Classical Solutions of Mixed Problem on the Plane for Hyperbolic Equation. The Method of Characteristic Parallelogram

V. Korzyuk [1,2], S. Naumavets [3]

[1] Institute of Mathematics, National Academy of Sciences of Belarus
Minsk, 220072 Belarus

[2] Belarusian State University
Minsk, 220030 Belarus
korzyuk@bsu.by

[3] Brest State Technical University
Brest, 224017 Belarus
e-cveta@tut.by

Abstract. *This paper regards the mixed problem for wave equation on the plane. The authors of the article prove that the usage of necessary and sufficient homogeneous compatibility conditions guarantees coming to the classical solution in the half-strip. The article gives the classical solution to the one-dimensional wave equation in analytical form if there are Dirichlet conditions at the side edges and conditions of Cauchy type at the bottom of the plane. The Classical solution means function which is defined in all points presented in the closure of the defined domain. This function must have all classical derivatives included in the equation and conditions of problem. In case of heterogeneous compatibility conditions correct problem is formulated. In this case the conjugation conditions are added to the problem. Then the classical solution is piecewise smooth in the corresponding subdomains of the half-strip. Also this paper obtains the classical solution of the problem using the method of the characteristic parallelogram.*

1 Introduction

Using the method of characteristics, the classical solution of problems for hyperbolic equations can get in an analytical form. For example it obtained for the problems for one-dimensional wave equation [3, 5-6], for the mixed problems for Klein-Gordon-Fock eqaution [8], for mixed problems for bi-wave equation [9]. The advantage of the method of characteristics is that the results are obtained for many problems with different boundary and integral conditions.

Here, using the example of the first mixed problem for a one-dimensional wave equation defined in a half-strip, it is shown that the solution or its derivatives have a discontinuity on a certain set of characteristics inside the domain of definition of the equation if there are no fully or partially compatibility conditions in the corner points for the given functions of the boundary conditions and the right part of the equation.

The problem is also considered in the case of non-uniform matching conditions. The formulation of the problem, especially in the case of non-uniform matching conditions, can be considered with the use of the conjugation conditions of the searched function and its derivatives.

Using the method of characteristic parallelogram, the solution of the problem is written in the form of a formula using the specified functions and a particular solution of the original equation.

2 Statement of the problem

In the closure $\overline{Q} = [0, \infty) \times [0, l]$ of the domain $Q = (0, \infty) \times (0, l)$ of two independent variables $\mathbf{x} = (x_0, x_1) \in \overline{Q} \subset \mathbb{R}^2$ we analyze the wave equation

$$\left(\partial_{x_0}^2 - a^2 \partial_{x_1}^2\right) u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} = (x_0, x_1) \in \overline{Q} \subset \mathbb{R}^2, \tag{1}$$

where a^2, l are positive real numbers, $\partial_{x_0}^2 = \partial^2/\partial {x_0}^2$, $\partial_{x_1}^2 = \partial^2/\partial {x_1}^2$ are partial derivatives of the second order. For equation (1) on the boundary ∂Q of the domain Q we add the initial conditions

$$u(0, x_1) = \varphi(x_1), \quad \partial_{x_0} u(0, x_1) = \psi(x_1), \quad x_1 \in [0, l], \tag{2}$$

and boundary conditions

$$u(x_0, 0) = \mu^{(1)}(x_0), \quad u(x_0, l) = \mu^{(2)}(x_0), \quad x_0 \in [0, \infty). \tag{3}$$

Here $f : \overline{Q} \ni \mathbf{x} \to f(\mathbf{x})$ is the given function in $\overline{Q}$, $\varphi : [0, l] \ni x_1 \to \varphi(x_1) \in \mathbb{R}$, $\psi : [0, l] \ni x_1 \to \psi(x_1) \in \mathbb{R}$ are functions on $[0, l]$, $\mu^{(j)} : [0, \infty) \ni x_0 \to \mu^{(j)}(x_0) \in \mathbb{R}$ are the given functions in $[0, \infty)$, the smoothness of which will be clarified below, $j = 1, 2$.

Functions f, φ, ψ, $\mu^{(j)}$, $j = 1, 2$ satisfy the following non-uniform compatibility conditions:

$$\begin{aligned} &\varphi(0) - \mu^{(1)}(0) = \delta^{(1)}, \quad \frac{1}{a}\left(d\mu^{(1)}(0) - \psi(0)\right) = \delta^{(2)}, \\ &\frac{1}{a^2}\left(a^2 d^2\varphi(0) - d^2\mu^{(1)}(0) + f(0,0)\right) = \delta^{(3)}, \end{aligned} \tag{4}$$

$$\begin{aligned} &\mu^{(2)}(0) - \varphi(l) = \sigma^{(1)}, \quad \frac{1}{a}\left(d\mu^{(2)}(0) - \psi(l)\right) = \sigma^{(2)}, \\ &\frac{1}{a^2}\left(d^2\mu^{(2)}(0) - a^2 d^2\varphi(l) - f(0,l)\right) = \sigma^{(3)}, \end{aligned} \tag{5}$$

where $d\mu^{(j)}$ and $d^2\mu^{(j)}$ – derivatives of functions $\mu^{(j)}$ of the first and second orders, $j = 1, 2$, $d^2\varphi$ – derivative of the second order of the function φ.

If all numbers $\sigma^{(j)} = \delta^{(j)} = 0$, $j = 1, 2, 3$ in the compatibility conditions (4) – (5), then the conditions (4) – (5) are the homogeneous compatibility conditions with respect to the given functions of the problem (1) – (3). Note that for sufficiently smooth given functions of the equation (1) in the set $\overline{Q}$, conditions (2) in the interval $[0,l]$, conditions (3) in the half-line $[0,\infty)$ there is the unique classical solution to this problem if and only if the coincidence conditions (4), (5) for these functions are homogeneous. Otherwise, on certain characteristics in the domain Q of solution u of the problem (1) – (3) along with derivatives, they have discontinuities. These discontinuities can be written in the form of pairing conditions, which will be done. Thus, in the general case, the problem (1) – (3) can be replaced with a problem (1) – (5) with conjugation conditions on characteristics, where the jumps of functions and theirs derivatives are expressed in terms of given real numbers $\delta^{(j)}$ and $\sigma^{(j)}$, $j=1,2,3$. The solution of the problem (1) – (5) will be written out in the analytical form through functions f, φ, ψ, $\mu^{(1)}$ and $\mu^{(2)}$ using the appropriate formulas.

3 Particular solutions of equation (1)

The general solution $u \in C^2(\overline{Q})$ of the equation (1) is a sum of the general solution $u^{(0)} \in C^2(\overline{Q})$ of the homogeneous equation

$$\left(\partial_{x_0}^2 - a^2\partial_{x_1}^2\right) u^{(0)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \overline{Q}, \tag{6}$$

and the particular solution $v_p \in C^2(\overline{Q})$ of the inhomogeneous equation (1). We now construct the particular solution of the equation (1) without continuing the function f in the variable x_1 outside the segment $[0,l]$.

Here the construction will be carried out locally. To do this, we divide the domain Q into subdomains $Q^{(m)} = \left(\frac{(m-1)l}{a}, \frac{ml}{a}\right) \times (0,l)$, where $m=1,2,\dots$ (see fig. 1). From the characteristics $x_1 - ax_0 = (1-m)l$ and $x_1 + ax_0 = ml$ of the domain $Q^{(m)}$ we integrate the equation (1).

As a result, we obtain in $Q^{(m)}$ the solution

$$\begin{gathered} v_p^{(m)}(\mathbf{x}) = f^{(1,m)}(x_1 - ax_0) + f^{(2,m)}(x_1 + ax_0) - \\ - \frac{1}{4a^2} \int\limits_{l-ml}^{x_1 - ax_0} dy \int\limits_{ml}^{x_1+ax_0} f\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz, \\ x_1 \in [0,l], \end{gathered} \tag{7}$$

where $f^{(j,m)}$ are from C^2 in all their domains of definition, $\mathbf{x} \in \overline{Q}$. The particular solution v_p of the equation (1) is determined by the following formula

$$v_p(\mathbf{x}) = v_p^{(m)}(\mathbf{x}), \mathbf{x} \in Q^{(m)}, m \in N. \tag{8}$$

Theorem 1. *Let the function f is from $C^1(\overline{Q})$, $\overline{Q} = [0,\infty) \times [0,l]$. Then the function v_p, defined by the formulas (7) and (8) with the certain choice of the functions $f^{(j,m)}$, $j=1,2$, $m \in \mathbb{N}$, is from the space $C^2\left(\overline{Q}\right)$, is the solution of the equation (1) and satisfies the conditions*

$$\begin{gathered} v_p(0,x_1) = \partial_{x_0} v_p(0,x_1) = 0, \\ \partial_{x_0}^2 v_p(0,x_1) = f(0,x_1), x_1 \in [0,l]. \end{gathered} \tag{9}$$

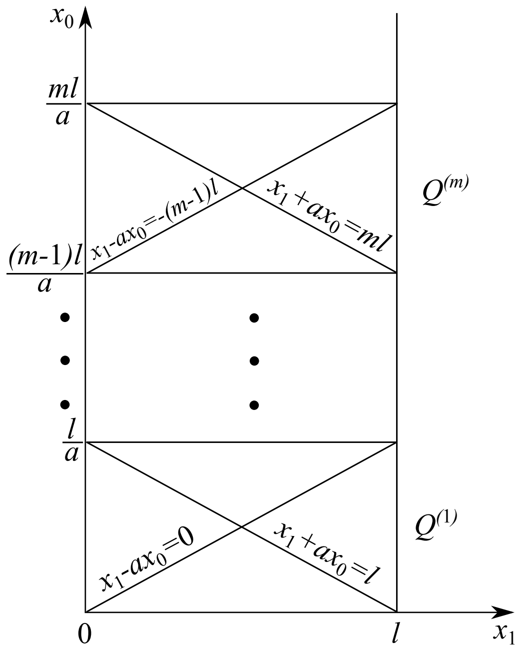

Figure 1: Dividing the domain Q into subdomains $Q^{(m)}$.

