\ y_3=-3e^{-t},\\ z_3=-5e^{-t}.\end{cases}$$

We extract the general solution of the system of the differential equations

$$\begin{cases}x(t)=C_1x_1+C_2x_2+C_3x_3,\\ y(t)=C_1y_1+C_2y_2+C_3y_3,\\ z(t)=C_1z_1+C_2z_2+C_3z_3,\end{cases}\Rightarrow\begin{cases}x(t)=C_1e^{t}+C_2e^{2t}+C_3e^{-t},\\ y(t)=C_1e^{t}-3C_3e^{-t},\\ z(t)=C_1e^{t}+C_2e^{2t}-5C_3e^{-t}.\end{cases}$$

Exercise Set 2

In Exercise 1 to 10, solve the differential equation.

1. a) $y''+4y'-12y=0$, b) $y''-4y'+4y=0$, c) $y''+6y'+13y=0$.
2. a) $y''-2y'-15y=0$, b) $y''+8y'+16y=0$, c) $y''-10y'+29y=0$.
3. a) $y''+2y'-8y=0$, b) $y''-14y'+49y=0$, c) $y''+6y'+34y=0$.
4. a) $y''-3y'-10y=0$, b) $y''+10y'+25y=0$, c) $y''-8y'+25y=0$.
5. a) $y''+6y'-16y=0$, b) $y''-6y'+9y=0$, c) $y''+4y'+20y=0$.
6. a) $y''-5y'-14y=0$, b) $y''+22y'+121y=0$, c) $y''-8y'+41y=0$.
7. a) $y''+y'-12y=0$, b) $y''-18y'+81y=0$, c) $y''+4y'+40y=0$.
8. a) $y''-y'-20y=0$, b) $y''+12y'+36y=0$, c) $y''-14y'+53y=0$.
9. a) $y''+4y'-21y=0$, b) $y''-20y'+100y=0$, c) $y''+8y'+20y=0$.
10. a) $y''-3y'-18y=0$, b) $y''+16y'+64y=0$, c) $y''-10y'+34y=0$.

In Exercise 11 to 15, solve the differential equation.

11. a) $y''+2y'-8y=-16x^2+16x-22$. b) $y''+4y'+4y=6e^{-2x}$.
12. a) $y''-y'-6y=-24x^2-2x-9$. b) $y''-2y'-8y=18e^{4x}$.
13. a) $y''-4y'-5y=-15x^2+6x+10$. b) $y''-8y'+16y=6e^{4x}$.
14. a) $y''-6y'-7y=-14x^2+4x+7$. b) $y''+3y'-10y=14e^{2x}$.
15. a) $y''+y'-6y=-24x^2-4x+16$. b) $y''-6y'+9y=10e^{3x}$.

In Exercise 16 to 18, solve the differential equation.

16. $y''+y=2\cos x$.
17. $y''+4y=4(\cos 2x+\sin 2x)$.
18. $y''+y=4\sin x-6\cos x$.

In Exercise 19, solve the initial-value problem.

19. $y''+9y=2\cos 4x-3\sin 4x,\ y(0)=0,\ y'(0)=12$.

In Exercise 20 to 22, solve the differential equation.

20. $y'' - 2y' + y = \frac{e^x}{x^2+1}$.

21. $y'' + 2y' + 2y = \frac{1}{e^x \sin x}$.

22. $y'' - y' = e^{2x} \cos e^x$.

In Exercise 23 to 32, find the general solution of the system of the differential equations.

23. $\begin{cases} x' = 2x + y, \\ y' = 3x + 4y. \end{cases}$

24. $\begin{cases} x' = 3x + y, \\ y' = x + 3y. \end{cases}$

25. $\begin{cases} x' = 4x - 8y, \\ y' = -8x + 4y. \end{cases}$

26. $\begin{cases} x' = 4x + 2y, \\ y' = 4x + 6y. \end{cases}$

27. $\begin{cases} x' = 3x + y, \\ y' = 8x + y. \end{cases}$

28. $\begin{cases} x' = 2x + y, \\ y' = -6x - 3y. \end{cases}$

29. $\begin{cases} x' = 4x - y, \\ y' = -x + 4y. \end{cases}$

30. $\begin{cases} x' = 6x - y, \\ y' = 3x + 2y. \end{cases}$

31. $\begin{cases} x' = x + 2y, \\ y' = 4x + 3y. \end{cases}$

32. $\begin{cases} x' = -2x + y, \\ y' = -3x + 2y. \end{cases}$

In Exercise 33 to 36, find the general solution of the system of the differential equations.

33. $\begin{cases} x' = 5x + 2y - 3z, \\ y' = 4x + 5y - 4z, \\ z' = 6x + 4y - 4z. \end{cases}$

34. $\begin{cases} x'(t) = -3x + 4y - 2z, \\ y'(t) = x + z, \\ z'(t) = 6x - 6y + 5z. \end{cases}$

35. $\begin{cases} x'(t) = 3x - y + z, \\ y'(t) = x + y + z, \\ z'(t) = 4x - y + 4z. \end{cases}$

36. $\begin{cases} x'(t) = x - 4y - z, \\ y'(t) = x + y, \\ z'(t) = 3x + z. \end{cases}$

II MULTIPLE INTEGRALS

In this chapter we extend the idea of a definite integral to double and triple integrals of functions of two or three variables. We also use double integrals to calculate probabilities when two random variables are involved.

We will see that polar coordinates are useful in computing double integrals over some types of regions. In a similar way, we will introduce two new coordinate systems in three-dimensional space — cylindrical coordinates and spherical coordinates — that greatly simplify the computation of triple integrals over certain commonly occurring solid regions.

2.1 Double Integrals Over Rectangles

We consider a function f of two variables defined on a closed rectangle

$$R=\left\{(x,y)\in\Re^2 \mid a\le x\le b, c\le y\le d\right\}$$

and we first suppose that $f(x,y)\ge 0$. The graph of f is a surface with equation $z=f(x,y)$.

Let *S* be the solid that lies above *R* and under the graph of f, that is,

$$S=\left\{(x,y,z)\in\Re^3 \mid 0\le z\le f(x,y),(x,y)\in\Re\right\}$$

(See Figure 1). Our goal is to find the volume of *S*.

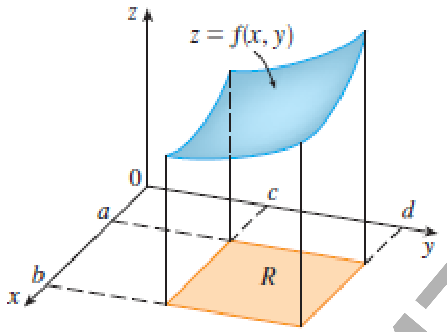


Figure 1

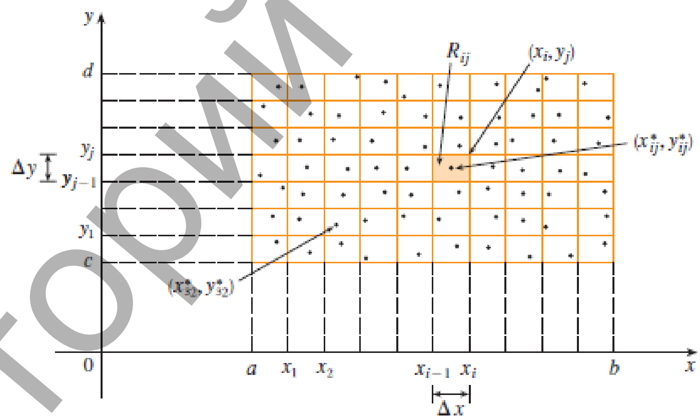


Figure 2

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1},x_i]$ of equal width $\Delta x=\dfrac{b-a}{m}$ and dividing $[c,d]$ into n subintervals $\left[y_{j-1},y_j\right]$ of equal width $\Delta y=\dfrac{d-c}{n}$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 2, we form the subrectangles

$$R_{ij}=\left\{(x,y) \mid x_{i-1}\le x\le x_i, y_{j-1}\le y\le y_j\right\}$$

each with area $\Delta S=\Delta x\Delta y$.

If we choose a **sample point** (x_{ij}^*,y_{ij}^*) in each R_{ij}, then we can approximate the part of *S* that lies above each R_{ij} by a thin rectangular box (or “column”) with base R_{ij} and height $f(x_{ij}^*,y_{ij}^*)$ as shown in Figure 3.The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\Delta S.$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of *S*:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S \qquad (1)$$

(See Figure 4.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

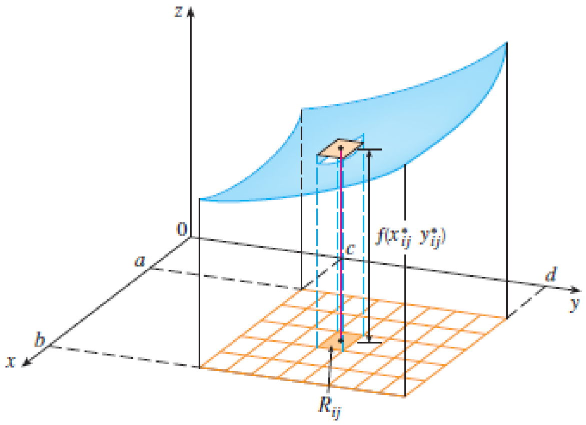


Figure 3

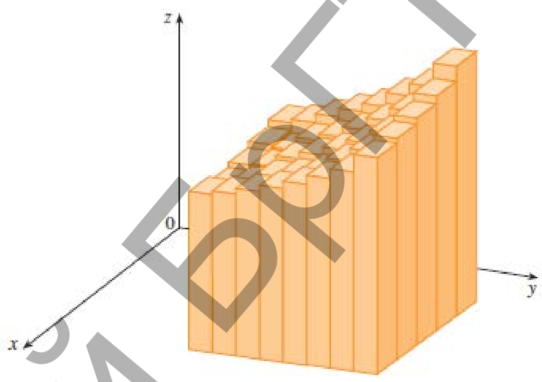


Figure 4

Our intuition tells us that the approximation given in (1) becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S. \qquad (2)$$

We use the expression in Equation 2 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R.

Limits of the type that appear in Equation 2 occur frequently, not just in finding volumes but in a variety of other situations as well even when f is not a positive function. So we make the following definition.

Definition. The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S$$

if this limit exists.

A function f is called **integrable** if the limit in Definition exists. It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is "not too discontinuous." In particular, if f is bounded, and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij} but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 2], then the expression for the double integral looks simpler:

$$\iint\limits_R f(x,y)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta S \qquad (3)$$

The sum in Definition,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 4, and is an approximation to the volume under the graph of f and above the rectangle R.

Properties of Double Integrals

1. $\iint\limits_R (\alpha f_1(x,y) \pm \beta f_2(x,y))dxdy = \alpha\iint\limits_R f_1(x,y)dxdy \pm \beta\iint\limits_R f_2(x,y)dxdy.$

2. If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 5), then

$$\iint\limits_R f(x,y)dxdy = \sum_{i=1}^{n}\iint\limits_{R_i} f(x,y)dxdy.$$

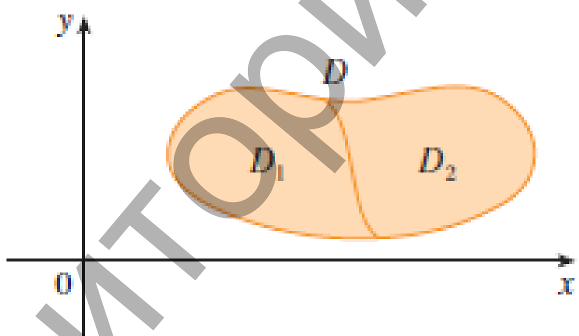


Figure 5

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then $\iint\limits_R f(x,y)dxdy \geq \iint\limits_R g(x,y)dxdy$.

4. If we integrate the constant function $f(x,y) = 1$ over a region D, we get the area of D:

$$S_D = \iint\limits_D 1dS$$

5. If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint\limits_R f(x,y)dS.$$

6. **Midpoint rule for double integrals.** If function $z = f(x,y)$ is continuous in the closed domain R, then there is a point $P_0(x_0, y_0)$ in this region such, that

$$\iint_R f(x,y)dS = f(P_0)\cdot S,\quad \iint_R f(x,y)dxdy = f(x_0,y_0)\cdot S,\quad f(P_0) = \frac{1}{S}\iint_R f(x,y)dS$$

is the average value of function $z = f(x,y)$ in the region R.

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 6. We suppose that D is a bounded region, which means that it can be enclosed in a rectangular region R as in Figure 7. Then we define a new function F with domain by

$$F(x,y) = \begin{cases} f(x,y);(x,y) \in D \\ 0;(x,y) \in R,(x,y) \notin D \end{cases}$$

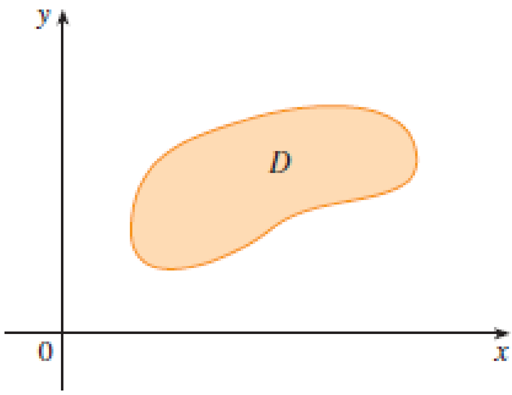


Figure 6

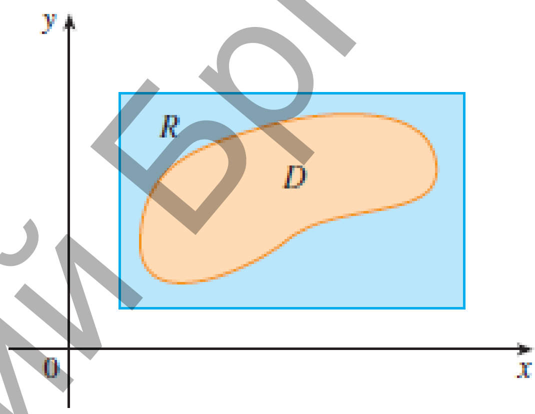


Figure 7

If F is integrable over R, then we define the **double integral of** f **over** D by

$$\iint_D f(x,y)dS = \iint_R F(x,y)dS.$$

In the case where $f(x,y) \geq 0$, we can still interpret $\iint_D f(x,y)dS$ as the volume of the solid that lies above D and under the surface $z = f(x,y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 8 and 9 and remembering that $\iint_R F(x,y)dS$ is the volume under the graph of F.

Figure 9 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is “well behaved” (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x,y)dS$ exists and therefore $\iint_D f(x,y)dS$ exists. In particular, this is the case for the following types of regions.

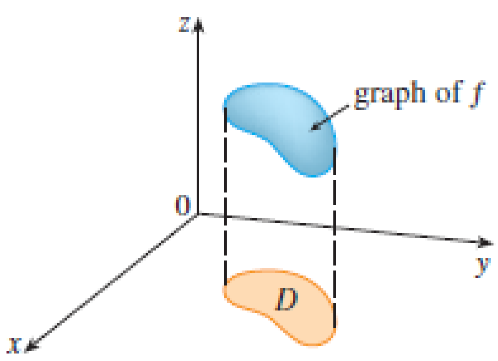


Figure 8

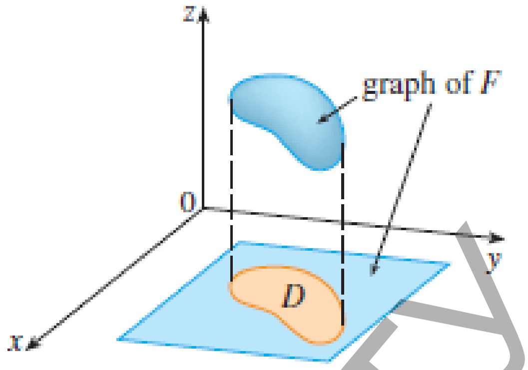


Figure 9

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is

$$D=\{(x,y)\,|\,a\le x\le b, g_1(x)\le y\le g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$. Some examples of type I regions are shown in Figure 10.