Proof. If $f \in C^1\left(\overline{Q}\right)$, $f^{(1,m)} \in C^2\left([-ml, 2l-ml]\right)$, $f^{(2,m)} \in C^2\left([ml-l, ml+l]\right)$, then according to the formula (7) $v_p^{(m)} \in C^2\left(\overline{Q^{(m)}}\right)$ for each $m \in \mathbb{N}$. We calculate the derived functions $v_p^{(m)}$, namely:

$$\partial_{x_0} v_p^{(m)}(\mathbf{x}) = -adf^{(1,m)}(x_1 - ax_0) + adf^{(2,m)}(x_1 + ax_0) +$$

$$+\frac{1}{4a}\int\limits_{ml}^{x_1+ax_0} f\left(\frac{z - x_1 + ax_0}{2a}, \frac{z + x_1 - ax_0}{2}\right) dz -$$

$$-\frac{1}{4a}\int\limits_{l-ml}^{x_1-ax_0} f\left(\frac{x_1 + ax_0 - y}{2a}, \frac{x_1 + ax_0 + y}{2}\right) dy,$$

$$\partial_{x_0}^2 v_p^{(m)}(\mathbf{x}) = a^2 d^2 f^{(1,m)}(x_1 - ax_0) + a^2 d^2 f^{(2,m)}(x_1 + ax_0) + \frac{1}{2} f(\mathbf{x}) +$$

$$+\frac{1}{8a}\int\limits_{ml}^{x_1+ax_0} \partial_{y_0} f\left(y_0 = \frac{z - x_1 + ax_0}{2a}, \frac{z + x_1 - ax_0}{2}\right) dz -$$

$$-\frac{1}{8}\int\limits_{ml}^{x_1+ax_0} \partial_{y_1} f\left(\frac{z - x_1 + ax_0}{2a}, y_1 = \frac{z + x_1 - ax_0}{2}\right) dz -$$

$$-\frac{1}{8a}\int\limits_{l-ml}^{x_1-ax_0}\partial_{y_0} f\left(y_0=\frac{x_1+ax_0-y}{2a},\frac{x_1+ax_0+y}{2}\right)dy-$$

$$-\frac{1}{8}\int\limits_{l-ml}^{x_1-ax_0}\partial_{y_1} f\left(\frac{x_1+ax_0-y}{2a},y_1=\frac{x_1+ax_0+y}{2}\right)dy,$$

$$\partial_{x_1} v_p^{(m)}(\mathbf{x})=df^{(1,m)}(x_1-ax_0)+df^{(2,m)}(x_1+ax_0)-$$

$$-\frac{1}{4a^2}\int\limits_{ml}^{x_1+ax_0} f\left(\frac{z-x_1+ax_0}{2a},\frac{z+x_1-ax_0}{2}\right)dz-$$

$$-\frac{1}{4a^2}\int\limits_{l-ml}^{x_1-ax_0} f\left(\frac{x_1+ax_0-y}{2a},\frac{x_1+ax_0+y}{2}\right)dy,$$

$$\partial_{x_1}^2 v_p^{(m)}(\mathbf{x})=\frac{1}{a^2}\partial_{x_0}^2 v_p^{(m)}(\mathbf{x})-\frac{1}{2a^2}f(\mathbf{x}).$$

The obtained ratios of derivatives are substituted into the equation (1). As a result, we are convinced that the function $v_p^{(m)}$ is the solution of this equation for $\mathbf{x}\in\overline{Q^{(m)}}$ and each number $m\in\mathbb{N}$.

According to the definition of the function v_p by the formula (8), requiring to fulfill the first two conditions from (9), for functions $f^{(j,1)}$ ($j=1,2$), substituting $v_p^{(1)}$ into these conditions, we obtain the system of equations

$$\begin{aligned}
&f^{(1,1)}(x_1)+f^{(2,1)}(x_1)=\frac{1}{4a^2}\int\limits_0^{x_1}dy\int\limits_l^{x_1} f\left(\frac{z-y}{2a},\frac{z+y}{2}\right)dz,\\
&-df^{(1,1)}(x_1)+df^{(2,1)}(x_1)=\\
&=\frac{1}{4a^2}\int\limits_l^{x_1} f\left(\frac{z-x_1}{2a},\frac{z+x_1}{2}\right)dz-\frac{1}{4a^2}\int\limits_0^{x_1} f\left(\frac{x_1-y}{2a},\frac{x_1+y}{2}\right)dy.
\end{aligned}\tag{10}$$

Solving the system (10), we obtain the values of the functions $f^{(j,1)}$ ($j=1,2$),

$$f^{(1,1)}(x_1)=\frac{1}{4a^2}\int\limits_0^{x_1}d\xi\int\limits_l^{\xi} f\Big(\frac{z-\xi}{2a},\frac{z+\xi}{2}\Big)dz,$$

$$f^{(2,1)}(x_1)=\frac{1}{4a^2}\int\limits_0^{x_1}d\eta\int\limits_0^{\eta} f\Big(\frac{\eta-y}{2a},\frac{y+\eta}{2}\Big)dy=\frac{1}{4a^2}\int\limits_0^{x_1}dy\int\limits_y^{x_1} f\Big(\frac{\eta-y}{2a},\frac{y+\eta}{2}\Big)d\eta,$$

for which

$$v_p^{(1)}(0,x_1)=\partial_{x_0}v_p^{(1)}(0,x_1)=0.\tag{11}$$

From the equation (1) and the conditions (11) the correlation follows

$$\partial^2_{x_0} v_p^{(1)}(0, x_1) = f(0, x_1) + a^2 \partial^2_{x_1} v_p^{(1)}(0, x_1) = f(0, x_1).$$

Thus, for a certain function v_p the ratio (8) with the appropriate choice of functions $f^{(j,1)}$ fulfills the requirements (9) of theorem 1.

Since $f \in C^1\left(\overline{Q}\right)$, the functions $v_p^{(m)}$, $m = 1, 2$, are from the space $C^2\left(\overline{Q^{(m)}}\right)$. On the line of contact $\{\mathbf{x} | \, x_0 = l/a\}$ between the two domains $Q^{(1)}$ and $Q^{(2)}$ we will demand that the continuity of two functions be fulfilled $v_p^{(1)}$ and $v_p^{(2)}$, namely

$$\begin{aligned} &v_p^{(2)}\left(\frac{l}{a}, x_1\right) = v_p^{(1)}\left(\frac{l}{a}, x_1\right), \\ &\partial_{x_0} v_p^{(2)}\left(\frac{l}{a}, x_1\right) = \partial_{x_0} v_p^{(1)}\left(\frac{l}{a}, x_1\right), \; x_1 \in [0, l]. \end{aligned} \tag{12}$$

We will consider the conditions (12) as Cauchy conditions for the function $v_p^{(2)}$. They can be achieved by selecting the functions $f^{(j,2)}$ accordingly. Substituting representations (7) into conditions (12), we obtain the system (10) for the functions $f^{(j,2)}$, $j = 1, 2$.

On the basis of the conditions (12) and the equation (1) we obtain the equality

$$\partial^2_{x_0} v_p^{(2)}(\frac{l}{a}, x_1) = \partial^2_{x_0} v_p^{(1)}\left(\frac{l}{a}, x_1\right), \; x_1 \in [0, l]. \tag{13}$$

From the previous arguments and equalities (12), (13) we conclude that the function $v_p^{(1,2)}$, defined by the relation

$$v_p^{(1,2)}(\mathbf{x}) = v_p^{(m)}(\mathbf{x}), \, \mathbf{x} \in \overline{Q^{(m)}}, m = 1, 2,$$

is from the space $C^2\left(\overline{Q^{(1)}} \cup \overline{Q^{(2)}}\right)$.

Moving along the proposed scheme due to the certain choice of functions $f^{(j,m)}$, $j = 1, 2$; $m = 3, 4, \ldots$, we prove that the function constructed v_p according to the formulas (7) and (8) is to the class $C^2\left(\overline{Q}\right)$ and satisfies the equation (1), for all $\mathbf{x} \in \overline{Q}$.

4 The solution of the problem (1) - (3). Uniform compatibility conditions

We begin the study and construction of the classical solution of the problem (1) – (3) with the general solution $u \in C^2(\overline{Q})$ of the equation (1). The general solution from $C^2(\overline{Q})$ this equation can be represented as

$$u(\mathbf{x}) = g^{(1)}(x_1 - a x_0) + g^{(2)}(x_1 + a x_0) + v_p(\mathbf{x}), \tag{14}$$

where $g^{(1)} \in C^2((-\infty, l])$, $g^{(2)} \in C^2([0, \infty))$, v_p – private solution of the equation (1) $v_p \in C^2(\overline{Q})$, $a > 0$ (for definiteness).

If $f \in C^1(\overline{Q})$, then the particular solution according to theorem 1 is determined by formulas (7) – (8). Here also $v_p \in C^2(\overline{Q})$, $C^1(\overline{Q}) \subset C^{0,1}(\overline{Q})$.

From the general representation (14) of the solution of the equation (1) we select the one that satisfies the condition (2) and (3). To do this, we will assume that $\varphi \in C^2([0,l]), \psi \in C^1([0,l]), \mu^{(j)} \in C^2([0,\infty)), j = 1,2$.

From (14) and Cauchy conditions (2) we have a system

$$\begin{aligned} & g^{(1)}(x_1) + g^{(2)}(x_1) = \varphi(x_1),\ x_1 \in [0,l], \\ & -a\,\mathrm{d}\,g^{(1)}(x_1) + a\,\mathrm{d}\,g^{(2)}(x_1) = \psi(x_1),\ x_1 \in [0,l]. \end{aligned} \tag{15}$$

Here conditions were taken (9).