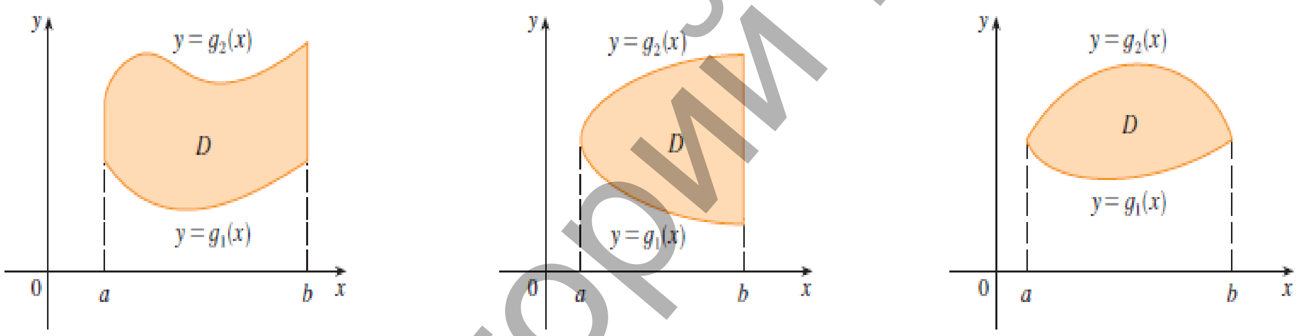


Figure 10

We also consider plane regions of **type II**, which can be expressed as

$$D=\{(x,y)\,|\,c\le y\le d, h_1(y)\le x\le h_2(y)\},$$

where $h_1(y)$ and $h_2(y)$ are continuous. Two such regions are illustrated in Figure 11.

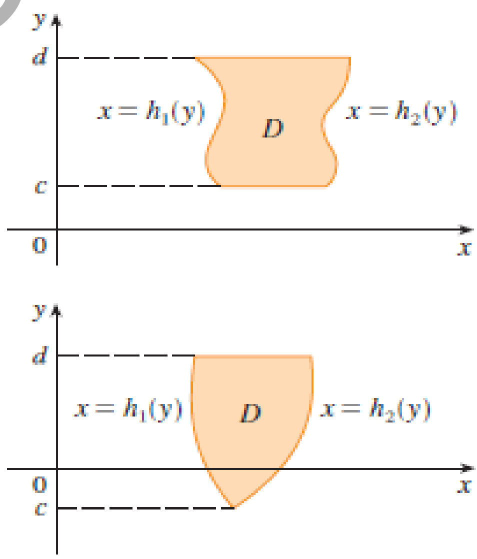


Figure 11

2.2 Iterated Integrals

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral, but the Fundamental Theorem of Calculus provides a much easier method. The evaluation of double integrals from first principles is even more difficult, but in this section we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle

$$R=\left\{(x,y)\in\Re^2 \mid a\le x\le b, c\le y\le d\right\}.$$

We use the notation $\int_c^d f(x,y)dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from $y=c$ to $y=d$. This procedure is called *partial integration with respect to* y. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x,y)dy$ is a number that depends on the value of x, so it defines a function of x:

$$S(x)=\int_c^d f(x,y)dy.$$

If we now integrate the function $S(x)$ with respect to x from $x=a$ to $x=b$, we get

$$\int_a^b S(x)dx=\int_a^b\left[\int_c^d f(x,y)dy\right]dx. \qquad (1)$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\int_a^b\int_c^d f(x,y)dydx=\int_a^b\left[\int_c^d f(x,y)dy\right]dx \qquad (2)$$

means that we first integrate with respect to y from $y=c$ to $y=d$ and then with respect to x from $x=a$ to $x=b$.

Similarly, the iterated integral

$$\int_c^d\int_a^b f(x,y)dydx=\int_c^d\left[\int_a^b f(x,y)dx\right]dy \qquad (3)$$

means that we first integrate with respect to x (holding y fixed) from $x=a$ to $x=b$ and then we integrate the resulting function of y with respect to y from $y=c$ to $y=d$. Notice that in both Equations 2 and 3 we work *from the inside out.*

Fubini's Theorem. If f is continuous on the rectangle

$$R=\left\{(x,y)\in\Re^2 \mid a\le x\le b, c\le y\le d\right\},$$

then

$$\iint_R f(x,y)dS = \int_c^d dy \int_a^b f(x,y)dx = \int_a^b dx \int_c^d f(x,y)dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 1. Evaluate the double integral

$$\iint_R (x-3y^2)dxdy\,,$$

where $R = \{(x,y) \in \Re^2 \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Solution 1. Fubini's Theorem gives

$$\iint_R (x-3y^2)dxdy = \int_0^2 dx \int_1^2 (x-3y^2)dy = \int_0^2 \Big[xy - y^3 \Big]_{y=1}^{y=2} dx = \int_0^2 (x-7)dx = \left[\frac{x^2}{2} - 7x \right]_0^2 = -12\,.$$

Solution 2. Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R f dxdy = \int_1^2 dy \int_0^2 (x-3y^2)dx = \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy = \int_1^2 (2-6y^2)dy = \Big[2y - 2y^3 \Big]_1^2 = -12\,.$$

If f is continuous on a type I region D such that $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then

$$\iint_D f(x,y)dS = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,. \qquad (4)$$

The integral on the right side of (4) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x,y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\} \qquad (5)$$

where $h_1(y)$ and $h_2(y)$ are continuous. Two such regions are illustrated in Figure 11.

Using the same methods that were used in establishing (4), we can show that

$$\iint_D f(x,y)dS = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)dx \qquad (6)$$

where D is a type II region given by Equation 5.

Example 2. Evaluate

$$\iint_D (x+2y)dxdy\,,$$

where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 12, is a type I region but not a type II region and we can write $D = \left\{(x,y) \mid -1 \le x \le 1, 2x^2 \le y \le 1 + x^2\right\}$.

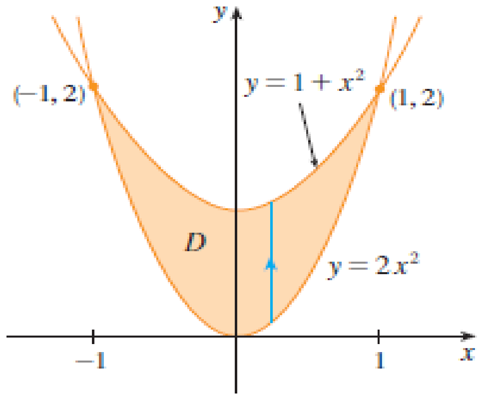


Figure 12

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 4 gives

$$\iint_D (x+2y)dxdy = \int_{-1}^{1} dx \int_{2x^2}^{1+x^2} (x+2y)dy = \int_{-1}^{1} \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx =$$

$$= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{1} = \frac{32}{15}.$$

Example 3. Evaluate

$$\iint_D xydxdy,$$

where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. The region D is shown in Figure 13. Again it is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x,y) \mid -2 \le y \le 4, \frac{1}{2}y^2 - 3 \le x \le y + 1\right\}$$

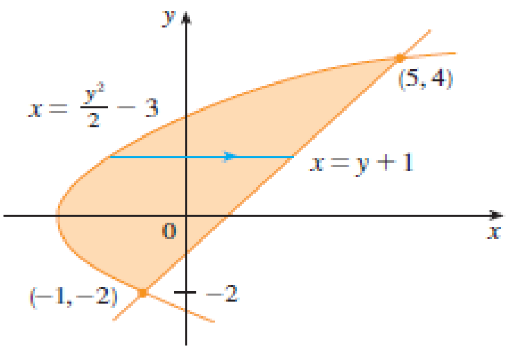


Figure 13

Then (6) gives

$$\iint_D xydxdy = \int_{-2}^{4} dy \int_{\frac{1}{2}y^2-3}^{1+y} xydx = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=1+y} dy = \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy =$$

$$=\frac{1}{2}\left[-\frac{y^6}{6}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4}=36.$$

2.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_D f(x,y)dS$, where D is one of the regions shown in Figure 14. In either case the description of D in terms of rectangular coordinates is rather complicated but D is easily described using polar coordinates.

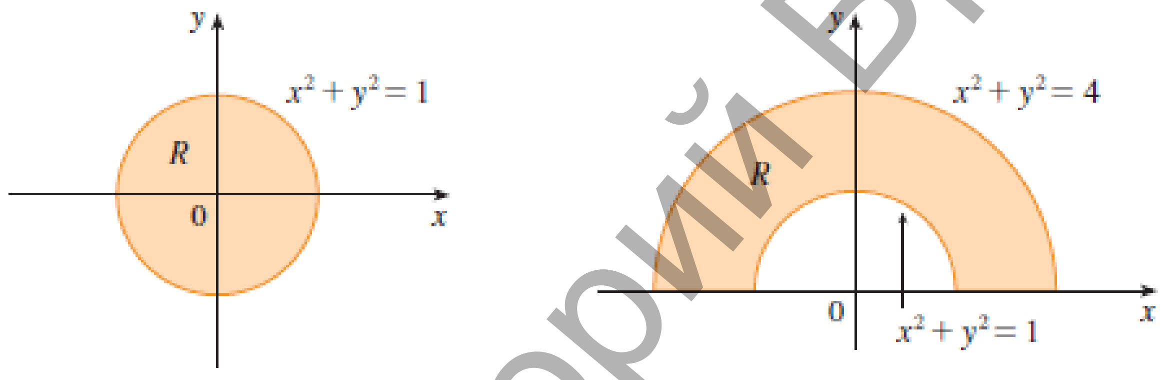


Figure 14

Recall from Figure 15 that the polar coordinates of a point are related to the rectangular coordinates by the equations

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

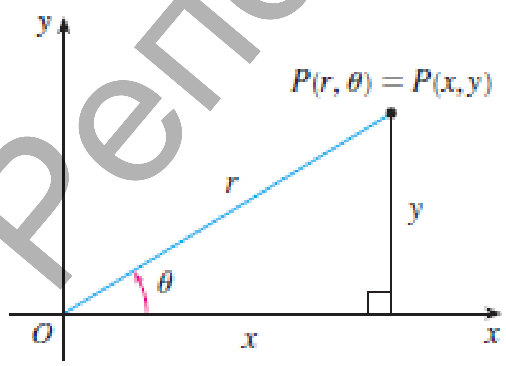


Figure 15

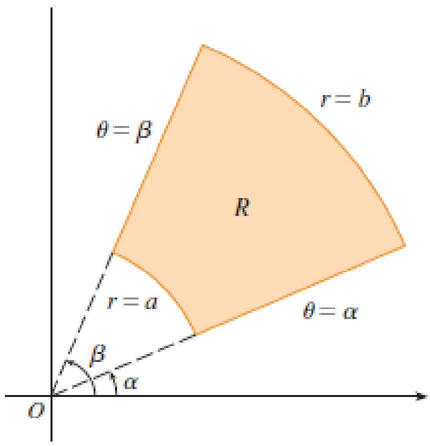


Figure 16

The regions in Figure 14 are special cases of a **polar rectangle** which is shown in Figure 16.

Change to Polar Coordinates in a Double Integral. If f is continuous on a polar rectangle D given by $r_1(\theta)\le r\le r_2(\theta)$, $\alpha\le\theta\le\beta$, then

$$\iint\limits_D f(x,y)\,dxdy=\iint\limits_D f(r\cos\theta,r\sin\theta)r\,dr\,d\theta=\int\limits_{\alpha}^{\beta}d\phi\int\limits_{r_1(\theta)}^{r_2(\theta)}f(r\cos\theta,r\sin\theta)r\,dr. \qquad (1)$$

The formula (1) says that we convert from rectangular to polar coordinates in a double integral by writing $x=r\cos\theta$ and $y=r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dS by $dr\,d\theta$. Be careful not to forget the additional factor r on the right side of Formula 1.

Example 1. Evaluate

$$\iint\limits_D \sin\pi\left(\frac{x^2}{4}+y^2\right)dxdy\ ,$$

where $D:\ \left\{(x,y)\ |\ \frac{x^2}{4}+y^2=1;\ \ \frac{x^2}{16}+\frac{y^2}{4}=1\right\}.$

Solution. $x=2r\cos\theta,\ y=r\sin\theta,\quad I=2r.$

$$\frac{x^2}{4}+y^2=1\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=1\Rightarrow r=1.$$

$$\frac{x^2}{16}+\frac{y^2}{4}=1\Rightarrow\frac{1}{4}\left(r^2\cos^2\theta+r^2\sin^2\theta\right)=1\Rightarrow\frac{r^2}{4}=1\Rightarrow r=2,\qquad 0\le\theta\le 2\pi.$$

In polar coordinates it is given by $1\le r\le 2,\ 0\le\theta\le 2\pi$. Therefore, by Formula 1:

$$\iint\limits_D \sin\pi\left(\frac{x^2}{4}+y^2\right)dxdy=\iint\limits_D \sin\left(\pi\cdot r^2\right)\cdot 2r\,dr\,d\theta=\int\limits_0^{2\pi}d\theta\int\limits_1^2\sin\left(\pi r^2\right)\cdot 2r\,dr=$$

$$=2\int\limits_1^2\sin\left(r^2\pi\right)d\left(r^2\pi\right)=-2\cos\left(r^2\pi\right)\Big|_1^2=-2(\cos 4\pi-\cos\pi)=-2(1+1)=-4.$$

Exercise Set 3

In Exercise 1 to 6, evaluate the iterated integral.

1. $\int\limits_0^2 dx\int\limits_0^1 (x^2+2y)dy.$

2. $\int\limits_{-3}^{8} dy\int\limits_{y^2-4}^{5}(x+2y)dx.$

3. $\int\limits_1^2 dx\int\limits_{1/x}^{x}\frac{x^2}{y^2}dy.$

4. $\int\limits_3^4 dx\int\limits_1^2\frac{dy}{(x+y)^2}.$

5. $\int\limits_1^2 dx\int\limits_x^{x^2}(2x-y)dy.$

6. $\int\limits_0^1 dx\int\limits_0^1\frac{x^2dy}{1+y^2}.$

In Exercise 7 to 12, sketch the region of integration and change the order of integration.

7. $\int\limits_0^1 dx\int\limits_{x^3}^{\sqrt{x}} f(x,y)dy.$

8. $\int\limits_0^1 dx\int\limits_{2x}^{3x} f(x,y)dy.$

9. $\int\limits_1^e dx\int\limits_0^{\ln x} f(x,y)dy.$

10. $\int\limits_0^1 dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x,y)dx.$ 11. $\int\limits_{-6}^{2} dx \int\limits_{\frac{x^2}{4}-1}^{2-x} f(x,y)dy.$ 12. $\int\limits_0^1 dy \int\limits_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx.$

In Exercise 13 to 16, evaluate the double integral $\iint\limits_D f(x,y)\,dx\,dy$, if $f(x,y)=1$ and D:

13. $\left\{(x,y) \mid x^2 = 2y,\ 5x-2y-6=0\right\}$. 14. $\left\{(x,y) \mid y=\sqrt{4-x^2},\ y=\sqrt{3}x, x\ge 0\right\}$.

15. $\left\{(x,y) \mid y=-x,\ y^2 = x+2\right\}$. 16. $\left\{(x,y) \mid y=\log_{0,5} x,\ y=1,\ y=-1,\ x\ge 0\right\}$.