As the functions $g^{(j)}, j = 1,2$, in the system (15) are determined partially, and not in all areas $\mathcal{D}(g^{(j)})$ of their definition, here we introduce the notation

$$g^{(j)}(z) = g^{(j,0)}(z),\ z \in [0,l].$$

Integrating the second equation from (9), and then solving the resulting algebraic system, we find the values $g^{(j,0)}(x_1)$, namely:

$$g^{(j)}(z) = g^{(j,0)}(z) = \frac{1}{2}\varphi(z) + (-1)^j \frac{1}{2a}\int_0^z \psi(\xi)d\xi + (-1)^j C,\ j = 1,2, \tag{16}$$

for $z \in [0,l]$, C is an arbitrary constant from the set $\mathbb{R}$, which appeared as the result of integrating the second equation of the system (15).

For other values of the argument z the values of the functions $g^{(j)}$ are determined in stages, using boundary conditions (3). Satisfying the solution (14) to the first boundary condition from (3), we obtain the equation

$$g^{(1)}(z) + g^{(2)}(-z) = \mu^{(1)}\Big(-\frac{z}{a}\Big) - v_p\Big(-\frac{z}{a}, 0\Big). \tag{17}$$

Since the function $g^{(2)}$ is already defined by the formula (16) for $-z = ax_0 \in [0,l]$, then

$$g^{(1)}(z) = g^{(1,1)}(z) = \mu^{(1)}\Big(-\frac{z}{a}\Big) - g^{(2,0)}(-z) - v_p\Big(-\frac{z}{a}, 0\Big),\ z \in [-l, 0]. \tag{18}$$

Similarly, by satisfying (14) the second boundary condition from (3), we obtain the relation

$$g^{(2)}(z) = \mu^{(2)}\Big(\frac{z-l}{a}\Big) - g^{(1)}(2l - z) - v_p\Big(\frac{z-l}{a}, l\Big), \tag{19}$$

and in particular

$$g^{(2)}(z) = g^{(2,1)}(z) = \mu^{(2)}\Big(\frac{z-l}{a}\Big) - g^{(1,0)}(2l - z) - v_p\Big(\frac{z-l}{a}, l\Big),\ z \in [l, 2l]. \tag{20}$$

Continuing this process further, for the k-th iteration through the previous values of the functions will be obtained using the relations

$$g^{(1)}(z) = g^{(1,k)}(z) = \mu^{(1)}\Big(-\frac{z}{a}\Big) - g^{(2,k-1)}(-z) - v_p\Big(-\frac{z}{a}, 0\Big), z \in [-kl, -(k-1)l], \quad (21)$$

$$g^{(2)}(z) = g^{(2,k)}(z) = \mu^{(2)}\Big(\frac{z-l}{a}\Big) - g^{(1,k-1)}(2l - z) - v_p\Big(\frac{z-l}{a}, l\Big), \ z \in [kl, (k+1)l], \quad (22)$$

$k = 1, 2, 3, \ldots$. From the formulas (21) – (22) it is clear that the values of the functions $g^{(j)}$, $j = 1, 2$, are defined piecewise on the corresponding segments through the values of the given functions $f, \varphi, \psi, \mu^{(j)}$ ($j = 1, 2$). Therefore, if we require sufficient smoothness of these functions, then the functions $g^{(j,k)}$ will also be smooth on the corresponding segments (see (21), (22)), for example, from the class C^2, $j = 1, 2$; $k = 1, 2, 3 \ldots$. Consequently, the solution (14) of the problem (1) – (3) will be piecewise smooth in Q. We need the function (14) to be from the class $C^2(\overline{Q})$ on the whole set $\overline{Q}$. According to theorem 1, $v_p \in C^2(\overline{Q})$, if $f \in C^1(\overline{Q})$. We require that the functions (16), (21), (22) and their derivatives of the first and second orders coincide at common points of contact. Calculating the derivatives of functions $g^{(j,k)}$ ($j = 1, 2$; $k = 1, 2, 3 \ldots$), represented by formulas (21) and (22), we get their expressions

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\, g^{(1,k)}(z) = -\frac{1}{a}\,\mathrm{d}\,\mu^{(1)}\Big(-\frac{z}{a}\Big) + \mathrm{d}\,g^{(2,k-1)}(-z) + \frac{1}{a}\frac{\partial}{\partial x_0}\, v_p\Big(x_0 = -\frac{z}{a}, 0\Big),\\
&\frac{\mathrm{d}^2}{\mathrm{d}z^2}\, g^{(1,k)}(z) = \frac{1}{a^2}\,\mathrm{d}^2\,\mu^{(1)}\Big(-\frac{z}{a}\Big) - \mathrm{d}^2 g^{(2,k-1)}(-z) - \frac{1}{a^2}\frac{\partial^2}{\partial x_0^2}\, v_p\Big(x_0 = -\frac{z}{a}, 0\Big),\\
&z \in [-kl, -(k-1)l],\ k = 1, 2, \ldots,
\end{aligned} \quad (23)$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\, g^{(2,k)}(z) = \frac{1}{a}\,\mathrm{d}\,\mu^{(2)}\Big(\frac{z-l}{a}\Big) + \mathrm{d}\,g^{(1,k-1)}(2l - z) -\\
&- \frac{1}{a}\frac{\partial}{\partial x_0}\, v_p\Big(x_0 = \frac{z-l}{a}, l\Big),\\
&\frac{\mathrm{d}^2}{\mathrm{d}z^2}\, g^{(2,k)}(z) = \frac{1}{a^2}\,\mathrm{d}^2\,\mu^{(2)}\Big(\frac{z-l}{a}\Big) - \mathrm{d}^2 g^{(1,k-1)}(2l - z) -\\
&- \frac{1}{a^2}\frac{\partial^2}{\partial x_0^2}\, v_p\Big(x_0 = \frac{z-l}{a}, l\Big),\\
&z \in [kl, (k+1)l],\ k = 1, 2, \ldots.
\end{aligned} \quad (24)$$

So that the function $g^{(1)}$, defined by the functions $g^{(1,k)}$, ($k = 0, 1, 2, \ldots$), is from $C^2((-\infty, l])$, and the function $g^{(2)}$, composed of $g^{(2,k)}$, ($k = 0, 1, 2, \ldots$), is from $C^2([0, \infty))$, if the equalities are valid

$$\begin{aligned}
&g^{(1,k+1)}(-kl) = g^{(1,k)}(-kl),\ \mathrm{d}\,g^{(1,k+1)}(-kl) = \mathrm{d}\,g^{(1,k)}(-kl),\\
&\mathrm{d}^2 g^{(1,k+1)}(-kl) = \mathrm{d}^2 g^{(1,k)}(-kl),\ k = 0, 1, 2, 3, \ldots,
\end{aligned} \quad (25)$$

$$\begin{aligned}
&g^{(2,k)}(kl) = g^{(2,k-1)}(kl),\ \mathrm{d}\,g^{(1,k)}(kl) = \mathrm{d}\,g^{(1,k-1)}(kl),\\
&\mathrm{d}^2 g^{(1,k)}(kl) = \mathrm{d}^2 g^{(1,k-1)}(kl),\ k = 1, 2, 3, \ldots.
\end{aligned} \quad (26)$$

In the ratios (16), (21), (22) we assume that $f \in C^1(\overline{Q})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, $\mu^{(j)} \in C^2([0,\infty))$, $(j = 1,2)$.

If we analyze the relationship (16) – (26), we can conclude the following.

Lemma 1. *Let $f \in C^1(\overline{Q})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, $\mu^{(j)} \in C^2([0,\infty))$, $(j = 1,2)$. Equalities* (25) *and* (26) *are fulfilled if and only if equalities* (25) *are fulfilled only for $k = 0$, and* (26) *are fulfilled for $k = 1$. In addition, equality* (25) *for $k = 0$, and equality* (26) *for $k = 1$ are valid if and only if homogeneous compatibility conditions are fulfilled* (4), (5), *i.e.*

$$\begin{aligned} &\varphi(0) - \mu^{(1)}(0) = 0,\ \mathrm{d}\,\mu^{(1)}(0) - \psi(0) = 0, \\ &a^2\,\mathrm{d}^2\,\varphi(0) - \mathrm{d}^2\,\mu^{(1)}(0) + f(0,0) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &\mu^{(2)}(0) - \varphi(l) = 0,\ \mathrm{d}\,\mu^{(2)}(0) - \psi(l) = 0, \\ &\mathrm{d}^2\mu^{(2)}(0) - a^2\,\mathrm{d}^2\psi(l) - f(0,l) = 0. \end{aligned} \tag{28}$$

Proof. From relations (16), (18) – (22) using mathematical induction method, it can be concluded that all equalities (25), (26) are satisfied if and only if some of them are valid for one number k, only for example, as stated in the lemma, the first for $k = 0$, and the second ones for $k = 1$.

And now we will check under what conditions the values of the functions and their derivative relations (16) – (24) for $k = 0$. We write equality (25) for $k = 0$ through the specified problem functions (1) – (3):

$$\begin{aligned} &g^{(1,1)}(0) = \mu^{(1)}(0) - g^{(2)}(0) - v_p(0,0) = \\ &= \mu^{(1)}(0) - \frac{1}{2}\varphi(0) - C = g^{(1,0)}(0) = \frac{1}{2}\varphi(0) - C, \\ &\mathrm{d}\,g^{(1,1)}(0) = -\frac{1}{a}\,\mathrm{d}\,\mu^{(1)}(0) - \,\mathrm{d}\,g^{(2)}(0) + \frac{1}{a}\frac{\partial}{\partial x_0}v_p(x_0 = 0,0) = \\ &= -\frac{1}{a}\,\mathrm{d}\,\mu^{(1)}(0) + \frac{1}{2}\,\mathrm{d}\,\varphi(0) + \frac{1}{2a}\psi(0) = \\ &= \mathrm{d}\,g^{(1,0)}(0) = \frac{1}{2}\varphi(0) - \frac{1}{2a}\psi(0), \\ &\mathrm{d}^2\,g^{(1,1)}(0) = \frac{1}{a^2}\,\mathrm{d}^2\,\mu^{(1)}(0) - \mathrm{d}^2\,g^{(2,0)}(0) - \frac{1}{a^2}\frac{\partial^2}{\partial x_0^2}v_p(x_0 = 0,0) = \\ &= \frac{1}{a^2}\,\mathrm{d}^2\,\mu^{(1)}(0) - \frac{1}{2}\,\mathrm{d}^2\,\varphi(0) - \frac{1}{2a}\,\mathrm{d}\,\psi(0) - \frac{1}{a^2}f(0,0) = \\ &= \mathrm{d}^2\,g^{(1,0)}(0) = \frac{1}{2}\,\mathrm{d}^2\varphi(0) - \frac{1}{2a}\,\mathrm{d}\,\psi(0). \end{aligned} \tag{29}$$

Equations (27) follow from the relations (29). Conversely, if the conditions (27) are satisfied, then the equalities (29) are fulfilled, and accordingly – and equalities (25) for $k = 0$.

In the detailed record, we represent the equalities (26) for $k = 1$. As a result, we obtain the ratio:

$$
\begin{aligned}
& g^{(2,1)}(l) = \mu^{(2)}(0) - g^{(1,0)}(l) - v_p(0,l) = \\
& = \mu^{(2)}(0) - \frac{1}{2}\varphi(l) + \frac{1}{2a}\int_0^l \psi(\xi)d\xi + C = \\
& = g^{(2,0)}(l) = \frac{1}{2}\varphi(l) + \frac{1}{2a}\int_0^l \psi(\xi)d\xi + C, \\
& \mathrm{d}\, g^{(2,1)}(l) = \frac{1}{a}\,\mathrm{d}\,\mu^{(2)}(0) + \mathrm{d}\, g^{(1,0)}(l) - \frac{1}{a}\frac{\partial}{\partial x_0} v_p(x_0=0,l) = \\
& = \frac{1}{a}\,\mathrm{d}\,\mu^{(2)}(0) + \frac{1}{2}\,\mathrm{d}\,\varphi(0) - \frac{1}{2a}\psi(l) = \mathrm{d}\, g^{(2,0)}(l) = \frac{1}{2}\,\mathrm{d}\,\varphi(0) + \frac{1}{2a}\psi(l), \\
& \mathrm{d}^2 g^{(2,1)}(l) = \frac{1}{a^2}\,\mathrm{d}^2\mu^{(2)}(0) - \mathrm{d}^2\, g^{(1,0)}(l) - \frac{1}{a^2}\frac{\partial^2}{\partial x_0^2} v_p(x_0=0,l) = \\
& = \frac{1}{a^2}\,\mathrm{d}^2\mu^{(2)}(0) - \frac{1}{2}\,\mathrm{d}^2\varphi(0) + \frac{1}{2a}\,\mathrm{d}\,\psi(l) - \frac{1}{a^2} f(0,l) = \\
& = \mathrm{d}^2 g^{(2,0)}(l) = \frac{1}{2}\,\mathrm{d}^2\varphi(l) + \frac{1}{2a}\,\mathrm{d}\,\psi(l).
\end{aligned}
\tag{30}
$$

It is clear from equalities (30) that equalities (26) for $k=1$ are fulfilled if and only if homogeneous compatibility conditions are fulfilled (28).

Constructed functions $g^{(j,k)} : D(g^{(j,k)}) = \Big\{ z|\, z \in [(-1)^j kl, (-1)^j kl + l] \Big\} \ni z \to g^{(j,k)}(z)$ define functions $g^{(j)}$ according to the formulas

$$
g^{(j)}(z) = g^{(j,k)}(z),\ z \in \mathcal{D}(g^{(j,k)}(z)),\ j=1,2;\ k=0,1,2,\dots. \tag{31}
$$

Lemma 2. *If the functions $f \in C^1(\overline{Q})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, $\mu^{(j)} \in C^2([0,\infty))$, $(j=1,2)$, and homogeneous compatibility conditions* (27) *and* (28) *are fulfilled, then the functions defined by the formulas* (31), (16), (21), (22) *have the representations*

$$
g^{(j)}(z) = \widetilde{g}^{(j)}(z) + (-1)^j C,\ j=1,2, \tag{32}
$$

where C is an arbitrary constant from $\mathbb{R}$. In addition, functions $\widetilde{g}^{(j)}$ are uniquely defined and $\widetilde{g}^{(1)} \in C^2((-\infty,l])$, $\widetilde{g}^{(2)} \in C^2([0,\infty))$.

Proof. The conditions of lemma 2 and the formulas (16), (21), (22) imply that the functions $g^{(i,k)}$ are to the corresponding classes $C^2(\mathcal{D}(g^{(i,k)}))$. According to the definition of functions $g^{(j)}$ by the relations (31) from lemma 1, equalities (25), (26) and homogeneous compatibility conditions (27) and (28) it follows $g^{(1)} \in C^2((-\infty,l])$, $g^{(2)} \in C^2([0,\infty))$.

Further, starting with the formulas (16) the functions for each $k=1,2,3,\dots$ are determined by relations (21), (22). Using mathematical induction, it is easy to show, that

$$
g^{(j,k)}(z) = \widetilde{g}^{(j,k)}(z) + (-1)^j C, \tag{33}
$$

where $\widetilde{g}^{(j,k)}$ for each pair of indices are determined uniquely, the arbitrary constant C in all relations (33) is the same. Thus, representations (33), definitions of functions $g^{(j)}$ by equalities (31) prove the representation (32). From this and the fact that $g^{(1)} \in C^2((-\infty,l])$, $g^{(2)} \in C^2([0,\infty))$ the second statement of lemma 2 follows.

Theorem 2. *If functions $f \in C^1(\bar{Q})$, $\varphi \in C^2([0,l]), \psi \in C^1([0,l]), \mu^{(j)} \in C^2([0,\infty]), j = 1,2$, then the function of the form* (14) *is the only classical solution from the class $C^2(\bar{Q})$ of the problem* (1) - (3) *if and only if homogeneous compatibility conditions* (27) *and* (28) *are fulfilled for the given functions of the problem, where the functions $g^{(j)}(j = 1,2)$ are determined by the relations* (31), (16), (21), (22).

Proof of the theorem 2 follows from the previous arguments, theorem 1, lemmas 1, 2.

5 Non-uniform compatibility conditions (4) - (5)

Let's suppose now, that the homogeneous compatibility conditions (27), (28) are partially or completely not fulfilled.

In other words, we have non-uniform compatibility conditions defined by the formulas (4), (5).

As in the previous case of homogeneous compatibility conditions, we start the study with the general solution (14) of the equation (1).

Repeating the previous arguments for functions $g^{(j)}$ $(j = 1,2)$, we obtain the formulas (16) – (22). Since the function v_p, defined by the relations (7), (8), according to theorem 1 is to the class $C^2(\overline{Q})$, if $f \in C^1(\overline{Q})$, for the functions $g^{(j,k)}(j = 1,2; k = 0,1)$ we obtain the following values through the compatibility conditions (4), (5) at the common points of their contact $z = 0$, $z = l$:

$$
\begin{aligned}
g^{(1,0)}(0) - g^{(1,1)}(0) &= \varphi(0) - \mu^{(1)}(0) = \delta^{(1)}, \\
g^{(2,1)}(l) - g^{(2,0)}(l) &= \mu^{(2)}(0) - \varphi(l) = \sigma^{(1)}, \\
dg^{(1,0)}(0) - dg^{(1,1)}(0) &= \frac{1}{a}\left(d\mu^{(1)}(0) - \psi(0)\right) = \delta^{(2)}, \\
dg^{(2,1)}(l) - dg^{(2,0)}(l) &= \frac{1}{a}\left(d\mu^{(2)}(0) - \psi(l)\right) = \sigma^{(2)}, \\
d^2g^{(1,0)}(0) - d^2g^{(1,1)}(0) &= \frac{1}{a^2}\left(a^2d^2\varphi(0) - d^2\mu^{(1)}(0) + f(0,0)\right) = \delta^{(3)}, \\
d^2g^{(2,1)}(l) - d^2g^{(2,0)}(l) &= \frac{1}{a^2}\left(d^2\mu^{(2)}(0) - a^2d^2\varphi(l) - f(0,l)\right) = \sigma^{(3)}.
\end{aligned}
\tag{34}
$$

Further, using the mathematical induction method, in the general case the relations are proved:

$$
d^pg^{(1,k-1)}(l-kl) - d^pg^{(1,k)}(l-kl) = \begin{cases} \delta^{(p+1)}, & k = 1,3,5,\dots, \\ (-1)^p\sigma^{(p+1)}, & k = 2,4,6,\dots, \end{cases} \tag{35}
$$

$$
d^pg^{(2,k)}(kl) - d^pg^{(2,k-1)}(kl) = \begin{cases} \sigma^{(p+1)}, & k = 1,3,5,\dots, \\ (-1)^p\delta^{(p+1)}, & k = 2,4,6,\dots, \end{cases} \tag{36}
$$

for $p = 0,1,2$.

Thus, the presence of non-uniform compatibility conditions violates the continuity of functions $g^{(j)}$ or their derivatives, or altogether. This conclusion can be formulated as the following statement.

Statement 1. *If homogeneous compatibility conditions* (27), (28) *are not fulfilled for given functions* $f, \varphi, \psi, \mu^{(j)}$ $(j = 1, 2)$, *then no matter how smooth these functions are, the problem* (1) – (3) *has no classical solution defined in* $\overline{Q} = [0, \infty) \times [0, l]$.