In Exercise 17 to 18, evaluate the double integral

17. $\iint\limits_D (x^3+3y)\,dx\,dy$, where $D:\left\{(x,y) \mid x+y=1,\ y=x^2-1,\ x\ge 0\right\}$.

18. $\iint\limits_D xy\,dx\,dy$, where $D:\left\{(x,y) \mid y=\sqrt{x},\ y=0,\ x+y=2\right\}$.

In Exercise 19 to 26, evaluate the given integral by changing to polar coordinates.

19. $\iint\limits_D \left(1-\frac{y^2}{x^2}\right)dxdy$, where $D:\left\{(x,y) \mid x^2+y^2\le \pi^2\right\}$.

20. $\iint\limits_D 6\,dx\,dy$, where $D:\left\{(x,y) \mid x^2+y^2=4x,\ x^2+y^2=6x,\ y=x,\ y=0\right\}$.

21. $\iint\limits_D \left(x^2+y^2\right)dx\,dy$, where $D:\left\{(x,y) \mid x^2+y^2\le 4x\right\}$.

22. $\iint\limits_D \frac{xy}{\sqrt{x^2+y^2}}\,dx\,dy$, where $D:\left\{(x,y) \mid 1\le x^2+y^2\le 4,\ y=x,\ y=0,\ x<0,\ y<0\right\}$.

23. $\iint\limits_D e^{-x^2-y^2}dxdy$, where $D:\left\{(x,y) \mid x^2+y^2\le R^2\right\}$.

24. $\iint\limits_D \left(x^2+y^2\right)dx\,dy$, where $D:\left\{(x,y) \mid x^2+y^2=4x, x^2+y^2=6x, y=x, y=\sqrt{3}x\ (y>0)\right\}$.

25. $\int\limits_0^1 dx \int\limits_0^{\sqrt{1-x^2}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dy$.

26. $\int\limits_{-2}^{0} dx \int\limits_0^{\sqrt{4-x^2}} \sin(x^2+y^2)\,dy$.

2.4 Applications of Double Integrals

We have already seen one application of double integrals: computing volumes. Another geometric application is finding areas of surfaces. In this section we explore physical applications such as computing mass, center of mass, and moment of inertia. We will see that these physical ideas are also important when applied to probability density functions of two random variables.

Areas of Figures and Volumes of Bodies

1. If we integrate the constant function $f(x,y)=1$ over a region D, we get the area of D:

$$S_D=\iint_D 1dS=\iint_D dS.$$

2. If region D is determined in the polar coordinates, we see that the area of the region D bounded by $\alpha\le\theta\le\beta,\quad r_1(\theta)\le r\le r_2(\theta)$, is

$$S=\iint_D r\,dr\,d\theta=\int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)} r\,dr.$$

3. If $f(x,y)\ge 0$, then the volume V of the solid that lies above the region D and below the surface $z=f(x,y)$ is

$$V=\iint_D f(x,y)dS.$$

Example 1. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

Solution. The solid lies above the disk D whose boundary circle has equation $x^2+y^2=2x$ or, after completing the square, $(x-1)^2+y^2=1$ (See Figures 17 and 18). In polar coordinates we have $x^2+y^2=r^2$ and $x=r\cos\theta$, so the boundary circle becomes $r^2=2r\cos\theta$, or $r=2\cos\theta$.

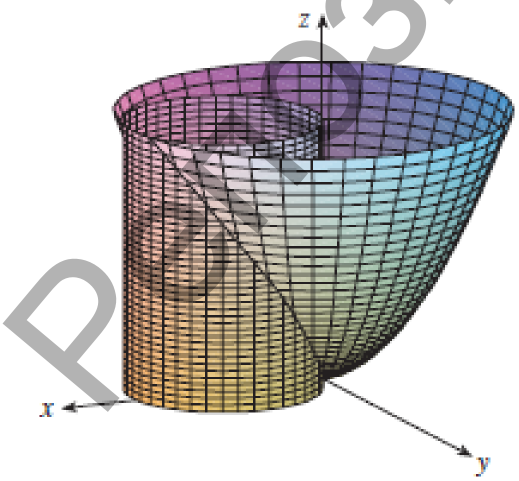


Figure 17

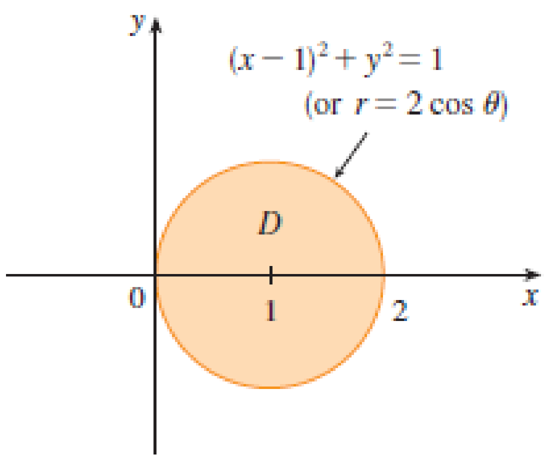


Figure 18

Thus the disk D is given by $D=\left\{(r,\theta)\,|\,-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}, 0\le r\le 2\cos\theta\right\}$ and we have

$$V=\iint_D (x^2+y^2)\,dS=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^2 r\,dr=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = 4\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = 2\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right) d\theta=$$

$$=2\cdot\left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{3\pi}{2}.$$

Example 2. Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$ and $z=0$.

Solution. In a question such as this, it's wise to draw two diagrams: one of the three dimensional solid and another of the plane region D over which it lies. Figure 19 shows the tetrahedron T bounded by the coordinate planes $x=0$, $z=0$, the vertical plane $x=2y$, and the plane $x+2y+z=2$. Since the plane $x+2y+z=2$ intersects the xy-plane (whose equation is $z=0$) in the line $x+2y=2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x=2y$, $x+2y=2$, and $x=0$ (See Figure 20).

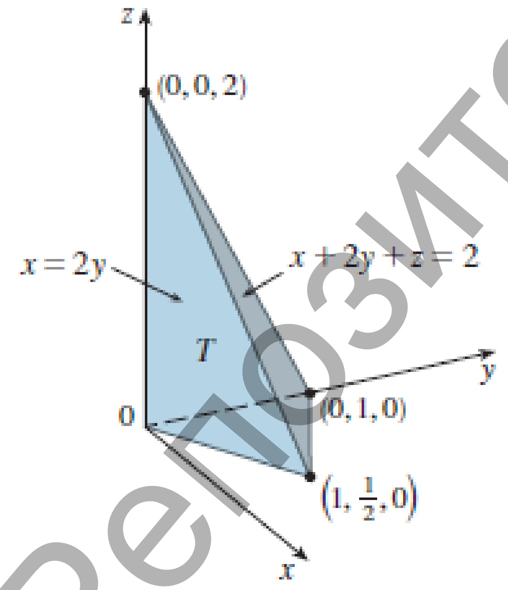


Figure 19

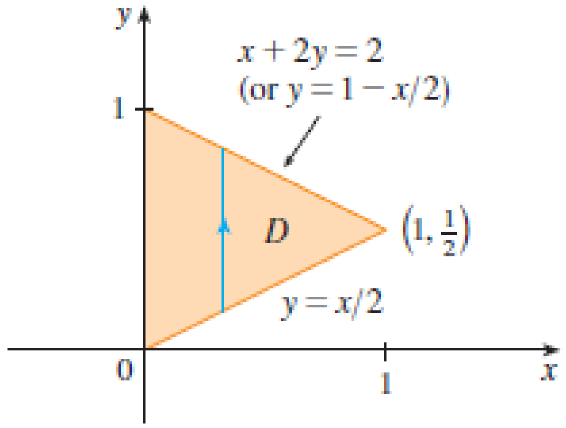


Figure 20

The plane $x+2y+z=2$ can be written as $z=2-x-2y$, so the required volume lies under the graph of the function $z=2-x-2y$ and above $D=\left\{(x,y)\,|\,0\le x\le 1, \frac{x}{2}\le y\le 1-\frac{x}{2}\right\}$.

Therefore

$$V=\iint\limits_D(2-x-2y)dS=\int\limits_0^1 dx\int\limits_{\frac{x}{2}}^{1-\frac{x}{2}}(2-x-2y)dy=\int\limits_0^1\left[2y-xy-y^2\right]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}}dx=$$

$$=\int\limits_0^1\left(2-x-x\left(1-\frac{x}{2}\right)-\left(1-\frac{x}{2}\right)^2-x+\frac{x^2}{2}+\frac{x^2}{4}\right)dx=\int\limits_0^1\left(x^2-2x+1\right)dx=\left[\frac{x^3}{3}-x^2+x\right]_0^1=\frac{1}{3}.$$

Example 3. Use a double integral to find the area enclosed by one loop of the four leaved rose $r=\cos 2\theta$.

Solution. From the sketch of the curve in Figure 21, we see that a loop is given by the region $D=\left\{(r,\theta)\,|\,-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}, 0\le r\le\cos 2\theta\right\}$.

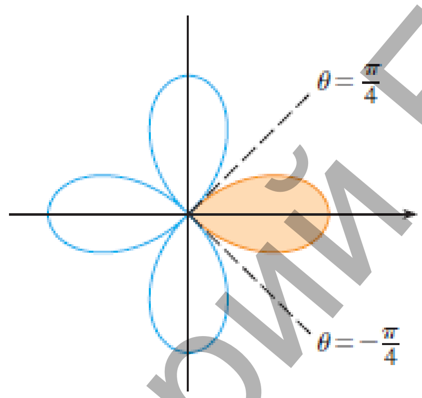


Figure 21

So the area is

$$S=\iint\limits_D r\,dr\,d\theta=\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\int\limits_0^{\cos 2\theta}r\,dr=\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{1}{2}r^2\right]_0^{\cos 2\theta}d\theta=\frac{1}{2}\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 2\theta\,d\theta=\frac{1}{4}\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 4\theta)d\theta=$$

$$=\frac{1}{4}\left[\theta+\frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{8}.$$

Moments and Centers of Mass

Consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x,y) in D is given by $\rho(x,y)$, where is a continuous function on D.

To find the total mass m of the lamina, we divide a rectangle R containing D into subrectangles R_{ij} of equal size (as in Figure 22) and consider $\rho(x,y)$ to be 0 outside. If we choose a point (x_{ij}^*,y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*,y_{ij}^*)\Delta S$, where ΔS is the area of R_{ij}.

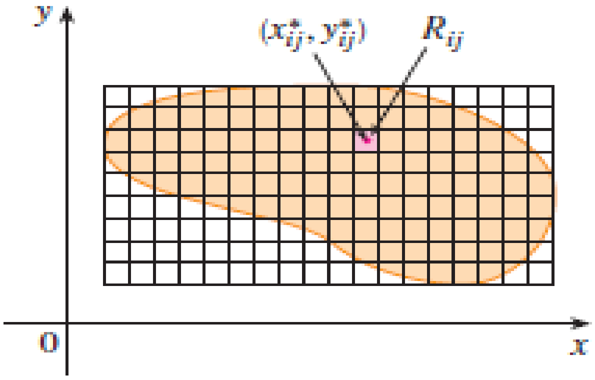


Figure 22

If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k}\sum_{j=1}^{l}\rho(x_{ij}^*, y_{ij}^*)\Delta S.$$

If we now increase the number of subrectangles, we obtain the total mass m of the lamina as the limiting value of the approximations:

$$m = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\rho(x_{ij}^*, y_{ij}^*)\Delta S = \iint_D \rho(x,y)dS.$$

Suppose the lamina occupies a region D and has density function $\rho(x,y)$. The **moment** of the entire lamina **about the** ***x*****-axis** is

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}y_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta S = \iint_D y\rho(x,y)dS.$$

Similarly, the **moment about the** ***y*****-axis** is

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}x_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta S = \iint_D x\rho(x,y)dS.$$

As before, we define the center of mass $(\overline{x}, \overline{y})$ so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$:

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}$$

The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass (see Figure 23).

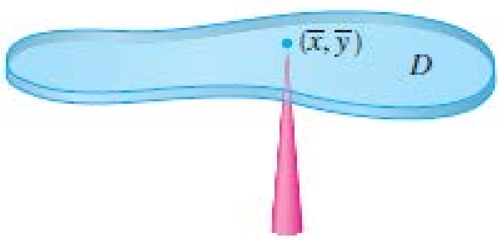


Figure 23

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x,y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the *x*-axis**:

$$I_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D y^2 \rho(x,y) dS.$$

Similarly, the **moment of inertia about the *y*-axis** is

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D x^2 \rho(x,y) dS.$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$I_0 = \iint_D (x^2 + y^2) \rho(x,y) dS.$$

Exercise Set 4

In Exercise 1 to 11, use a double integral to find the area of the region.

1. $y = x^2,\ y = \frac{3}{4}x^2.$
2. $y^2 = 4 + x,\ x + 3y = 0.$
3. $x = y^2,\ x = \sqrt{2 - y^2}.$
4. $r = a\sin 2\theta,\ a > 0.$
5. $r\cos\theta = 1,\ r = 2.$
6. $r = 4(1 + \cos\theta).$
7. $y = 2 - x,\ y^2 = 4x + 4.$
8. $x = y^2 - 2y,\ x + y = 0.$
9. $r = a\cos 5\theta,\ a > 0.$
10. $r = a\sin 3\theta,\ a > 0.$
11. $y = 4x - x^2,\ y = 2x^2 - 5x.$

In Exercise 12 to 20, use polar coordinates to find the volume of the given solid.

12. $x^2 + y^2 = R^2,\ x^2 + z^2 = R^2.$
13. $z = x^2 + y^2,\ z = x + y + 10,\ z = 0.$
14. $x^2 + y^2 = 4x,\ 2z = x^2 + y^2,\ z = 0.$
15. $6z = x^2 + y^2,\ x^2 + y^2 + z^2 = 27,\ z > 0.$
16. $z = x^2 + y^2,\ y = x^2, y = 1,\ z = 0.$
17. $x^2 + y^2 = 9,\ x^2 + y^2 - z^2 = -9.$
18. $z = 4 - x^2, 2x + y = 4, x = 0, y = 0, z = 0.$
19. $2(x^2 + y^2) - z^2 = 0,\ x^2 + y^2 - z^2 = -16.$
20. $z = 1 + x^2 + y^2, x = 0,\ y = 0,\ z = 0, x = 4, y = 4.$

In Exercise 21 to 23, find the mass and center of mass of the lamina that occupies the region D and has the given density function $\rho(x,y)$.

21. $\rho(x,y) = 1;$ $D: \{(x,y) \mid x + y = 2, x = 2, y = 2\}.$
22. $\rho(x,y) = 3{,}5;$ $D: \{(x,y) \mid x^2 + y^2 - 2x = 0\}.$
23. $\rho(x,y) = x^2 y;$ $D: \{(x,y) \mid y = x^2, y = 1\}.$

2.5 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$B=\left\{(x,y,z)\in\Re^3 \mid a\le x\le b, c\le y\le d, r\le z\le s\right\} \quad (1)$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a,b]$ into l subintervals $[x_{i-1},x_i]$ of equal width Δx, $[c,d]$ dividing into m subintervals $[y_{j-1},y_j]$ of width Δy, and dividing $[r,s]$ into n subintervals $[z_{k-1},z_k]$ of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes which are shown in Figure 24. Each sub-box has volume $\Delta V=\Delta x\Delta y\Delta z$.

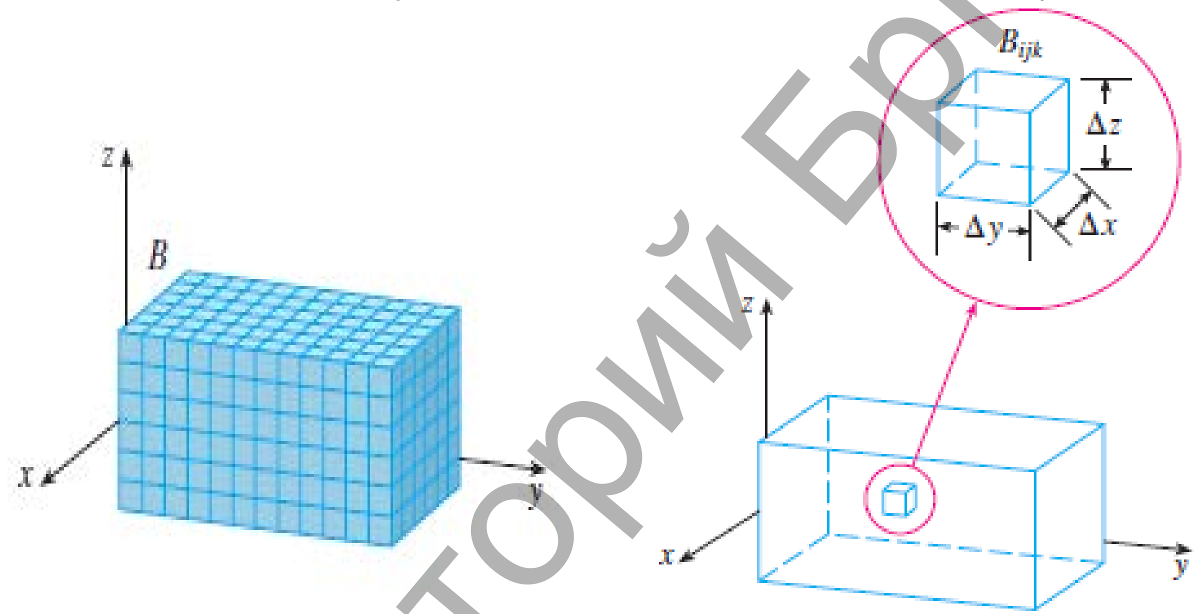


Figure 24

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*,y_{ijk}^*,z_{ijk}^*)\Delta V. \quad (2)$$

Definition. The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)dV=\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*,y_{ijk}^*,z_{ijk}^*)\Delta V \quad (3)$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i,y_j,z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x,y,z)dV=\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i,y_j,z_k)\Delta V.$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for triple integrals. If f is continuous on the rectangular box B, then

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz. \quad (4)$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x,y,z)dV = \int_a^b \int_r^s \int_c^d f(x,y,z)dydzdx. \quad (5)$$

Now we define the **triple integral over a general bounded region E** in three dimensional space (a solid) by much the same procedure that we used for double integrals. We enclose E in a box B of the type given by Equation 1. Then we define a function F so that it agrees with f on E but is 0 for points in that are outside E. By definition,

$$\iiint_B f(x,y,z)dV = \iiint_B F(x,y,z)dV.$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth". The triple integral has essentially the same properties as the double integral.

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\} \quad (6)$$

where D is the projection of E onto the xy-plane as shown in Figure 25. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x,y)$, while the lower boundary is the surface $z = u_1(x,y)$.

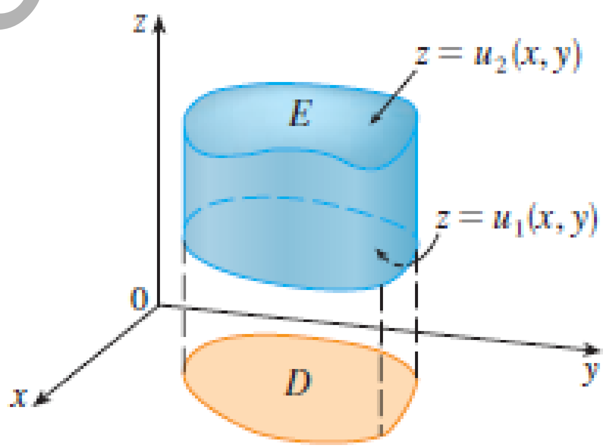


Figure 25

If E is a type 1 region given by Equation 6, then

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left[\int\limits_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right] dS. \qquad (7)$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 26), then

$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$$

and Equation 7 becomes

$$\iiint\limits_E f(x,y,z)dV = \int\limits_a^b \int\limits_{g_1(x)}^{g_1(x)} \int\limits_{u_1(x,y)}^{u_1(x,y)} f(x,y,z)dzdydx. \qquad (8)$$

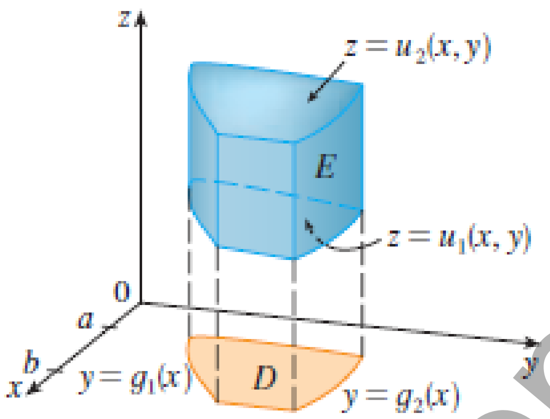


Figure 26

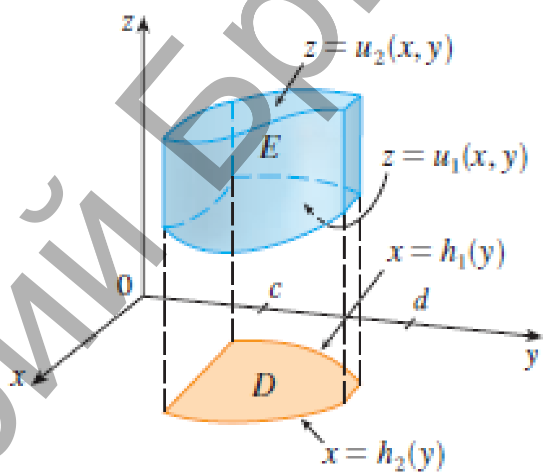


Figure 27

If, on the other hand, D is a type II plane region (as in Figure 27), then

$$E = \{(x,y,z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\}$$

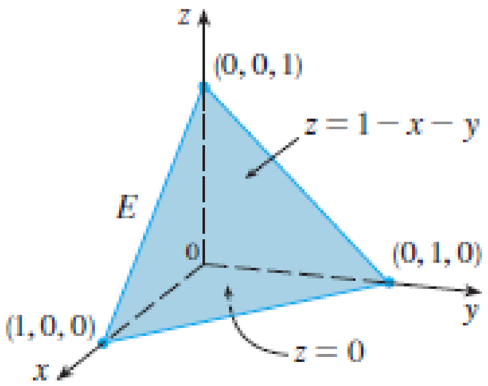
and Equation 7 becomes

$$\iiint\limits_E f(x,y,z)dV = \int\limits_c^d \int\limits_{h_1(y)}^{h_1(y)} \int\limits_{u_1(x,y)}^{u_1(x,y)} f(x,y,z)dzdxdy. \qquad (9)$$

Example 1. Evaluate $\iiint\limits_E zdV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Solution. When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 28) and one of its projection D on the xy-plane (see Figure 29). The lower boundary of the tetrahedron is the plane $z=0$ and the upper boundary is the plane $x+y+z=1$ (or $z=1-x-y$), so we use $u_1(x,y)=0$ and $u_2(x,y)=1-x-y$ n Formula 8. Notice that the planes $x+y+z=1$ and $z=0$ intersect in the line $x+y=1$ (or $y=1-x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 29, and we have

$$E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y\}.$$



igure 28

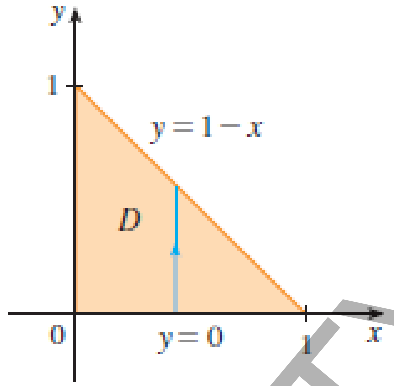


Figure 29

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint\limits_E z\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz=\int_0^1\int_0^{1-x}\left[\frac{z^2}{2}\right]_0^{1-x-y}dy=\frac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)^2dy=$$

$$=\frac{1}{2}\int_0^1\left[-\frac{(1-x-y)^3}{3}\right]_0^{1-x}dx=\frac{1}{6}\int_0^1(1-x)^3dx=\frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1=\frac{1}{24}.$$

A solid region E is of **type 2** if it is of the form $E=\{(x,y,z)\,|\,(y,z)\in D, u_1(y,z)\le x\le u_2(y,z)\}$ where, this time, D is the projection of onto the yz -plane (see Figure 30). The back surface is $x=u_1(y,z)$, the front surface is $x=u_2(y,z)$, and we have

$$\iiint\limits_E f(x,y,z)dV=\iint\limits_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right]dS. \qquad (10)$$

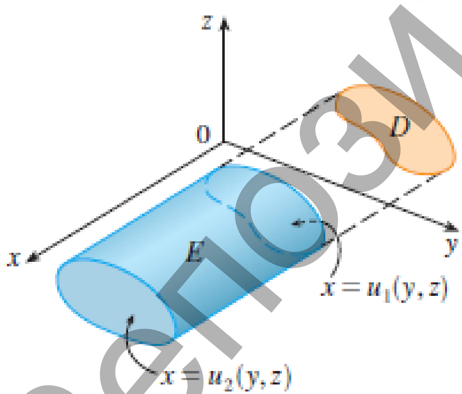


Figure 30

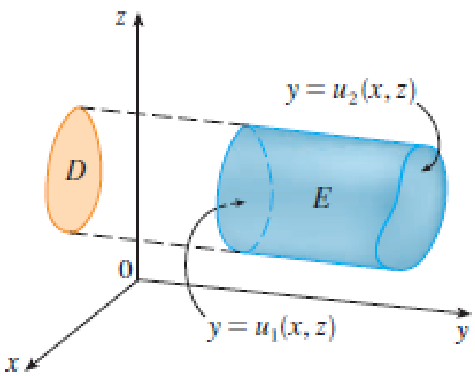


Figure 31

Finally, a **type 3** region is of the form $E=\{(x,y,z)\,|\,(x,z)\in D, u_1(x,z)\le y\le u_2(x,z)\}$ where D is the projection of onto the xz -plane, $y=u_1(x,z)$ is the left surface, and $y=u_2(x,z)$ is the right surface (see Figure 31). For this type of region we have

$$\iiint\limits_E f(x,y,z)dV=\iint\limits_D\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right]dS. \qquad (11)$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 8 and 9).

Applications of Triple Integrals

1. Let's begin with the special case where $f(x,y,z)=1$ for all points in E. Then the triple integral does represent the volume of E:

$$\iiint_E dV = V_E .$$

2. All the applications of double integrals in Section 2.4 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit volume, at any given point (x,y,z), then its **mass** is

$$m=\iiint_E \rho(x,y,z)dV$$

and its **moments** about the three coordinate planes are

$$M_{xy}=\iiint_E z\rho(x,y,z)dV ; \quad M_{yz}=\iiint_E x\rho(x,y,z)dV ; \quad M_{xz}=\iiint_E y\rho(x,y,z)dV .$$

The **center of mass** is located at the point $(\overline{x},\overline{y},\overline{z})$, where

$$\overline{x}=\frac{M_{yz}}{m}, \quad \overline{y}=\frac{M_{xz}}{m}, \quad \overline{z}=\frac{M_{xy}}{m}.$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

$$I_x=\iiint_E (z^2+y^2)\rho(x,y,z)dV ; \quad I_y=\iiint_E (x^2+z^2)\rho(x,y,z)dV ; \quad I_z=\iiint_E (y^2+x^2)\rho(x,y,z)dV .$$

2.6 Triple Integrals in Cylindrical Coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r,θ,z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P (See Figure 32).

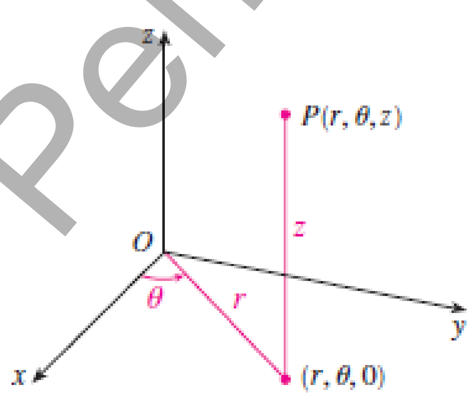


Figure 32

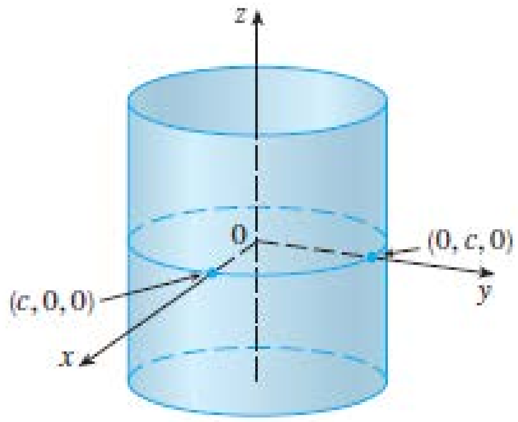


Figure 33

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z, \qquad r \ge 0,\ 0 \le \theta \le 2\pi,\ z \in R.$$

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure 33.) This is the reason for the name "cylindrical" coordinates.

Suppose that E is a type 1 region whose projection D on the xy-plane is conveniently described in polar coordinates (see Figure 34). In particular, suppose that f is continuous and $E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$ where D is given in polar coordinates by $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$.

We also know how to evaluate double integrals in polar coordinates. We obtain

$$\iiint\limits_E f(x,y,z)\,dxdydz = \iiint\limits_E f(r\cos\theta, r\sin\theta, z)\,r\,drd\theta dz = \int_\alpha^\beta d\phi \int_{h_1(\theta)}^{h_2(\theta)} r\,dr \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz.$$

The last formula is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dr\,d\theta\,dz$.

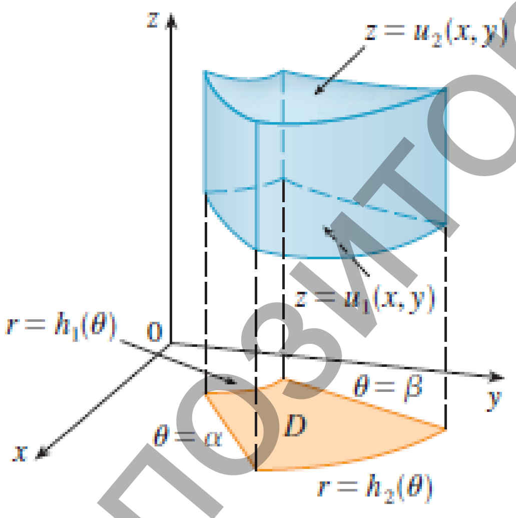


Figure 34

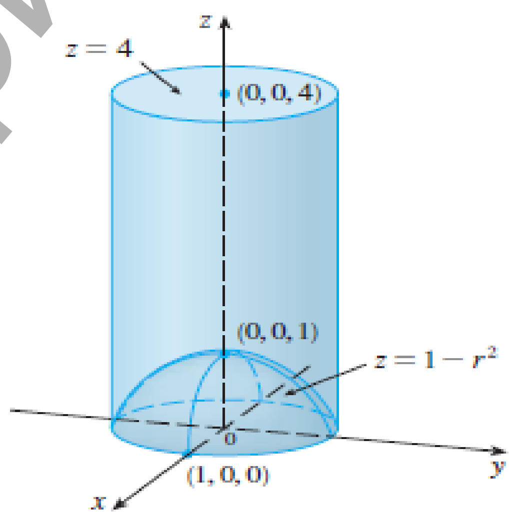


Figure 35

Example 1. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 35.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution. In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 1 - r^2 \le z \le 4\}$. Since the density at (x,y,z) is proportional to the distance from the z-axis, the density function is

$$f(x,y,z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, the mass of E is

$$m=\iiint_E \rho(x,y,z)dV=\iiint_E K\sqrt{x^2+y^2}dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)rdzdrd\theta=\int_0^{2\pi}\int_0^1 Kr^2\left[4-(1-r^2)\right]drd\theta=$$

$$=K\int_0^{2\pi}d\theta\int_0^1\left(3r^2+r^4\right)dr=2\pi K\left[r^3+\frac{r^5}{5}\right]_0^1=\frac{12\pi K}{5}.$$

Example 2. Evaluate

$$\int_{-2}^{2\pi}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{y^2+x^2}}^{2}(x^2+y^2)dzdydx.$$

Solution. This iterated integral is a triple integral over the solid region

$$E=\left\{(x,y,z)\,|\,-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\sqrt{x^2+y^2}\le z\le 2\right\}$$

and the projection of E onto the xy -plane is the disk $x^2+y^2\le 4$. The lower surface of E is the cone $z=\sqrt{x^2+y^2}$ and its upper surface is the plane $z=2$. (See Figure 36.)