Let the given functions of the equation (1), the boundary conditions (2), (3) be sufficiently smooth and such as in theorem 2: $f \in C^1(\overline{Q})$, $\varphi \in C^2[0, l]$, $\psi \in C^1[0, l]$, $\mu^{(j)} \in C^2[0, \infty)$, $j = 1, 2$. Under given assumptions, the given functions are defined $g^{(j)} (j = 1, 2)$. Since the compatibility conditions (4), (5) are non-uniform, for $\delta^{(p+1)}$ and $\sigma^{(p+1)}$, $p = 0, 1, 2$, not partially equal to zero or completely, we obtain discontinuous functions $g^{(j)}$ or their derivatives according to the expressions (35) – (36).

Let us consider first, according to the formula (14) the partially defined solutions

$$u^{(k,m)}(\mathbf{x}) = g^{(1,k)}(x_1 - ax_0) + g^{(2,m)}(x_1 + ax_0) + v_p(\mathbf{x})$$

for $\mathbf{x} \in Q$, where $z = (x_1 - ax_0) \in (-kl, -(k-1)l)$, $\tilde{z} = (x_1 + ax_0) \in (ml, (m+1)l)$. Let us introduce the notation of new functions $r^{(1,k)} \colon \mathbb{R}^2 \supset Q \supset D(r^{(1,k)}) \ni \mathbf{x} \to r^{(1,k)}(\mathbf{x}) = g^{(1,k)}(x_1 - ax_0) \in \mathbb{R}$, $r^{(2,m)} \colon \mathbb{R}^2 \supset Q \supset D(r^{(2,m)}) \ni \mathbf{x} \to r^{(2,m)}(\mathbf{x}) = g^{(2,m)}(x_1 + ax_0) \in \mathbb{R}$. Thus,

$$u^{(k,m)}(\mathbf{x}) - v_p(\mathbf{x}) = r^{(1,k)}(\mathbf{x}) + r^{(2,m)}(\mathbf{x}), \tag{37}$$

where the domain $D(u^{(k,m)})$ of the function $u^{(k,m)}$ is the intersection of the domains $D(r^{(1,k)})$ and $D(r^{(2,m)})$, i.e. $D(u^{(k,m)}) = D(r^{(1,k)}) \cap D(r^{(2,m)})$. Since the values of an arguments $\mathbf{x}$ are within the domain Q, for most indices k and m $D(u^{(k,m)}) = \varnothing$, $\varnothing$ is an empty set. We are interested in functions $u^{(k,m)}$ for which $D(u^{(k,m)}) \neq \varnothing$, $k, m \in \{0, 1, \ldots\}$.

If we analyze the domains of definition of functions $g^{(j,k)}$, $j = 1, 2$, $k = 0, 1, 2, \ldots$, then in (37) $D(u^{(k,m)}) = \varnothing$ under the condition that $|k - m| > 1$. This means the following: $D(u^{(k,m)}) \neq \varnothing$ for $k = 1, 2, \ldots$, if, accordingly $m = k-1, k, k+1$, and $D(u^{(0,m)}) \neq \varnothing$, if $m = 0, 1$.

We denote through $Q^{(k,m)} = D(u^{(k,m)})$ subdomains the domain Q, $k = 0, 1, 2, \ldots$, $m = k-1, k, k+1$, $Q^{(0,-1)} = \varnothing$. Note that the boundary points $z = x_1 - ax_0 = -kl$, $z = x_1 - ax_0 = -(k-1)l$ of the functions $g^{(1,k)}$ determine the characteristics of the equation (1) in the domain Q, $k = 1, 2, \ldots$, of one family, and the boundary points $z = x_1 + ax_0 = ml$, $z = x_1 + ax_0 = (m+1)l$, $m = 1, 2, \ldots$, in Q define the characteristics of another family. These non-intersecting subdomains $Q^{(k,m)}$ of the domain Q in the Cartesian coordinate system of the plane $\mathbb{R}^2$ of variables x_0, x_1 are described by the following relations:

$$Q^{(0,-1)} = \varnothing,$$

$$Q^{(0,0)} = \left\{ \mathbf{x} \in Q \mid x_1 \in \left(0, \frac{l}{2}\right], 0 < x_0 < \frac{x_1}{a} \right\} \cup$$
$$\cup \left\{ \mathbf{x} \in Q \mid x_1 \in \left[\frac{l}{2}, l\right), 0 < x_0 < \frac{l - x_1}{a} \right\},$$

$$Q^{(k,k-1)} = \left\{ \mathbf{x} \in Q \mid x_1 \in \left(0, \frac{l}{2}\right], \frac{x_1 + (k-1)l}{a} < x_0 < \frac{kl - x_1}{a} \right\},\ k = 1, 2, \dots,$$

$$Q^{(k-1,k)} = \left\{ \mathbf{x} \in Q \mid x_1 \in \left(\frac{l}{2}, l\right), \frac{kl - x_1}{a} < x_0 < \frac{x_1 + (k-1)l}{a} \right\},\ k = 1, 2, \dots,$$

$$Q^{(k,k)} = \left\{ \mathbf{x} \in Q \mid x_1 \in \left(0, \frac{l}{2}\right], \frac{kl - x_1}{a} < x_0 < \frac{x_1 + kl}{a} \right\} \cup$$

$$\cup \left\{ \mathbf{x} \in Q \mid x_1 \in [\frac{l}{2}, l), \frac{x_1 + (k-1)l}{a} < x_0 < \frac{(k+1)l - x_1}{a} \right\},\ k = 1, 2, 3, \dots.$$

For clarity, the subdomains $Q^{(k,m)}$ are shown in the figure 2 in the Cartesian coordinate system of variables x_0, x_1.

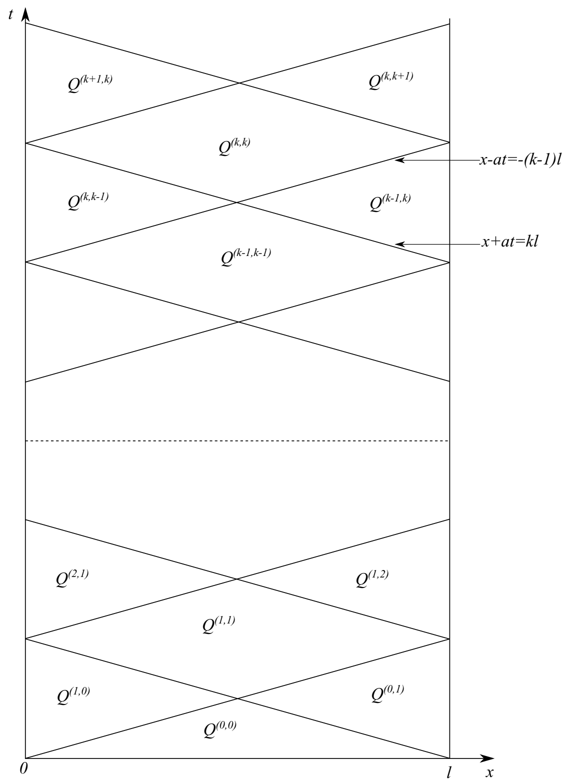

Figure 2: Dividing the domain Q into subdomains $Q^{(k,m)}$.

We denote by $\widetilde{Q}$ way of unification $Q^{(k,m)}$, namely: $\widetilde{Q} = \bigcup\limits_{k=0}^{\infty} \bigcup\limits_{m=k-1}^{k+1} Q^{(k,m)}$. Clearly $\widetilde{Q} \subset Q$.

Along with the subdomains $Q^{(k,m)}$ of the domain Q , we denote $\mathcal{M}^{(j,k)}$ intervals the characteristics of the equation

$$\partial_{x_0}^2 u(\mathbf{x}) - a^2 \partial_{x_1}^2 u(\mathbf{x}) = f\left(\mathbf{x}\right),$$

is Q and passing through the points $\left(k\frac{l}{a}, 0\right)$, $k = 0, 1, \ldots$, $j = 1, 2$. Notations $\mathcal{M}^{(j,k)}$ can be viewed as a subset of the domain Q and $\mathcal{M}^{(j,k)} = \{\mathbf{x} \in Q \mid x_1 = (-1)^{j-1}ax_0 + (k-2+j)l, x_1 \in (0,l), k = 1, 2, \ldots, j = 1, 2\}$.

Thus $Q = \widetilde{Q} \cup (\bigcup\limits_{k=1}^{\infty} \bigcup\limits_{j=1}^{2} \mathcal{M}^{(j,k)})$, $\overline{Q} = \overline{\widetilde{Q}}$.

Let us denote $\tilde{u}$ the function defined on the set $\widetilde{Q}$ in the following way:

$$\begin{aligned} \tilde{u}(\mathbf{x}) = u^{(k,m)}(\mathbf{x}) = g^{(1,k)}(x_1 - ax_0) + g^{(2,m)}(x_1 + ax_0) + v_p(\mathbf{x}) = \\ = r^{(1,k)}(\mathbf{x}) + r^{(2,m)}(\mathbf{x}) + v_p(\mathbf{x}), \end{aligned} \tag{38}$$

where $k = 0, 1, 2, \ldots$, $m = k-1, k, k+1$; $u^{(2,-1)}(t,x) \equiv 0$.

The following theorem is valid.

Theorem 3. *Let the functions be the following* $f \in C^1(\overline{Q})$, $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$, $\mu^{(j)} \in C^2[0,\infty)$, $j = 1, 2$, *and* $\sum\limits_{p=1}^{3} \left[\left(\delta^{(p)}\right)^2 + \left(\sigma^{(p)}\right)^2\right] \neq 0$. *Then the function* $\tilde{u}$ *from the class* $C^2(\widetilde{Q})$ *is the only solution to the problem* (1) – (3) *if and only if the compatibility conditions* (4), (5) *are fulfilled .*

Proof of theorem 3 follows from the previous arguments.