This region has a much simpler description in cylindrical coordinates:

$$E=\left\{(r,\theta,z)\,|\,0\le\theta\le 2\pi,0\le r\le 2,r\le z\le 2\right\}.$$

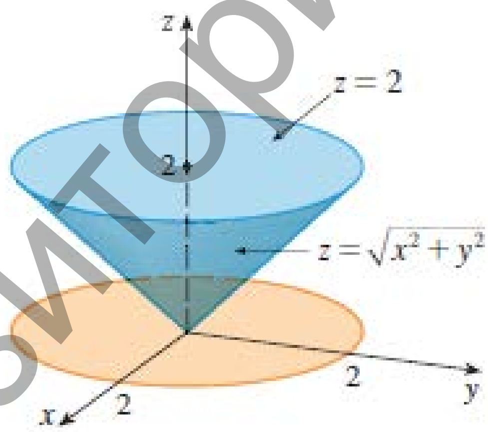


Figure 36

Therefore, we have

$$\int_{-2}^{2\pi}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{y^2+x^2}}^{2}(x^2+y^2)dzdydx=\int_0^{2\pi}\int_0^2\int_r^2 r^2rdzdrd\theta=\int_0^{2\pi}d\theta\int_0^2 r^3(2-r)dr=$$

$$=2\pi\left[\frac{1}{2}r^4-\frac{1}{5}r^5\right]_0^2=\frac{16}{5}\pi.$$

2.7 Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the *spherical coordinate system*. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

The **spherical coordinates** (r,θ,ϕ) of a point P in space are shown in Figure 37, where $\rho=|OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive z-axis and the line segment OP.

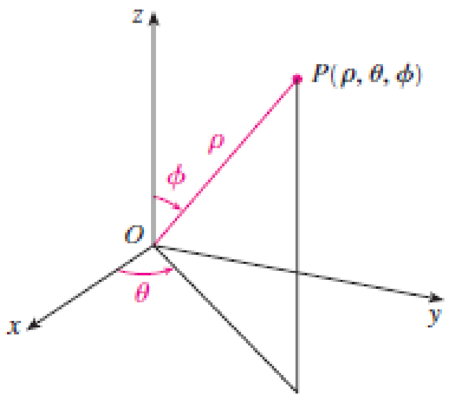


Figure 37

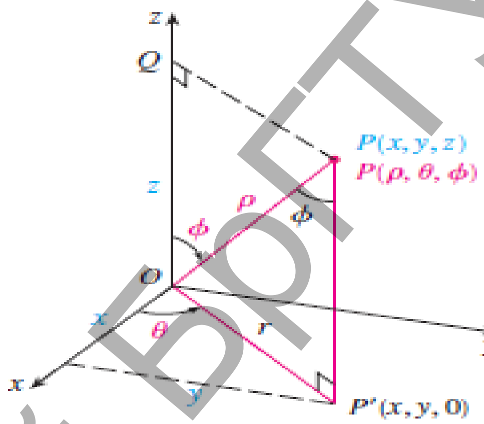


Figure 38

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

The relationship between rectangular and spherical coordinates can be seen from Figure 38. From triangles OQP and OPP' we have $r=\rho\sin\varphi$, $z=\rho\cos\varphi$.

But $x=r\cos\theta$ and $y=r\sin\theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$x=\rho\sin\varphi\cos\theta,\quad y=\rho\sin\varphi\sin\theta,\ z=\rho\cos\varphi.$$

We have arrived at the following **formula for triple integration in spherical coordinates**.

$$\iiint\limits_E f(x,y,z)dxdydz=\iiint\limits_E f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\theta\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi.$$

Example 1. Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2\le z$. (See Figure 39.)

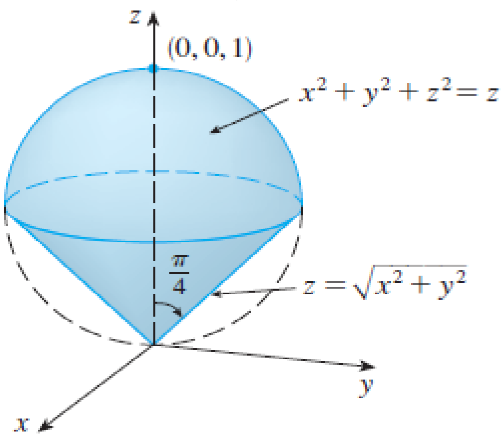
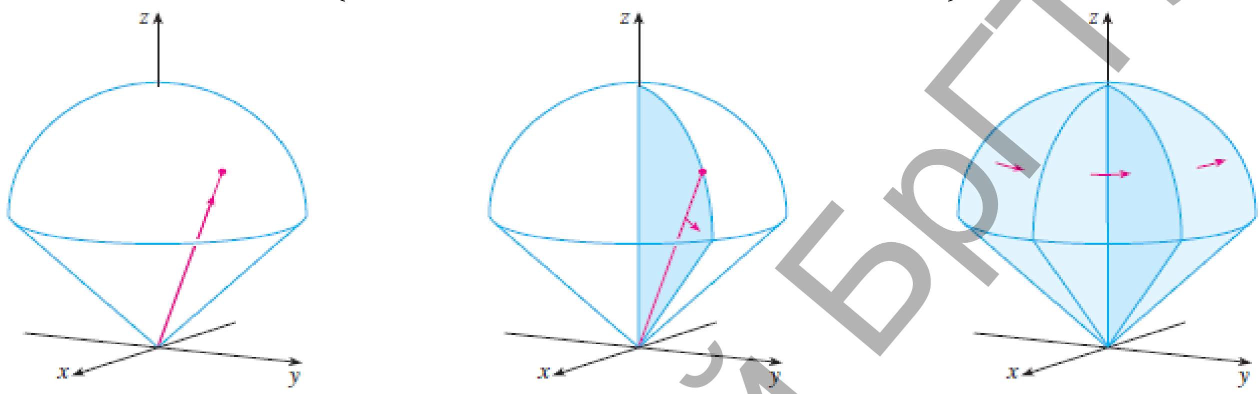


Figure 39

Solution. Notice that the sphere passes through the origin and has center $\left(0,0,\frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as $\rho^2=\rho\cos\varphi$ or $\rho=\cos\varphi$.

The equation of the cone can be written as this gives $\sin\varphi=\cos\varphi$, or $\varphi=\frac{\pi}{4}$. Therefore the description of the solid E in spherical coordinates is

$$E=\left\{(\rho,\theta,\varphi)\,|\,0\le\theta\le 2\pi, 0\le\varphi\le\frac{\pi}{4}, 0\le\rho\le\cos\varphi\right\}.$$



ρ varies from 0 to cos ϕ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Figure 40

Figure 40 shows how E is swept out if we integrate first with respect to ρ, then φ, and then θ. The volume of E is

$$V=\iiint_E dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}\rho^2\sin\varphi\, d\varphi d\rho d\theta=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\left[\frac{\rho^3}{3}\right]_0^{\cos\varphi}d\varphi=$$

$$=\frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\sin\varphi\cos^3\varphi\, d\varphi=\frac{2\pi}{3}\left[-\frac{\cos^4\varphi}{4}\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}.$$

Exercise Set 5

In Exercise 1 to 5, evaluate the triple integral.

1. $\iiint_V x^3y^2z\,dxdydz,\quad V:\{(x,y,z)|0\le x\le 1, 0\le y\le x, 0\le z\le xy\}.$
2. $\iiint_V x^2y^2\,dxdydz,\quad V:\{(x,y,z)|z=x^2+y^2, x^2+y^2=1, z=0\}.$
3. $\iiint_V (2x-y+4z)dV,\quad V:\{(x,y,z)|\, x+2y+z=2, x>0, y>0, z>0\}.$
4. $\iiint_V (2x-y)dV,\quad V:\{(x,y,z)|\, z=x+y+4, y^2=4x, x=4, z=0, y>0\}.$

5. $\iiint\limits_V (x-y+z)dV,\ V:\left\{(x,y,z)\,|\,y^2=4x,\ z=4-x,\ z=0\right\}$.

In Exercise 6 to 11, use a triple integral to find the volume of the given solid.

6. $V:\left\{(x,y,z)\,|\,x^2+y^2=10x,\ x^2+y^2=13x,\ z=\sqrt{x^2+y^2},\ z=0,\ y\ge 0\right\}$.

7. $V:\left\{(x,y,z)\,|\,x^2+y^2+z^2=1,\ z^2=x^2+y^2\right\}$.

8. $V:\left\{(x,y,z)\,|\,z=x^2+y^2,\ y=x^2,\ y=1,\ z=0\right\}$.

9. $V:\left\{(x,y,z)\,|\,z^2=x^2+y^2,\ z=6\right\}$.

10. $V:\left\{(x,y,z)\,|\,x^2+z^2=4,\ y=-1,\ y=3\right\}$.

11. $V:\left\{(x,y,z)\,|\,z={}^2+y^2,\ x=y^2,\ x=4,\ z=0\right\}$.

In Exercise 12 to 14, find the mass and center of mass of the solid V with the given density function $\rho(x,y,z)=1$.

12. $V:\left\{(x,y,z)\,|\,z=8\left(x^2+y^2\right),\ z=32\right\}$.

13. $V:\left\{(x,y,z)\,|\,z=9\sqrt{x^2+y^2},\ z=36\right\}$.

14. $V:\left\{(x,y,z)\,|\,y=3\sqrt{x^2+z^2},\ y=9\right\}$.

2.8 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a,b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve C given by the parametric equations

$$x=x(t), y=y(t), a\le t\le b \tag{1}$$

If we divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1},t_i]$ of equal width and we let $x_i=x(t_i)$ and $y_i=y(t_i)$, then the corresponding points $P_i(x_i,y_i)$ divide C into n subarcs with lengths $\Delta s_1,\Delta s_2,\Delta s_3,...,\Delta s_n$. (See Figure 41.)

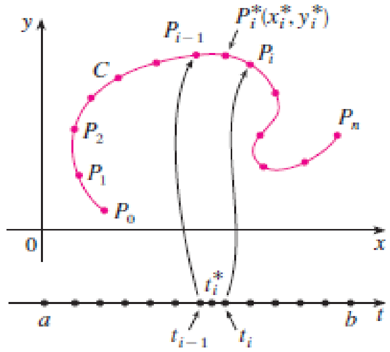


Figure 41

We choose any point $P_i^*(x_i^*,y_i^*)$ in the i-th subarc. (This corresponds to a point t_i^* in $[t_{i-1},t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point $P_i^*(x_i^*,y_i^*)$, multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

Definition. If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta s_i \qquad (2)$$

if this limit exists.

We found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition always exists and the following formula can be used to evaluate the line integral:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \qquad (3)$$

The value of the line integral does not depend on the parameterization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x,y)\geq 0$, $\int_C f(x,y)ds$ represents the area of one side of the "fence" or "curtain" in Figure 42, whose base is C and whose height above the point (x,y) is $f(x,y)$.

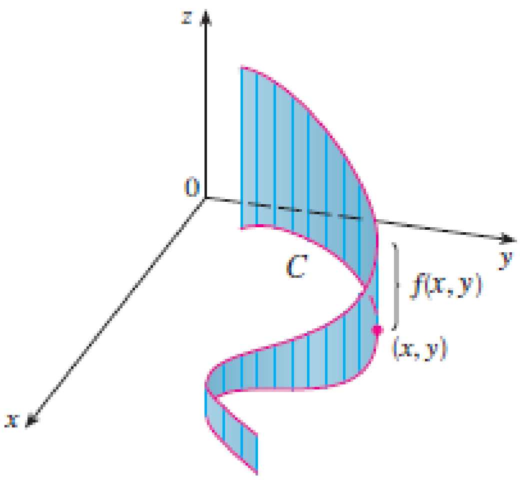


Figure 42

Example 1. Evaluate $\int_C (2+x^2y)ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

Solution. In order to use Formula 3, we first need parametric equations to represent C. Recall that the unit circle can be parameterized by means of the equations $x=\cos t, y=\sin t$ and the upper half of the circle is described by the parameter interval $0 \le t \le \pi$. (See Figure 43.)

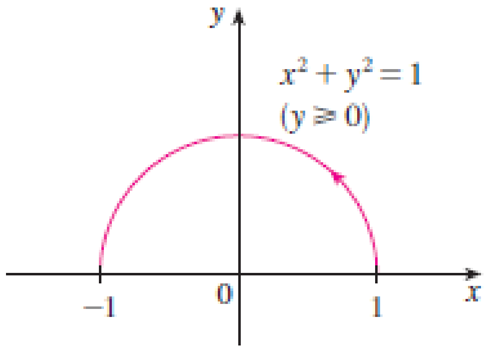


Figure 43

Therefore Formula 3 gives

$$\int_C (2+x^2y)ds = \int_0^{\pi}(2+\cos^2 t\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt = \int_0^{\pi}(2+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}dt =$$

$$= \int_0^{\pi}(2+\cos^2 t\sin t)dt = \left[2t-\frac{\cos^3 t}{3}\right]_0^{\pi} = 2\pi+\frac{2}{3}.$$

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, C_3, ..., C_n$ where, as illustrated in Figure 44, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + ... + \int_{C_n} f(x,y)ds.$$

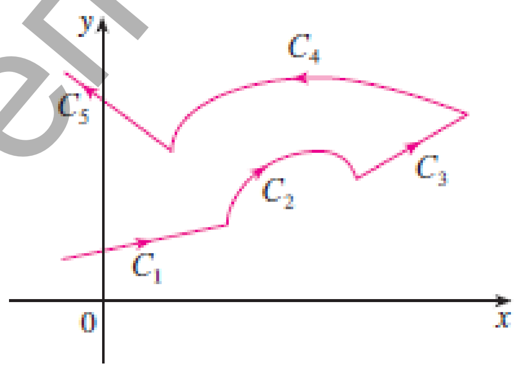


Figure 44

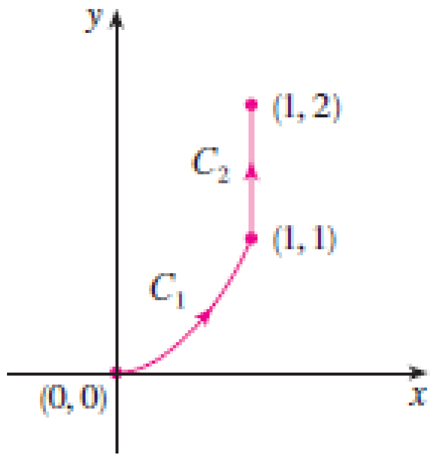


Figure 45

Example 2. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$ C_1.