Numerous characteristics $\mathcal{M}^{(j,k)}$ can be divided into two classes:

$$\mathcal{M}(\sigma) = (\bigcup_{s=1}^{\infty} \mathcal{M}^{(1,2s-1)}) \cup (\bigcup_{s=1}^{\infty} \mathcal{M}^{(2,2s)}),$$

$$\mathcal{M}(\delta) = (\bigcup_{s=1}^{\infty} \mathcal{M}^{(1,2s)}) \cup (\bigcup_{s=1}^{\infty} \mathcal{M}^{(2,2s-1)}).$$

Corollary 1. *If* $\delta^{(1)} \neq 0$ *and* $\sigma^{(1)} \neq 0$, *then in terms of the relations* (35), *(36) the function* $\tilde{u}$ *for each* $k = 1, 2, \ldots$ *and* $j = 1, 2$ *on* $\mathcal{M}^{(j,k)}$ *is discontinued when passing through the given characteristic interval* $x_1 = (-1)^{j-1}x_0 + (k-2+j)l$. *In addition, on the set* $\mathcal{M}(\delta)$ *for each of theirs points, the gap is equal to the same number* $\delta^{(1)}$, *on* $\mathcal{M}(\sigma)$ *and the gap is equal to* $\sigma^{(1)}$. *The derivatives of the first order function* $\tilde{u}$ *on sets* $\mathcal{M}(\delta)$ *and* $\mathcal{M}(\sigma)$ *suffer the gap equal to* $\delta^{(2)}$ *and* $\sigma^{(2)}$ *respectively. Similarly, second-order derivatives of solution* $\tilde{u}$ *on these sets suffer gaps equal to the numbers* $\delta^{(3)}$ *and* $\sigma^{(3)}$ *respectively.*

Theorem 4. *Let the functions be the following* $f \in C^1(\overline{Q})$, $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$, $\mu^{(j)} \in C^2[0,\infty)$, $j = 1, 2$, $\delta^{(1)} = \sigma^{(1)} = 0$. *Then a function* $\tilde{u}$ *from the domain* $C(\bar{Q}) \cap C^2(\tilde{Q})$ *is the only solution to the problem* (1) – (3) *if and only if the compatibility conditions* (4), (5) *are fulfilled.*

Proof follows in fact from the theorem 3 and corollary 1. Indeed, if $\delta^{(1)} = \sigma^{(1)} = 0$ then the solution $\tilde{u}$ on the sets $\mathcal{M}(\delta)$ and $\mathcal{M}(\sigma)$ is continuous. Consequently, besides the solution $\tilde{u} \in C^2(\tilde{Q})$, it is a continuous function in the domaim $\overline{Q}$, $\tilde{u} \in C(\overline{Q})$.

Theorem 5. *Let the conditions of theorems 3 and 4 be fulfilled and, moreover* $\delta^{(2)} = \sigma^{(2)} = 0$. *Then the solution* $\tilde{u}$ *of the problem* (1) – (3) *is to the class* $C^1(\overline{Q}) \cap C^2(\tilde{Q})$ *and is unique if and only if the compatibility conditions* (4), (5) *are fulfilled.*

Proof follows easily from theorems 3, 4 and corollary 1, since in this case $\tilde{u}$ it is not only continuous on $\mathcal{M}(\delta) \cup \mathcal{M}(\sigma)$, but by virtue (35) has of continuous derivatives of first order.

Remark 1. *If $\delta^{(p)} = 0$, $p = 1, 2, 3$, then the solution $\tilde{u}$, defined by the formula* (38)*, of the problem* (1) – (5) *can have gaps together with its derivatives of the first and second orders only on $\mathcal{M}(\sigma)$. In this case $\tilde{u}$ is from the domain $C^2(\tilde{Q} \cup \mathcal{M}(\delta))$. We can formulate analogous theorems, where in theorems 3 - 4 the set $C^2(\tilde{Q})$ should be replaced by the set $C^2(\tilde{Q} \cup \mathcal{M}(\delta))$.*

If $\sigma^{(p)} = 0$, $p = 1, 2, 3$, then a similar remark is valid, where $C^2(\tilde{Q} \cup \mathcal{M}(\delta))$ in this case instead $C^2(\tilde{Q} \cup \mathcal{M}(\sigma))$.

Remark 2. The problem (1) – (3) with conjunction conditions on characteristics can be formulated as follows.

We need find the classical solution of the equation (1), with the Cauchy conditions (2), the boundary conditions (3) and conjugation conditions

$$\left[\left(\partial^p_{x_1}\tilde{u}\right)^+ - \left(\partial^p_{x_1}\tilde{u}\right)^-\right](x_0, x_1 = ax_0 - (k-1)l) = \\ = \begin{cases} \delta^{(p+1)}, & k = 3, 5, \ldots, \\ (-1)^p \sigma^{(p+1)}, & k = 2, 4, 6, \ldots, \end{cases} \quad ax_0 \in [kl, (k-1)l], \tag{39}$$

$$\left[\left(\partial^p_{x_1}u\right)^+ - \left(\partial^p_{x_1}u\right)^-\right](x_0, x_1 = -ax_0 + kl) = \begin{cases} \sigma^{(p+1)}, & k = 1, 3, 5, \ldots, \\ (-1)^p \delta^{(p+1)}, & k = 2, 4, 6, \ldots, \end{cases} \tag{40}$$

where the numbers $\delta^{(j)}$ and $\sigma^{(j)}$, $j = 1, 2, 3$, from the conditions (4) – (5), $()^{\pm}$ are the limiting values of the function u and its derivatives $\partial_{x_1}u$, $\partial^2_{x_1}u$ from different sides on the characteristics $x_1 - ax_0 = -(k-1)l$ and $x_1 + ax_0 = kl$, i.e.

$$\left(\partial^p_{x_1}\tilde{u}\right)^{\pm}(x_0, x_1 = ax_0 - (k-1)l) = \lim_{\Delta x_1 > 0, \Delta x_1 \longrightarrow 0} \left(\partial^p_{x_1}\tilde{u}\right)(x_0, x_1 \pm \Delta x_1 = ax_0 - (k-1)l),$$

$$\left(\partial^p_{x_1}u\right)^{\pm}(x_0, x_1 = -ax_0 + kl) = \lim_{\Delta x_1 > 0, \Delta x_1 \longrightarrow 0} \left(\partial^p_{x_1}\right)(x_0, x_1 \pm \Delta x_1 = -ax_0 + kl).$$

Note that such the formulation of the considered problem with conjugation conditions is more acceptable for its numerical implementation.

6 The method of characteristic parallelogram. The formula for solving the first mixed problem for a one-dimensional wave equation

In the domain Q we consider rectangles $Q^{(m)} = \left((m-1)\frac{l}{a}, m\frac{l}{a}\right)$, $m = 1, 2, ...$ (see. fig. 1). It is obvious that, $\bar{Q} = \bigcup\limits_m \bar{Q}^{(m)}$.

Let the point $\boldsymbol{x} = A = A^{(m)}$ with coordinates (x_0, x_1) is to $\bar{Q}^{(m)}$, $m \geqslant 2$. Through A the conduct of the characteristic $y_0 = \frac{y_1}{a} + x_0 - \frac{x_1}{a}$, $y_0 = -\frac{y_1}{a} + x_0 - \frac{x_1}{a}$ to the intersection with the side parts of the boundary ∂Q of the area Q at the points B and C respectively (fig. 3). The points B and C have coordinates $B = (x_0 - \frac{x_1}{a}, 0), C = (x_0 + \frac{x_1}{a} - \frac{l}{a}, l)$, where x_0 and x_1 - coordinates $A = A^{(m)}$. Through B and C we make new characteristics that intersect at the point $A^{(m-1)} = (x_0 - \frac{l}{a}, l - x_1)$. We get the characteristic parallelogram $A^{(m)}BCA^{(m-1)}$.

According to the formulas 14, (21) and (22) the solution of the problem (1) - (3) at the point $\boldsymbol{x} = A^{(m)}$ is written in the following way:

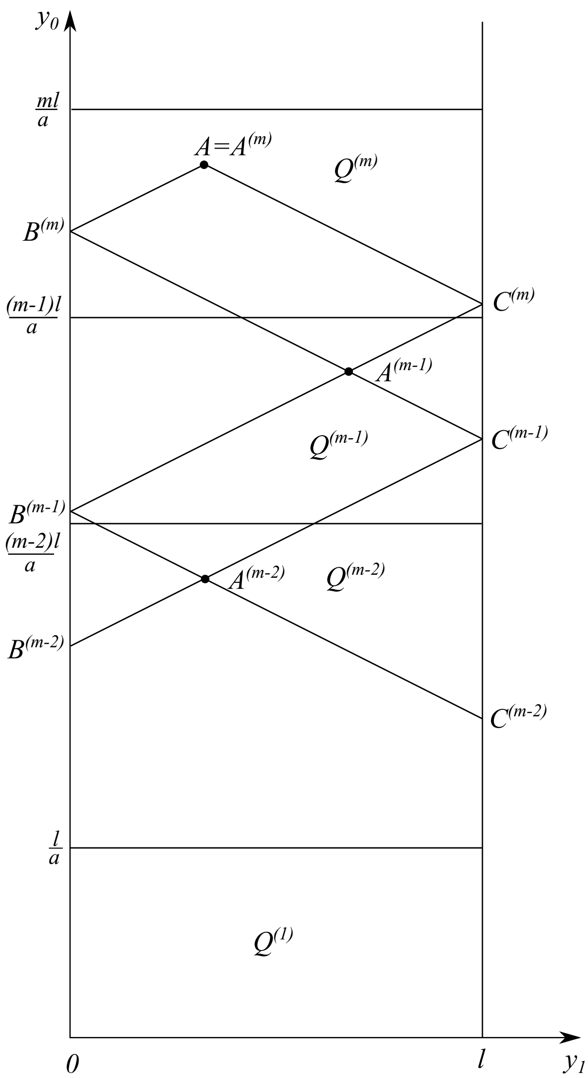

Figure 3:

$$
\begin{aligned}
u(\boldsymbol{x}) = u(A^{(m)}) = \mu^{(1)}(x_0 - \frac{x_1}{a}) + \mu^{(2)}(x_0 + \frac{x_1}{a} - \frac{l}{a}) +& \\
+ g^{(1,m-1)}(2l - x_1 - ax_0)g^{(2,m-1)}(ax_0 - x_1) +& \\
+ v_p(\boldsymbol{x}) - v_p(x_0 - \frac{x_1}{a}, 0) - v_p(x_0 + \frac{x_1}{a} - \frac{l}{a}, l).&
\end{aligned} \tag{41}
$$