Solution. The curve C is shown in Figure 45. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become $x = x, y = x^2, 0 \le x \le 1$.

Therefore

$$\int_C 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1+4x^2}\,dx = \frac{1}{4}\cdot\frac{2}{3}\left(1+4x^2\right)^{\frac{3}{2}}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}.$$

On C_2 we choose y as the parameter, so the equations of C_2 are $x = 1, y = y, 1 \le y \le 2$ and

$$\int_C 2x\,ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = 2,$$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5}-1}{6} + 2.$$

Any physical interpretation of a line integral $\int_C f(x,y)\,ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the mass m of C is

$$m = \int_C \rho(x,y)\,ds.$$

The **center of mass** of the wire with density function $\rho(x,y)$ is located at the point $(\overline{x},\overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds; \qquad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds.$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition. They are called the **line integrals of f along C with respect to x and y**:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta x_i; \quad \int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta y_i. \qquad (4)$$

When we want to distinguish the original line integral $\int_C f(x,y)\,ds$ from those in Equation 4, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of $t: x = x(t), y = y(t), dx = x'(t), dy = y'(t)$.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt.$$

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))yx(t)dt\,. \qquad (5)$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy\,. \qquad (6)$$

Example 3. Evaluate $\int_C y^2dx + xdy$, where (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$. (See Figure 46.)

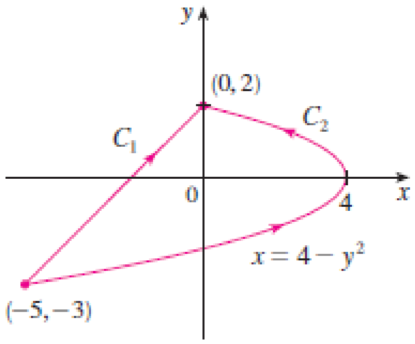


Figure 46

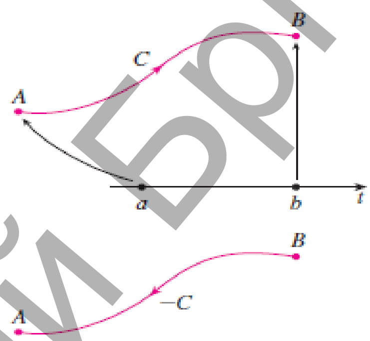


Figure 47

Solution.

(a) A parametric representation for the line segment is $x = 5t - 5, y = 5t - 3, 0 \le t \le 1$. Then $dx = 5dt, dy = 5dt$,and Formula 6 give

$$\int_C y^2dx + xdy = \int_0^1 (5t-3)^2(5dt) + (5t-5)(5dt) = 5\int_0^1 (25t^2 - 25t + 4)dt =$$

$$= 5\left(\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right)\Bigg|_0^1 = -\frac{5}{6}.$$

(b) Since the parabola is given as a function o y, let's take as the parameter and write C_2 as $x = 4 - y^2, y = y, -3 \le y \le 2$. Then $dx = -2ydy$ and by Formulas 6 we have

$$\int_C y^2dx + xdy = \int_{-3}^2 y^2(-2y)dy + (4 - y^2)dy = \int_{-3}^2 (-2y^3 - y^2 + 4)dy =$$

$$= \left(-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right)\Bigg|_{-3}^2 = 40\frac{5}{6}.$$

In general, a given parameterization $x = x(t), y = y(t), a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 47, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to b.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 47), then we have

$$\int_{-C} f(x,y)dx = -\int_{C} f(x,y)dx\,;\ \int_{-C} f(x,y)dy = -\int_{C} f(x,y)dy \qquad (7)$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x,y)ds = \int_{C} f(x,y)ds.$$

We now suppose that is a smooth space curve given by the parametric equations

$$x = x(t), y = y(t), z = z(t), a \le t \le b.$$

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_{C} f(x,y,z)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i.$$

We evaluate it using a formula similar to Formula 3:

$$\int_{C} f(x,y,z)ds = \int_{a}^{b} f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \qquad (8)$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_{C} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \qquad (9)$$

by expressing everything (x,y,z,dx,dy,dz) in terms of the parameter t.

Example 4. Evaluate $\int_{C} y\sin z\,ds$, where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t, 0 \le t \le 2\pi$. (See Figure 48.)

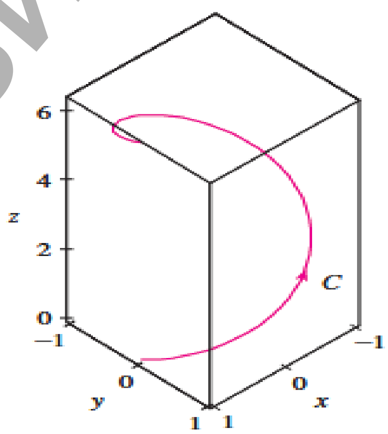


Figure 48

Solution. Formula 8 gives

$$\int_{C} y\sin z\,ds = \int_{0}^{2\pi} (\sin t)\sin t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_{0}^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}dt =$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)dt = \frac{\sqrt{2}}{2}\left[t-\frac{1}{2}\sin 2t\right]_0^{2\pi} = \sqrt{2}\pi .$$

Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 49. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\vec{r}(t), a \le t \le b$, then the region D is always on the left as the point $\vec{r}(t)$ traverses C. (See Figure 50.)

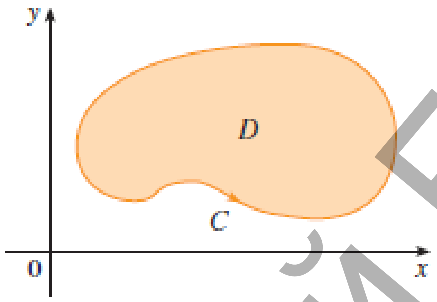
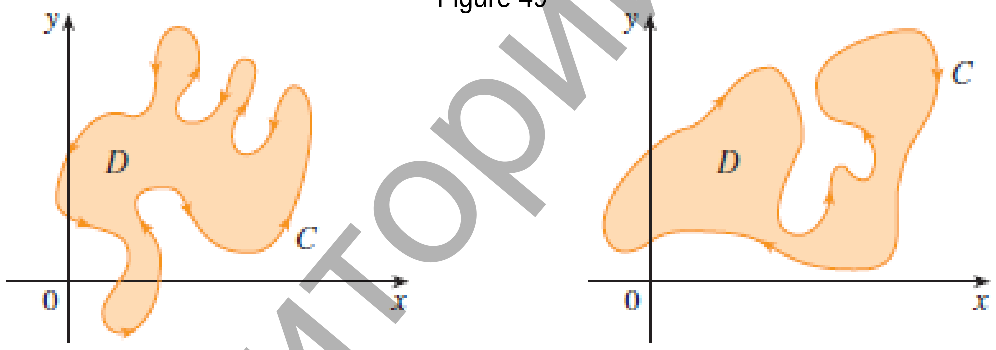


Figure 49



(a) Positive orientation (b) Negative orientation

Figure 50

Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dS . \qquad (10)$$

Then Green's Theorem gives the following formulas for the area of D:

$$S = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx . \qquad (11)$$

Example 5. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution. The ellipse has parametric equations $x = a\cos t, y = b\sin t$, where $0 \le t \le 2\pi$. Using the third Formula 11, we have

$$S=\frac{1}{2}\int_C xdy-ydx=\frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)dt-(b\sin t)(-a\sin t)dt=\frac{ab}{2}\int_0^{2\pi}dt=\pi ab.$$

Exercise Set 6

In Exercise 1 to 11, evaluate the line integral, where L is the given curve.

1. $\int_L x\,dl$, if L is line segment from $A(0;0)$ to $B(1,2)$.

2. $\int_L \frac{dl}{x+y}$, if L is line segment, $y=x+2$, from $A(2;4)$ to $B(1,3)$.

3. $\int_L \sqrt{2y}\,dl$, if L $\begin{cases}x=a(t-\sin t),\\ y=a(1-\cos t),\end{cases}$ $(a>0)$.

4. $\int_L (x+y)\,dl$, if L $\rho^2=a^2\cos^2\theta$.

5. $\int_L \left(x^2-2xy\right)dx+\left(y^2-2xy\right)dy$, if L is $y=x^2$ from $A(-1,1)$ to $B(1,1)$.

6. $\int_L 2xy\,dx-x^2\,dy$, if L OAB: $O(0,0)$, $B(2,0)$, $A(2,1)$.

7. $\int_L x\,dy-y\,dx$, if L $\begin{cases}x=2\cos^3 t,\\ y=2\sin^3 t,\end{cases}$ from $A(2,0)$ to $B(0,2)$.

8. $\int_{L_{AB}} 2xy\,dx+y^2\,dy+z^2\,dz$, if L_{AB} $\begin{cases}x=\cos t,\\ y=\sin t,\\ z=2t,\end{cases}$ from $A(1,0,0)$ to $B(1,0,4\pi)$.

9. $\oint_L y\,dx-x\,dy$, if L $\begin{cases}x=a\cos t,\\ y=b\sin t.\end{cases}$

10. $\oint_L x\,dy$, if L is triangle bounded by $y=x$, $x=2$, $y=0$.

11. $\oint_L \left(x^2+y^2\right)dx+\left(x^2-y^2\right)dy$, if L is triangle with vertices $A(0,0)$, $B(1,0)$, $C(0,1)$.

III INFINITE SEQUENCES AND SERIES

3.1 Series

If we try to add the terms of an infinite sequence $\{a_n\}$ we get an expression of the form

$$a_1 + a_2 + a_3 + ... + a_n + ... \qquad (1)$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n .$$

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the **partial sums**

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$S_3 = a_1 + a_2 + a_3,$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{i=1}^{n} a_n . \qquad (2)$$

These partial sums form a new sequence $\{S_n\}$, which may or may not have a limit. If $\lim\limits_{n\to\infty} S_n = S$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\sum a_n$.

Definition. Given a series $a_1 + a_2 + a_3 + ... + a_n + ... = \sum\limits_{n=1}^{\infty} a_n$, let S_n denote its n-th partial sum: $S_n = a_1 + a_2 + a_3 + ... + a_n = \sum\limits_{i=1}^{n} a_n$.

If the sequence $\{S_n\}$ is convergent and $\lim\limits_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write $a_1 + a_2 + a_3 + ... + a_n + ... = S$.

The number S is called the **sum** of the series. Otherwise, the series is called **divergent**. Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum\limits_{n=1}^{\infty} a_n = S$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i .$$

Example 1. An important example of an infinite series is the **geometric series**

$$a + aq + aq^2 + aq^3 + aq^4 + ... + aq^{n-1} + ... = \sum_{n=1}^{\infty} aq^{n-1} .$$

Solution. Each term is obtained from the preceding one by multiplying it by the common ratio q.

If $q=1$, then $S_n = a+a+a+...+a = na \to \pm\infty$. Since $\lim\limits_{n\to\infty} S_n$ doesn't exist, the geometric series diverges in this case.

If $q \neq 1$, we have

$$S_n = a + aq + aq^2 + aq^3 + aq^4 + ... + aq^{n-1}$$
$$qS_n = aq + aq^2 + aq^3 + aq^4 + ... + aq^n.$$

Subtracting these equations, we get

$$S_n - qS_n = a - aq^n,$$

$$S_n = \frac{a(1-q^n)}{1-q}. \tag{3}$$

If $-1 < q < 1$, we know from that $q^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \frac{a}{1-q}.$$

Thus when $|q| < 1$ the geometric series is convergent and its sum is $S = \dfrac{a}{1-q}$.

If $|q| > 1$ or $q = -1$, the sequence $\{q^n\}$ is divergent and so, by Equation 3, $\lim\limits_{n\to\infty} S_n$ does not exist. Therefore the geometric series diverges in those cases.

We summarize the results of Example 1 as follows.

The geometric series

$$a + aq + aq^2 + aq^3 + aq^4 + ... + aq^{n-1} + ... = \sum_{n=1}^{\infty} aq^{n-1}$$

is convergent if $|q| < 1$ and its sum is $S = \dfrac{a}{1-q}$.

If $|q| > 1$, the geometric series is divergent.

Example 2. Find the sum of the geometric series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$.

Solution. The first term is $a_1 = \dfrac{3}{5}$ and the common ratio is $q = \dfrac{3}{5}$. Since $q = \dfrac{3}{5} < 1$, the series is convergent and its sum is $S = \dfrac{a}{1-q} = \dfrac{\frac{3}{5}}{1-\frac{3}{5}} = \dfrac{3}{2}$.

Example 3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Solution. This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)}.$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

Thus we have

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right);$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$$

and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Example 4. Show that the **harmonic series**

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

is divergent.

Solution. For this particular series it's convenient to consider the partial sums $S_2, S_4, S_8, \ldots, S_{2^n}, \ldots$ and show that they become large.

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$S_{16} > 1 + \frac{4}{2}$$

$$S_{32} > 1 + \frac{5}{2}$$

$$S_{64} > 1 + \frac{6}{2}$$

…

$$S_{2^n} > 1 + \frac{n}{2}.$$

This shows that $S_{2^n} \to \infty$ as $n \to \infty$ and so is divergent. Therefore the harmonic series diverges.

Theorem 1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Note 1. The converse of Theorem 1 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that is convergent. Observe that for the harmonic series $\sum \frac{1}{n}$ we have $\lim_{n\to\infty} a_n = 0$, but we showed in Example 4 that is divergent.

The test for divergence. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5. Show that the series $\sum_{n=1}^{\infty} \frac{2n^2+3}{n^2+5}$ diverges.

Solution. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2+3}{n^2+5} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 \neq 0$. So the series diverges by the Test for Divergence.

Theorem 2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where *c* is a constant), $\sum_{n=1}^{\infty} (a_n + b_n)$, and $\sum_{n=1}^{\infty} (a_n - b_n)$, and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n ,$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum a_n \pm \sum b_n .$$

Example 6. Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \left(\frac{3}{5}\right)^n \right)$.

Solution. The series $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ is a geometric series with $a_1 = \frac{3}{5}$ and $q = \frac{3}{5}$, so $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{3}{2}$.

In Example 3 we found that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

So, by Theorem 2, the given series is convergent and

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\left(\frac{3}{5}\right)^n\right)=3\sum\frac{1}{n(n+1)}+\sum\left(\frac{3}{5}\right)^n=3\cdot 1+\frac{3}{2}=\frac{9}{2}.$$

Exercise Set 7

In Exercise 1 to 15, determine whether the series $\sum a_n$ is convergent or divergent. If it is convergent, find its sum.

1. $\sum_{n=1}^{\infty}\frac{2n+3}{n+5}$.
2. $\sum_{n=1}^{\infty}\frac{3n^2-1}{2n^2+5}$.
3. $\sum_{n=1}^{\infty}\frac{2n^3+6}{5n^3+5n}$.
4. $\sum_{n=1}^{\infty}\frac{n^3+3}{n^2+9}$.
5. $\sum_{n=1}^{\infty}\frac{2n+3}{4n+5}$.
6. $\sum_{n=1}^{\infty}\frac{n^2+3n-1}{n^2+5n+9}$.
7. $\sum_{n=1}^{\infty}\left(\frac{7}{5}\right)^n$.
8. $\sum_{n=1}^{\infty}\left(\frac{3}{25}\right)^n$.
9. $\sum_{n=1}^{\infty}4\left(\frac{13}{5}\right)^n$.
10. $\sum_{n=1}^{\infty}\frac{5^n+2^n}{10^n}$.
11. $\sum_{n=1}^{\infty}\frac{3^n+4^n}{12^n}$.
12. $\sum_{n=1}^{\infty}\frac{5^n-15^n}{25^n}$.
13. $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$.
14. $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$.
15. $\sum_{n=1}^{\infty}\frac{5}{n(n+4)}$.

3.2 The Integral Test. The Comparison Tests

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ because in each of those cases we could find a simple formula for the n-th partial sum S_n. But usually it isn't easy to compute $\lim_{n\to\infty} S_n$. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots.$$

There's no simple formula for the sum S_n of the first *n* terms. We can confirm this impression with a geometric argument. Figure 1 shows the curve $y=\frac{1}{x^2}$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y=\frac{1}{x^2}$ at the right endpoint of the interval. So the sum of the areas of the rectangles is

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+...$$

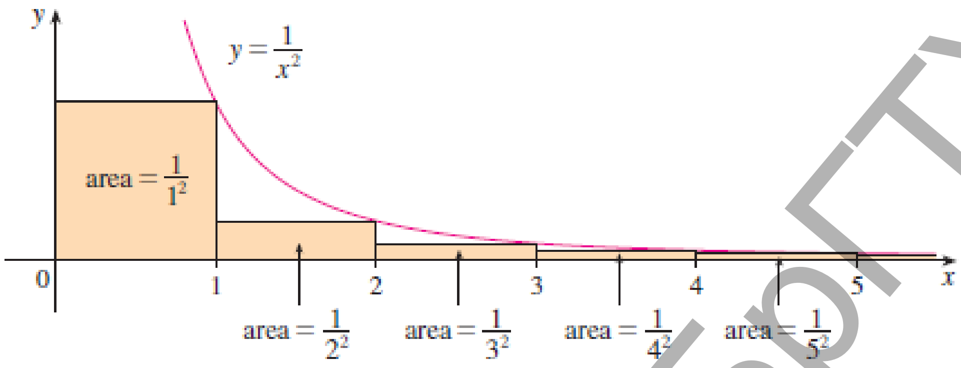


Figure 1

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y=\frac{1}{x^2}$ for $x\geq 1$, which is the value of the integral $\int_1^{\infty}\frac{dx}{x^2}$. We discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than $\frac{1}{1^2}+\int_1^{\infty}\frac{dx}{x^2}=2$.

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+...<2.$$

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1,+\infty)$ and let $a_n=f(n)$. Then the series $\sum_{n=1}^{\infty}a_n$ is convergent if and only if the improper integral $\int_1^{\infty}f(x)dx$ is convergent. In other words:

(a) If $\int_1^{\infty}f(x)dx$ is convergent, then $\sum_{n=1}^{\infty}a_n$ is convergent;

(b) If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 1. For what values of α is the series

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}=1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+...+\frac{1}{n^\alpha}+...$$

convergent?

Solution. If $\alpha<0$, then $\lim_{n\to\infty}\frac{1}{n^\alpha}=\infty$. If $\alpha=0$, then $\lim_{n\to\infty}\frac{1}{n^\alpha}=1$. In either case $\lim_{n\to\infty}\frac{1}{n^\alpha}\neq 0$, so the given series diverges by the Test for Divergence.

If $\alpha<0$, then the function $y=\frac{1}{x^\alpha}$ is clearly continuous, positive, and decreasing on $[1,+\infty)$. We found that $\int_1^\infty \frac{1}{x^\alpha}dx$ converges if $\alpha>1$ and diverges if $\alpha\leq 1$.

It follows from the Integral Test that the series $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ converges if $\alpha>1$ and diverges if $\alpha\leq 1$.

The series in Example 1 is called the α**-series**. It is important in the rest of this chapter, so we summarize the results of Example 1 for future reference as follows.

The α -series $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}=1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+...+\frac{1}{n^\alpha}+...$ is convergent if $\alpha>1$ and divergent if $\alpha\leq 1$.

Example 2. Determine whether the series

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^4(n+1)}=\frac{1}{2\ln^4 2}+\frac{1}{3\ln^4 3}+\frac{1}{4\ln^4 4}+...$$

converges or diverges.

Solution. The function $f(x)=\frac{1}{(x+1)\ln^4(x+1)}$ is positive and continuous for $x\geq 1$ because the logarithm function is continuous.

So we can apply the Integral Test:

$$\int_1^\infty f(x)\,dx=\int_1^\infty\frac{dx}{(x+1)\,\ln^4(x+1)}dx=\int_1^\infty\frac{d(\ln(x+1))}{\ln^4(x+1)}dx=\begin{vmatrix}\ln(x+1)=t\\ d(\ln(x+1))=dt\\ x=1\Rightarrow t=\ln 2\\ x=\infty\Rightarrow t=\infty\end{vmatrix}=\int_{\ln 2}^{+\infty}\frac{dt}{t^4}=$$

$$= \lim_{N\to\infty}\int_{\ln 2}^{N} t^{-4}dt = \lim_{N\to\infty}\left(\frac{t^{-3}}{-3}\right)\Bigg|_{\ln 2}^{N} = \lim_{N\to\infty}\left(-\frac{1}{3N^3}+\frac{1}{3\ln^3 2}\right) = 0+\frac{1}{3\ln^3 2} = \frac{1}{3\ln^3 2} = 1{,}001.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^4(n+1)}$ is also convergent by the Integral Test.

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent.

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms:

(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent;

(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Example 3. Determine whether the series $\sum \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution. For large the dominant term in the denominator is $2n^2$ so we compare the given series with the series $\sum \frac{5}{2n^2}$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know that

$$\sum \frac{5}{2n^2} = \frac{5}{2}\sum \frac{1}{n^2}$$

is convergent because it's a constant times α-series with $\alpha = 2 > 1$. Therefore

$$\sum \frac{5}{2n^2+4n+3}$$

is convergent by part (a) of the Comparison Test.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example 4. Determine whether the series $\sum_{n=1}^{\infty} \frac{3n^2+4n+7}{n^5+6}$ converges or diverges.

Solution. The dominant part of the numerator is $3n^2$ and the dominant part of the denominator is n^5. This suggests taking

$$a_n = \frac{3n^2+4n+7}{n^5+6}, \qquad b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^2+4n+7)n^3}{n^5+6} = \lim_{n\to\infty} \frac{3+\frac{4}{n^2}+\frac{7}{n^3}}{1+\frac{6}{n^5}} = 3.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (α -series with $\alpha = 3 > 1$), the given series converges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series by keeping only the highest powers in the numerator and denominator.

Exercise Set 8

In Exercise 1 to 20, determine whether the series $\sum a_n$ is convergent or divergent, where a_n is

1. $\frac{n^2+n}{n^4+1}$.
2. $\frac{4n+1}{n^3+2}$.
3. $\frac{1}{(n+1)\ln^2(n+1)}$.
4. $\frac{n+2}{n^3+3}$.
5. $\frac{1}{\sqrt{n^3+2n}}$.
6. $\frac{1}{(n+2)\ln(n+2)}$.
7. $\frac{n}{n^3+2}$.
8. $\frac{1}{(n+1)\ln^3(n+1)}$.
9. $\frac{3n+2}{n^3+6}$.
10. $\frac{2n+7}{\sqrt{n^5+3}}$.
11. $\frac{5n+1}{n^4\sqrt{n}}$.
12. $\frac{3n+2}{n\sqrt{n}+1}$.
13. $\frac{n^2}{n^3+5}$.
14. $\frac{1}{4n+\sqrt{n}}$.
15. $\frac{2n+1}{n\sqrt{n}+3}$.
16. $\frac{n}{(2n+1)(3n+2)}$.
17. $\frac{n+1}{\sqrt{n^6+5}}$.
18. $\frac{1}{\sqrt[3]{n^6+8}}$.
19. $\frac{1}{(n+5)\ln^2(n+5)}$.
20. $\frac{1}{(n+5)\ln^6(n+5)}$.

3.3 The Ratio and Root Tests

The following test is very useful in determining whether a given series is convergent.

The Ratio Test.

(a) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = A < 1$, then the series $\sum a_n$ is convergent.

(b) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = A > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = A = \infty$, then the series $\sum a_n$ is divergent.

(c) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = A = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Example 1. Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{3^n}{(n^2+1)\cdot n!}$ converges or diverges.

Solution. We use the Ratio Test with

$$a_n = \frac{3^n}{(n^2+1)\cdot n!},$$

$$a_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)\cdot(n+1)!} = \frac{3^{n+1}}{(n^2+2n+2)\cdot(n+1)!},$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}\cdot(n^2+1)\cdot n!}{(n^2+2n+2)\cdot n!(n+1)\cdot 3^n} = \lim_{n\to\infty} \frac{3\cdot(n^2+1)}{(n^2+2n+2)(n+1)} =$$

$$= 3\cdot \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^3}}{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)} = 3\cdot 0 = 0.$$

Since $A = 0 < 1$, the given series is convergent by the Ratio Test.

The following test is convenient to apply when n-th powers occur. Its proof is similar to the proof of the Ratio Test.

The Root Test.

(a) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A < 1$, then the series $\sum a_n$ is convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A = \infty$, then the series $\sum a_n$ is divergent.

(c) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A = 1$, the Root Test is inconclusive.

If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A = 1$, then part (c) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge. (If $A = 1$ in the Ratio Test, don't try the Root Test because A will again be 1. And if $A = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

Example 2. Test the convergence of the series $\sum\limits_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Solution. We use the Root Test with $a_n = \left(\frac{2n+3}{3n+2}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{2n}{3n} = \frac{2}{3}.$$

Since $A=\frac{2}{3}<1$, the given series is convergent by the Root Test.

Exercise Set 9

In Exercise 1 to 15, determine whether the series $\sum a_n$ is convergent or divergent, where a_n is

1. $\frac{n(n+2)}{4^n}$.
2. $\frac{3^{n+1}}{7^n\cdot n^4}$.
3. $\frac{3^{n+1}}{(n+2)!}$.
4. $\left(\frac{n^2+2n+3}{2n^2+1}\right)^n$.
5. $\frac{3^{n-1}}{5^n\cdot n^2}$.
6. $\frac{3n+7}{8^n}$.
7. $\frac{4^n\cdot n^4}{n!}$.
8. $\left(\frac{3n+5}{5n+3}\right)^n$.
9. $\frac{n+7}{(n+1)!}$.
10. $\frac{2^n}{5^n(2n+1)}$.
11. $\frac{n^3+3}{5^n}$.
12. $\frac{4^n}{7^n(3n+1)}$.
13. $\frac{4n+3}{6^n}$.
14. $\frac{5^n}{4^n(6n+5)}$.
15. $\left(\frac{2n-1}{2n+1}\right)^{n^2}$.

3.4 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}\frac{1}{n}+\ldots,$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}=-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\ldots+(-1)^n\frac{n}{n+1}+\ldots.$$

We see from these examples that the n-th term of an alternating series is of the form

$$a_n=(-1)^n b_n \quad\text{or}\quad a_n=(-1)^{n-1}b_n,$$

where b_n is a positive number. (In fact, $b_n=|a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

The Alternating Series Test. If the alternating series

$$a_1-a_2+a_3-a_4+\ldots+(-1)^{n-1}a_n+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}a_n,$$

satisfies

(a) $a_1>a_2>a_3>\ldots>a_n>a_{n+1}>\ldots,$

$$\text{(b)} \lim_{n\to\infty} a_n = 0$$

then the series is convergent.

Example 1. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ for convergence or divergence.

Solution. The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}\frac{1}{n} + \ldots$$

satisfies:

(a) $a_1 > a_2 > a_3 > \ldots > a_n > a_{n+1} > \ldots$, because $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots > \frac{1}{n} > \ldots$;

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

So the series is convergent by the Alternating Series Test.

Example 2. Test the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+2}$ for convergence or divergence.

Solution. The given series is alternating so we try to verify conditions (a) and (b) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $a_n = \frac{n}{n^2+2}$ is decreasing. However, if we consider the related function $f(x) = \frac{x}{x^2+2}$, we find that

$$f'(x) = \frac{2 - x^2}{(x^2+2)^2}.$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $x > \sqrt{2}$. Thus f is decreasing on the interval $(\sqrt{2};+\infty)$. This means that $f(n+1) < f(n)$ and therefore $a_n > a_{n+1}$ when $n \geq 2$. (The inequality $a_1 > a_2$ can be verified directly but all that really matters is that the sequence $\{a_n\}$ is eventually decreasing.)

Condition (b) is readily verified:

$$\lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus the given series is convergent by the Alternating Series Test.

Estimating Sums

A partial sum S_n of any convergent series can be used as an approximation to the total sum S, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $S \approx S_n$ is the remainder $R_n = S - S_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than a_{n+1}, which is the absolute value of the first neglected term.

Alternating Series Estimation Theorem

If $S=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is the sum of an alternating series that satisfies:

$$\text{(a) } a_1 > a_2 > a_3 > ... > a_n > a_{n+1} > ...$$

and

$$\text{(b) } \lim_{n\to\infty} a_n = 0,$$

then $|S-S_n| = |R_n| \le a_{n+1}$.

Example 3. Find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4n+1}{6^n}$ correct to three decimal places.

Solution. We first observe that the series is convergent by the Alternating Series Test

(a) $a_1 > a_2 > a_3 > ... > a_n > a_{n+1} > ...,$ because $\frac{5}{6} > \frac{9}{36} > \frac{13}{216} > \dots,$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{4n+1}{6^n} = \lim_{n\to\infty}\frac{4}{6^n \ln 6} = 0$.

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$S = \frac{5}{6} - \frac{9}{36} + \frac{13}{216} - \frac{17}{6^4} + \frac{21}{6^5} - \frac{25}{6^6} + \cdots.$$

Notice that

$$a_6 = \frac{25}{6^6} = 0{,}0005358 < 0{,}001$$

and

$$S \approx S_5 = \frac{5}{6} - \frac{9}{36} + \frac{13}{216} - \frac{17}{1296} + \frac{21}{7776} = 0{,}8333 - 0{,}2500 + 0{,}0602 - 0{,}0131 + 0{,}0003 =$$
$$= 0{,}6307 \approx 0{,}631.$$

Absolute Convergence

Given any series $\sum_{n=1}^{\infty} a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty}|a_n| = |a_1| + |a_2| + ... + |a_n| + ...$$

whose terms are the absolute values of the terms of the original series.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Notice that if $\sum_{n=1}^{\infty} a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Example 4. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+3)}$ for absolute convergence.

Solution. We use the Limit Comparison Test with $a_n = \frac{1}{n(n+3)}$. The dominant part of the numerator is 0 and the dominant part of the denominator is n^2. This suggests taking

$$a_n = \frac{1}{n(n+3)} \text{ and } b_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty}\left(\frac{1}{n(n+3)} : \frac{1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n(n+3)} \cdot \frac{n^2}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{3}{n}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (α -series with $\alpha = 2 > 1$), the given series converges by the Limit Comparison Test. Thus, the given series is absolutely convergent and therefore convergent.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent, but not absolutely convergent.

Theorem. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example 5. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+2}$ is absolutely convergent, conditionally convergent, or divergent.

Solution. We use the Limit Comparison Test with $a_n = \frac{n}{n^2+2}$.

The dominant part of the numerator is n and the dominant part of the denominator is n^2. This suggests taking

$$a_n = \frac{n}{n^2+2} \text{ and } b_n = \frac{1}{n}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+1} : \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is convergent (α -series with $\alpha = 1 \geq 1$), the given series diverges by the Limit Comparison Test.

We try to verify conditions (a) and (b) of the Alternating Series Test:

(a) $a_1 = a_2 > a_3 > a_4 > a_5 > a_6 > \ldots$;

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n}{n^2+2} = \lim\limits_{n\to\infty}\frac{n}{n^2\left(1+\frac{2}{n^2}\right)} = \lim\limits_{n\to\infty}\frac{1}{n\cdot\left(1+\frac{2}{n^2}\right)} = 0$.