If we compare the obtained formula (41) with the coordinates of the vertices of the characteristic parallelogram $A^{(m)}BCA^{(m-1)}$, then we can write it in the following form

$$\begin{gathered} u(\boldsymbol{x}) = u(A^{(m)}) = u(B) + u(C) + v_p(A^{(m)}) - \\ -v_p(B) - v_p(C) + v_p(A^{(m-1)}) - u(A^{(m-1)}). \end{gathered} \tag{42}$$

This process can be continued further. The point $A^{(m-1)}$ is the point from the area $Q^{(m-1)}$. If $Q^{(m-1)}$ it is not $Q^{(1)}$, through $A^{(m-1)}$ and $A^{(m-2)} = (x_0 - \frac{2l}{a}, x_1)$ then we draw characteristics again and build a characteristic parallelogram $A^{(m-1)}B^{(m-1)}C^{(m-1)}A^{(m-2)}$, where $B^{(m-1)} = (x_0 + \frac{x_1}{a} - \frac{2l}{a}, 0)$, $C^{(m-1)} = (x_0 - \frac{x_1}{a} - \frac{l}{a}, l)$. Similarly to the formula (42) we write the values of the solution of the problem (1) - (3) at the point $A^{(m-1)} = (x_0 - \frac{l}{a}, l - x_1)$ through the values at the points $B^{(m-1)}$, $C^{(m-1)}$, $A^{(m-2)}$. We continue this construction until we reach the value of the solution of the problem at the point of the set $\bar{Q}^{(1)}$.

Consider the rectangle $\bar{Q}^{(1)}$. It is divided into four parts $\Omega^{(1,k)}, k = 1,2,3,4$. Subdomains $\Omega^{(1,k)}$ can be analytically described in the following way:

$$\Omega^{(1,1)} = \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left(0, \frac{l}{2}\right], \quad 0 < y_0 < \frac{y_1}{a} \right\} \bigcup \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left[\frac{l}{2}, l\right), \right.$$
$$\left. 0 < y_0 < \frac{l - y_1}{a} \right\},$$
$$\Omega^{(1,2)} = \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left(0, \frac{l}{2}\right), \quad \frac{y_1}{a} < y_0 < \frac{l - y_1}{a} \right\},$$
$$\Omega^{(1,3)} = \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left(\frac{l}{2}, l\right), \quad \frac{l - y_1}{a} < y_0 < \frac{y_1}{a} \right\},$$
$$\Omega^{(1,4)} = \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left(0, \frac{l}{2}\right], \quad \frac{l - y_1}{a} < y_0 < l \right\} \bigcup \left\{ \boldsymbol{y} \in Q^{(1)} : y_1 \in \left[\frac{l}{2}, l\right), \right.$$
$$\left. \frac{y_1}{a} < y_0 < l \right\}.$$

If we consider the circuits, then $\bar{Q}^{(1)} = \bigcup\limits_{k=1}^{4} \bar{\Omega}^{(1,k)}$.

For $\boldsymbol{z} \in \bar{Q}^{(1)}$ the completely characteristic parallelogram it does not work, but only its part, where $\boldsymbol{z}$ - is one of its vertices. The configuration of the parts of each characteristic parallelogram differs from each other depending on the fact that the point $\boldsymbol{z}$ has a set $\bar{Q}^{(1,k)}$ for each $k = 1,2,3,4$. Therefore, the formulas for solving the problem (1) - (3) will also be different. Let us consider each case $\boldsymbol{z} \in \bar{\Omega}^{(1,k)}, k = 1,2,3,4$.

1. $\boldsymbol{z} \in \bar{\Omega}^{(1,1)}$. According to the formulas (14) and (16) the solution value of the problem (1) - (3) at the point $\boldsymbol{z} \in \bar{\Omega}^{(1,1)}$ is represented as

$$u(\boldsymbol{z}) = u(A^{(1)}) = g^{(1,0)}(z_1 - az_0) + g^{(2,0)}(z_1 + az_0) + v_p(\boldsymbol{z}) =$$
$$= \frac{1}{2}[\varphi(z_1 - az_0) + \varphi(z_1 + az_0)] + \frac{1}{2a}\int\limits_{z_1 - az_0}^{z_1 + az_0} \psi(\xi)d\xi + v_p(\boldsymbol{z}) = \tag{43}$$
$$= g^{(1,0)}(E^{(0)}) + g^{(2,0)}(F^{(0)}) + v_p(\boldsymbol{z}), \quad A^{(1)} \in \bar{\Omega}^{(1,1)}.$$

where $E^{(0)} = (0, z_1 - az_0)$, $F^{(0)} = (0, z_1 + az_0)$ (fig. 4).

2. $\boldsymbol{z} \in \bar{\Omega}^{(1,2)}$. According to the formulas (14), (16) and (21) in this case, the value $u(\boldsymbol{z})$ at the point $\boldsymbol{z} = A^{(1)} \in \bar{\Omega}^{(1,2)}$ is determined by the formula

$$u(\boldsymbol{z}) = u(A^{(1)}) = g^{(1,1)}(z_1 - az_0) + g^{(2,0)}(z_1 + az_0) + v_p(\boldsymbol{z}) =$$
$$= \frac{1}{2}[\varphi(z_1 + az_0) - \varphi(az_0 - z_1)] + \frac{1}{2a}\int\limits_{az_0 - z_1}^{z_1 + az_0} \psi(\xi)d\xi + \mu^{(1)}(z_0 - \frac{z_1}{a}) + \tag{44}$$
$$+ v_p(\boldsymbol{z}) = u(B^{(1)}) + g^{(2,0)}(F^{(0)}) - g^{(2,0)}(E^{(1)}) + v_p(\boldsymbol{z}) - v_p(B^{(1)}),$$
$$\boldsymbol{z} \in \bar{\Omega}^{(1,2)},$$

where $B^{(1)} = (z_0 - \frac{z_1}{a}, 0)$, $E^{(1)} = (0, az_0 - z_1)$ (fig.5).

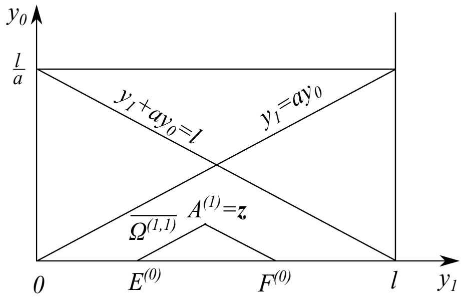

Figure 4:

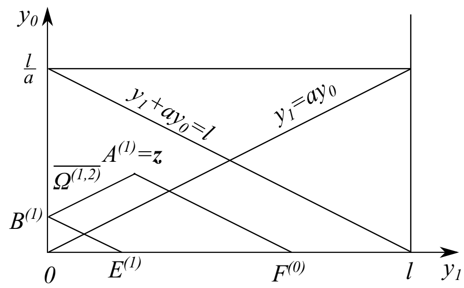

Figure 5:

3. $\boldsymbol{z} \in \bar{\Omega}^{(1,3)}$. Let $\boldsymbol{z} = A^{(1)} \in \bar{\Omega}^{(1,3)}$. According to the formulas (14), (16), (22) the value of the function u at the point $\boldsymbol{z} \in \bar{\Omega}^{(1,3)}$ is determined by the formula

$$\begin{aligned}
&u(\boldsymbol{z}) = u(A^{(1)}) = g^{(1,0)}(z_1 - az_0) + g^{(2,1)}(z_1 + az_0) + v_p(\boldsymbol{z}) = \\
&= \frac{1}{2}[\varphi(z_1 - az_0) - \varphi(2l - z_1 - az_0)] + \frac{1}{2a} \int\limits_{z_1 - az_0}^{2l - z_1 - az_0} \psi(\xi) d\xi + \\
&+ v_p(\boldsymbol{z}) + \mu^{(2)}(z_0 + \frac{z_1}{a} - \frac{l}{a}) - v_p(z_0 + \frac{z_1}{a} - \frac{l}{a}, l) = \\
&= u(C^{(1)}) + g^{(1,0)}(E^{(0)}) - g^{(1,0)}(F^{(1)}) + v_p(\boldsymbol{z}) - v_p(z_0 + \frac{z_1}{a} - \frac{l}{a}, l),
\end{aligned} \tag{45}$$

where $C^{(1)} = (z_0 + \frac{z_1}{a} - \frac{l}{a}, l)$, $E^{(0)} = (0, z_1 - az_0)$, $F^{(1)} = (0, az_0 - z_1 + 2l)$ (fig. 6).