Thus the given series is conditionally convergent by the Alternating Series Test.

Exercise Set 10

In Exercise 1 to 20, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$.
2. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^2+1}$.
3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$.
4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$.
5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{3n-2}}$.
6. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{(n+1)^2}$.
7. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^2-1}$.
8. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}$.
9. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+4}}$.
10. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[3]{n}}$.
11. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^n}{n+2}$.
12. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4^n(2n+3)}$.
13. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2(n+1)}$.
14. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}5^n}{(n+2)^2}$.
15. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{2n-1}}$.
16. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n\sqrt{n+1}}$.
17. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}(2n-1)}$.
18. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{3n+1}}$.
19. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{4n+5}}$.
20. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6^n(n+1)}$.

In Exercise 21 to 25, approximate the sum of the series correct to three decimal places.

21. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(0{,}2)^n}{(n+1)\cdot(4n+3)}$.
22. $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n\cdot n^2}$.
23. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n\cdot n^3}$.
24. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\cdot 3^n}{(2n+1)\cdot 7^n}$.
25. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3+3}{7^n}$.

3.5 Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad (1)$$

where x is a variable and the a_n 's are constants called the **coefficients** of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

More generally, a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n + \ldots, \qquad (2)$$

is called a **power series in** $(x - x_0)$ or a **power series centered at** x_0 or a **power series about** x_0. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - x_0)^n = 1$ even when $x = x_0$. Notice also that when $x = x_0$, all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x = x_0$.

Example 1. For what values of x is the series $\sum_{n=1}^{\infty} \frac{3n+2}{n+1} \cdot (x+4)^n$ convergent?

Solution. We use the Ratio Test. If we let a_n, as usual, denote the n-th term of the series, then $|u_n(x)| = \frac{3n+2}{(n+1)\cdot 7^n} \cdot |x+4|^n$.

If $x \neq -4$, we have $|u_{n+1}(x)| = \frac{3n+5}{(n+2)\cdot 7^{n+1}} \cdot |x+4|^{n+1}$:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \frac{(3n+5)\cdot |x+4|^{n+1} \cdot (n+1)\cdot 7^n}{(n+2)\cdot 7^{n+1} \cdot (3n+2)\cdot |x+4|^n} = \frac{|x+4|}{7} \cdot \lim_{n\to\infty} \frac{(3n+5)(n+1)}{(n+2)(3n+2)} =$$

$$= \frac{|x+4|}{7} \lim_{n\to\infty} \frac{3n^2}{3n^2} = \frac{|x+4|}{7}.$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $\frac{|x+4|}{7} < 1$ and divergent when $\frac{|x+4|}{7} > 1$. Now

$$\frac{|x+4|}{7} < 1 \Leftrightarrow |x+4| < 7 \Leftrightarrow -7 < x+4 < 7 \Leftrightarrow -11 < x < 3,$$

so the series converges when $x \in (-11;3)$ and diverges when $x \in (-\infty;-11) \cup (3;+\infty)$.

The Ratio Test gives no information when $\frac{|x+4|}{7}=1$ so we must consider $x=-11$ and $x=3$ separately.

If we put $x=3$ in the series, it becomes

$$\sum_{n=1}^{\infty}\frac{(3n+2)\cdot(3+4)^n}{(n+1)\cdot 7^n}=\sum_{n=1}^{\infty}\frac{3n+2}{n+1},$$

which is divergent by the test for divergence.

If $x=-11$, the series is

$$\sum_{n=1}^{\infty}\frac{(3n+2)\cdot(-11+4)^n}{(n+1)\cdot 7^n}=\sum_{n=1}^{\infty}\frac{(3n+2)}{(n+1)}\cdot(-\frac{7}{7})^n=\sum_{n=1}^{\infty}(-1)^n\frac{3n+2}{n+1},$$

which diverges by the Alternating Series Test ($\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\frac{3n+2}{n+1}=3\neq 0$). Thus the given power series converges for $-11<x<3$.

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval, the infinite interval or a collapsed interval]. The following theorem says that this is true in general.

Theorem. For a given power series $\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$, there are only three possibilities:

(a) The series converges only when $x=x_0$;

(b) The series converges for all x;

(c) There is a positive number R such that the series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

The number in case (c) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R=0$ in case (a) and $R=\infty$ in case (b). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (a) the interval consists of just a single point x_0. In case (b) the interval is $(-\infty;+\infty)$. In case (c) note that the inequality $|x-x_0|<R$ can be rewritten as $x_0-R<x<x_0+R$. When x is an *endpoint* of the interval, that is, $x=x_0\pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. The situation is illustrated in Figure 2.

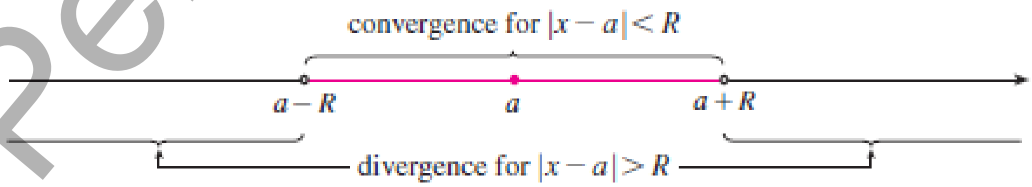


Figure 2

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

Series	Radius of convergence	Interval of convergence
$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1;1)$
$\sum_{n=0}^{\infty} n!x^n$	$R=0$	$\{0\}$
$\sum_{n=1}^{\infty} \frac{3n+2}{n+1}\cdot(x+4)^n$	$R=7$	$-11<x<3.$
$\sum_{n=0}^{\infty} \frac{(x-3)^{2n}}{n!(2n+3)}$	$R=\infty$	$(-\infty;+\infty)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Example 2. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2\cdot 3^{n-1}}.$$

Solution. Let $a_n=\frac{1}{n^2\cdot 3^{n-1}}$. Then $a_{n+1}=\frac{1}{(n+1)^2\cdot 3^n}$.

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2\cdot 3^n}{n^2\cdot 3^{n-1}}\right|=3\cdot\lim_{n\to\infty}\frac{(n+1)^2}{n^2}=3.$$

So it converges if $|x|<3$ and diverges if $|x|>3$. Thus the radius of convergence is $R=3$.

The inequality $|x|<3$ can be written as $-3<x<3$, so we test the series at the endpoints $x=-3$ and $x=3$. When $x=3$, the series is $\sum_{n=1}^{\infty}\frac{3}{n^2}=3\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (α -series with $\alpha=2>1$), the given series converges by the Limit Comparison Test.

When $x=-3$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 3}{n^2}=3\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which is absolutely convergent and therefore convergent.

Thus the series converges only when $-3\le x\le 3$, so the interval of convergence is $x\in[-3;3]$.

Exercise Set 11

In Exercise 1 to 15, find the radius of convergence and interval of convergence of the series.

1. $\sum_{n=1}^{\infty}\frac{(2n-1)^n}{2^n\cdot n^n}(x+1)^n$.
2. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}(x+6)^n$.
3. $\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n\cdot(n+1)}(x-3)^n$

4. $\sum_{n=1}^{\infty}(n+2)(x+3)^n$.

5. $\sum_{n=1}^{\infty}\frac{1}{n(2n+3)}(x-4)^n$.

6. $\sum_{n=1}^{\infty}\frac{n!}{6^n}(x-6)^{2n}$.

7. $\sum_{n=1}^{\infty}\frac{4^n}{n!}(x+1)^n$.

8. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n\cdot n^n}(x-3)^n$.

9. $\sum_{n=1}^{\infty}\frac{1}{9^n\cdot n^n}(x-1)^{2n}$.

10. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}(x+1)^{n+1}$.

11. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{3^n}(x-1)^n$.

12. $\sum_{n=1}^{\infty}\frac{2^n}{n(n+1)}(x-1)^n$.

13. $\sum_{n=1}^{\infty}\frac{1}{4^n\cdot n!}(x+3)^{n-1}$.

14. $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n+2}}(x+2)^n$.

15. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n\cdot(n+3)}(x+1)^n$

3.6 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. (Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.)

We start with an equation that we have seen before:

$$\frac{1}{1-x}=1+x+x^2+...+x^{n-1}+...=\sum_{n=0}^{\infty}x^n,\quad -1<x<1. \qquad (1)$$

We now regard Equation 1 as expressing the function as a sum of a power series.

Example 1. Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution. Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1+(-x^2)+(-x^2)^2+(-x^2)^3+...=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Because this is a geometric series, it converges when $\left|-x^2\right|<1$, that is, $x^2<1$, or $-1<x<1$. Therefore the interval of convergence is $(-1;1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

Example 2. Find a power series representation for $\frac{1}{x+2}$.

Solution. In order to put this function in the form of the left side of Equation 1 we first factor a 2 from the denominator:

$$\frac{1}{x+2}=\frac{1}{2\cdot\left(1+x/2\right)}=\frac{1}{2}\cdot\frac{1}{1-\left(-x/2\right)}=\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}\frac{(-1)^n x^n}{2^{n+1}}.$$

This series converges when $\left|-\frac{x}{2}\right| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2;2)$.

Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots,\quad |x-x_0| < R. \quad (2)$$

Let's try to determine what the coefficients a_n must be in terms of f.

Theorem. If f has a power series representation (expansion) at x_0, that is, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, |x-x_0| < R,$$

then its coefficients are given by the formula $a_n = \frac{f^{(n)}(x_0)}{n!}$.

Substituting this formula for a_n back into the series, we see that if f has a power series expansion at x_0, then it must be of the following form:

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots \quad (3).$$

The series in Equation 3 is called the **Taylor series of the function f at** x_0 (or **about** x_0 or **centered at** x_0). For the special case the Taylor series becomes

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \quad (4)$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Note 1. We have shown that if f can be represented as a power series about x_0, then is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series.

Theorem. If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the n th-degree Taylor polynomial of f at x_0 and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-x_0| < R$, then f is equal to the sum of its Taylor series on the interval $|x-x_0| < R$.

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

Table 1

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$	$-\infty < x < \infty$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + ...$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + ...$	$-\infty < x < \infty$
$\frac{1}{1-x} = 1 + x + x^2 + ... + x^{n-1} + ...$	$-1 < x < 1$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1}\frac{x^n}{n} + ...$	$-1 < x < 1$
$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n + ...$	$-1 < x < 1$

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, in the introduction to this chapter we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term. The function $f(x) = e^{-x^2}$ can't be integrated by techniques discussed so far because its antiderivative is not an elementary function. In the following example we use Newton's idea to integrate this function.

Example 3. Evaluate $\int_0^1 e^{-x^2}dx$ correct to within an error of 0.001.

Solution. First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it simply by replacing x with x^2 in the series for e^x given in Table 1. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$$

Now we integrate term by term:

$$\int_0^1 e^{-x^2}dx = \int_0^1\left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + ...\right)dx = \left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!}...\right)_0^1 =$$

$$= 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1}{216}.. \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0{,}748.$$

(This series converges for all because the original series for e^{-x^2} converges for all x.)

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1^9}{9\cdot 4!} = \frac{1}{216} < 0{,}01.$$

Example 4. Use power series to solve the initial-value problem

$$y' = 4xy^2 - x^3,\ y(0) = 2.$$

Solution. We assume there is a solution of the form

$$y(x)=y(0)+\frac{y'(0)}{1!}\cdot x+\frac{y''(0)}{2!}\cdot x^2+\frac{y'''(0)}{3!}\cdot x^3+\frac{y^{(4)}(0)}{4!}\cdot x^4+\dots .$$

$y'(0)=4\cdot 0\cdot 2^2-0^3=0$.

We can differentiate power series term by term, so

$$y''(x)=\left(4xy^2-x^3\right)'=\left(4xy^2\right)'-\left(x^3\right)'=4y^2\cdot(x)'+4x\cdot\left(y^2\right)'-3x^2=$$

$$=4y^2\cdot 1+4x\cdot 2y\cdot y'-3x^2=4y^2+8xyy'-3x^2.$$

Let $x=0$, $y=2$, $y'(0)=0$ then

$$y''(0)=4\cdot 2^2+8\cdot 0\cdot 2\cdot 0-3\cdot 0^2=16.$$

$$y'''(x)=\left(4y^2+8xyy'-3x^2\right)'=\left(4y^2\right)'+\left(8xyy'\right)'-\left(3x^2\right)'=$$

$$=8y\cdot y'+8yy'+8x\left(yy'\right)'-6x=16yy'+8xyy''+8x\left(y'\right)^2-6x.$$

Let $x=0$, $y=2$, $y'(0)=0$, $y''(0)=16$ then

$$y'''(0)=16\cdot 2\cdot 0+8\cdot 0\cdot 2\cdot 16+8\cdot 0\cdot (0)^2-6\cdot 0=0.$$

$$y^{(4)}(x)=\left(16yy'+8xyy''+8x(y')^2-6x\right)'=\left(16yy'\right)'+\left(8xyy''\right)'+\left(8x\left(y'\right)^2\right)'-(6x)'=$$

$$=16y'y'+16yy''+8yy''+8x\left(yy''\right)'+8\left(y'\right)^2+8x\cdot 2y'\cdot y''-6=$$

$$=24\left(y'\right)^2+24yy''+24xy'y''+8xyy'''-6.$$

Let $x=0$, $y=2$, $y'(0)=0$, $y''(0)=16$, $y'''(0)=0$ then

$$y^{(4)}(0)=24\cdot 0^2+24\cdot 2\cdot 16+24\cdot 0\cdot 0\cdot 16+8\cdot 0\cdot 2\cdot 0-6=768-6=762.$$

Substituting the obtained coefficients in the Maclaurin series, we will obtain the solution of the initial differential equation

$$y(x)=2+\frac{0}{1!}\cdot x+\frac{16}{2!}\cdot x^2+\frac{0}{3!}\cdot x^3+\frac{762}{4!}\cdot x^4+\dots=2+\frac{16}{2}\cdot x^2+\frac{762}{24}\cdot x^4+\dots=$$

$$=2+8x^2+31{,}75x^4+\dots .$$

Exercise Set 12

In Exercise 1 to 8, find a power series representation for the function and determine the interval of convergence.

1. $f(x)=\dfrac{1}{3+x}$.
2. $f(x)=\dfrac{2}{1-3x^2}$.
3. $f(x)=\cos\dfrac{2x^3}{3}$.
4. $f(x)=e^{4x}$.
5. $f(x)=\dfrac{x^2}{1+x}$.
6. $f(x)=\ln(1-4x)$.
7. $f(x)=\sin\dfrac{x^2}{3}$.
8. $f(x)=e^{-2x^3}$.

In Exercise 9 to 12, use a power series to approximate the definite integral to three decimal places.

9. $\int_{0}^{1} \cos\sqrt{x}\,dx$.   10. $\int_{0}^{0,5} \frac{dx}{1+x^6}$.   11. $\int_{0.1}^{0.5} \frac{e^x - 1}{x}dx$.   12. $\int_{0.01}^{0.1} \frac{\ln(1+x)}{x}dx$.

In Exercise 13 to 20, use power series to solve the initial-value problem.

13. $y' = 3\cos x + y^2$, $y(0) = 1$.
14. $y' = 3xy - e^x + 4$, $y(0) = 0$.
15. $y' = 2y + y^2$, $y(0) = 3$.
16. $y' = 2\sin x - x^2 y$, $y(0) = 1$.
17. $y' = 4\sin x + y^2$, $y(0) = 1$.
18. $y' = xy^3 - 2x$, $y(0) = 2$.

Literature

1 Stewart James Calculus Early Transcendental. 2008. pp. 1038.

Contents

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