4. $\boldsymbol{z} \in \bar{\Omega}^{(1,4)}$. In this case, according to the formulas (14), (16), (21), (22) , the solution of the problem (1) - (3) is determined by the formula (fig. 7)

$$\begin{aligned}
&u(\boldsymbol{z}) = v_p(\boldsymbol{z}) + g^{(2,1)}(z_1 + az_0) + g^{(1,1)}(z_1 - az_0) = \\
&= -\frac{1}{2}[\varphi(az_0 - z_1) + \varphi(2l - z_1 - az_0)] + \frac{1}{2a} \int\limits_{az_0 - z_1}^{2l - z_1 - az_0} \psi(\xi) d\xi + \\
&+ \mu^{(1)}(z_0 - \frac{z_1}{a}) + \mu^{(2)}(z_0 + \frac{z_1}{a} - \frac{l}{a}) + v_p(\boldsymbol{z}) - v_p(z_0 - \frac{z_1}{a}, 0) - \\
&- v_p(z_0 + \frac{z_1}{a} - \frac{l}{a}, l) = u(B^{(1)}) + u(C^{(1)}) - g^{(2,0)}(E^{(1)}) - \\
&- g^{(1,0)}(F^{(1)}) + v_p(\boldsymbol{z}) - v_p(z_0 - \frac{z_1}{a}, 0) - v_p(z_0 + \frac{z_1}{a} - \frac{l}{a}, l).
\end{aligned} \tag{46}$$

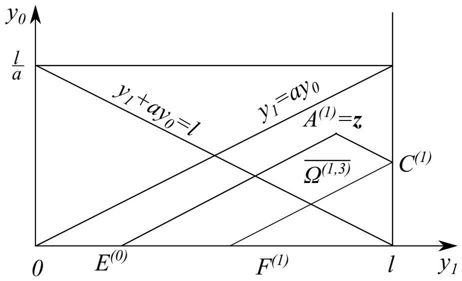

Figure 6:

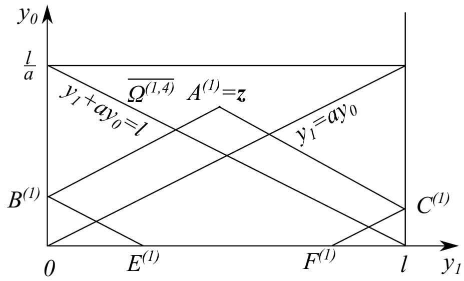

Figure 7:

Thus, to obtain the solution to the problem (1) - (3) using the method of characteristic parallelogram, we find this solution in $\bar{Q}^{(1)}$ using the formulas (43) - (46). In this case, we use the given conditions (2) and (3), as well as the values of the function on the right side of the equation (1), which determines the particular solution v_p by the formulas (8), (7), (10). Then, using the formula (42), we find the values of the solution of the problem (1) - (3) on $\bar{Q}^{(2)}$, on $\bar{Q}^{(3)}$ etc. Moving on $\bar{Q}^{(j)}$, $j = 1, 2, \ldots$, you can find the solution values and in $\bar{Q}^{(m)}$ for any number m.

This process of finding the solution to a problem (1) - (3) can be somewhat generalized. Let $\boldsymbol{x} = A = A^{(m)}$, $B = B^{(m)}$, $C = C^{(m)}$. Considering the constructed parallelograms for each rectangle $\bar{Q}^{(j)}$, we determine the coordinates of all the vertices. If $\boldsymbol{x} = A^{(m)}$, then through the coordinates x_0, x_1, the numbers a and l determine the coordinates of all the other vertices. Namely:

$$
\begin{gathered}
A^{(j)} = \begin{cases} \left(x_0 - \frac{(m-j)l}{a}, x_1\right), & j = m, m-2, m-4, \ldots, \\ \left(x_0 - \frac{(m-j)l}{a}, l - x_1\right), & j = m-1, m-3, \ldots, \end{cases} \\
B^{(j)} = \begin{cases} \left(x_0 - \frac{x_1}{a} - \frac{(m-j)l}{a}, 0\right), & j = m, m-2, m-4, \ldots, \\ \left(x_0 + \frac{x_1}{a} - \frac{(m-j+1)l}{a}, 0\right), & j = m-1, m-3, \ldots, \end{cases} \\
C^{(j)} = \begin{cases} \left(x_0 + \frac{x_1}{a} - \frac{(m-j-1)l}{a}, l\right), & j = m, m-2, m-4, \ldots, \\ \left(x_0 - \frac{x_1}{a} - \frac{(m-j)l}{a}, l\right), & j = m-1, m-3, \ldots. \end{cases}
\end{gathered}
\tag{47}
$$

Note that the sequences (47) end in the numbering $j = 1$. The coordinates of a point $A^{(1)}$ in the chain of sequences (47) depend on parity m. If m is the even number, then $A^{(1)} = \left(x_0 - \frac{m-1}{a}l, l - x_1\right)$, if m is the odd number, then $A^{(1)} = \left(x_0 - \frac{m-1}{a}l, x_1\right)$.

Taking into account (47) the representation of the solution by the formula (42) at the point $\boldsymbol{x} = A^{(m)}$ can be proceeded further. To do this, according to the same formula, we replace the value $u(A^{(m-1)})$ through $u(A^{(m-2)})$ and the specified values u through the boundary conditions (3) $u(B^{(m-1)})$, $u(C^{(m-1)})$, the corresponding values of the particular solution v_p, then $u(A^{(m-2)})$ through $u(A^{(m-3)})$ etc. Continue this procedure until we get in the analytical representation $u(A^{(m)})$ through $u(A^{(1)})$, the boundary terms and the values of the particular solution. But the specific form of this representation depends again on the parity of the number m. Let us consider each case separately and write down the solution formulas.

Let m be an even number, $\boldsymbol{x} \in \bar{Q}^{(m)}$. If we trace in this case the construction of the solution of the problem (1) - (3), then we obtain its value $u(\boldsymbol{x})$ at the point $\boldsymbol{x}$ as a formula

$$u(\boldsymbol{x}) = u(A^{(m)}) = \sum_{j=1}^{m}(-1)^j[u(B^{(j)}) - v_p(B^{(j)})]+$$
$$+\sum_{j=1}^{m}(-1)^j[u(C^{(j)}) - v_p(C^{(j)})] + v_p(\boldsymbol{x}) + v_p(A^{(1)}) - u(A^{(1)}), \quad (48)$$

where $A^{(1)} = \left(x_0 - \frac{m-1}{a}l, l - x_1\right)$. If m is the odd number, then

$$u(\boldsymbol{x}) = u(A^{(m)}) = \sum_{j=1}^{m}(-1)^j[u(B^{(j)}) - v_p(B^{(j)})]+$$
$$+\sum_{j=1}^{m}(-1)^j[u(C^{(j)}) - v_p(C^{(j)})] + v_p(\boldsymbol{x}) - v_p(A^{(1)}) + u(A^{(1)}), \quad (49)$$

where $A^{(1)} = \left(x_0 - \frac{m-1}{a}l, x_1\right)$. The value $u(A^{(1)})$ is determined by one of the formulas (43) - (46) depending on the coordinates of the point $\boldsymbol{x} = (x_0, x_1)$ and the parity of the number m. Note that the values $u(B^{(j)})$ are the values of the function $\mu^{(1)}$ from the first coordinates of points $B^{(j)}$ from (47), and $u(C^{(j)})$ similarly, the values of the function $\mu^{(2)}$ from the first coordinates of points $C^{(j)}$ from (47).

Statement 2. *The formulas* (48) *and* (49) *show that the problem solution* (1) - (3) *value* $u(\boldsymbol{x})$ *is determined for any rectangle* $\bar{Q}^{(m)}(m = 1, 2, \ldots)$ *through the boundary values* (2) *of the given functions* φ *and* ψ, *the boundary values* (3) *of the given functions* $\mu^{(1)}$ *and* $\mu^{(2)}$, *the values of the particular solution* v_p *at the corresponding boundary points* $B^{(j)}$ *and* $C^{(j)}$, *at the point* $\boldsymbol{x}$ *and* $A^{(1)}$, *the value of the searched solution at the point* $A^{(1)} \subset \bar{Q}^{(1)}$ *bypassing its calculations on the sets* $\bar{Q}^{(m-1)}$ *etc* $(m \geqslant 2)$.

Like splitting the rectangle $Q^{(1)} = \left(0, \frac{l}{a}\right) \times \left(0, l\right)$ into subdomains $\Omega^{(1,j)}$, $j = 1, 2, 3, 4$, we divide the rectangle $Q^{(m)} = \left((m-1)\frac{l}{a}, m\frac{l}{a}\right) \times (0, l)$ into characteristics $y_1 - ay_0 = -(m-1)l$, $y_1 + ay_0 = ml$ into corresponding subdomains $\Omega^{(m,j)}$, $j = 1, 2, 3, 4$ (see fig.8). Correspondence here should be understood in the sense that if $\boldsymbol{x} = (x_1, x_0) \in \Omega^{(m,j)}$, $j \in \{1, 2, 3, 4\}$, then the point $\tilde{\boldsymbol{x}}$ with coordinates $\tilde{\boldsymbol{x}} = \left(x_0 - \frac{m-1}{a}l, x_1\right) = (\widetilde{x_0}, x_1)$ is $\Omega^{(1,j)}$ for the same $j \in \{1, 2, 3, 4\}$.

Remark 3. *If the compatibility conditions* (4) *and* (5) *are non-uniform, i.e.* $\sum_{j=1}^{3}[(\delta^{(j)})^2 + (\sigma^{(j)})^2] \neq 0$, *then the solution of the problem* (1) - (3) *or its derivatives suffers the discontinuity of the form* (35) *or* (36) *on some or all of the characteristics* $y_1 - ay_0 = (1-j)l$, $y_1 + ay_0 = jl, j = 1, 2, \ldots, m$. *Therefore, in this case, the solution* *on the characteristics $y_1 - ay_0 = (1-m)l$, $y_1 + ay_0 = ml$ is considered as a limit value on the characteristic from inside the corresponding area $\Omega^{(m,s)}$, $s \in \{1,2,3,4\}$.*

In the formula (48) (m is the even number) the point $A^{(1)}$ has coordinates $A^{(1)} = (x_0 - \frac{m-1}{a}l, l - x_1)$. This means the following. If $\boldsymbol{x} \in \bar{\Omega}^{(m,1)}$, then $A^{(1)} \in \bar{\Omega}^{(1,1)}$, if $\boldsymbol{x} \in \bar{\Omega}^{(m,4)}$, then also $A^{(1)} \in \bar{\Omega}^{(1,4)}$. If $\boldsymbol{x} \in \bar{\Omega}^{(m,2)}$, then $A^{(1)} \in \bar{\Omega}^{(1,3)}$, and vice versa if $\boldsymbol{x} \in \bar{\Omega}^{(m,3)}$, then $A^{(1)} \in \bar{\Omega}^{(1,2)}$.(see fig. 9)

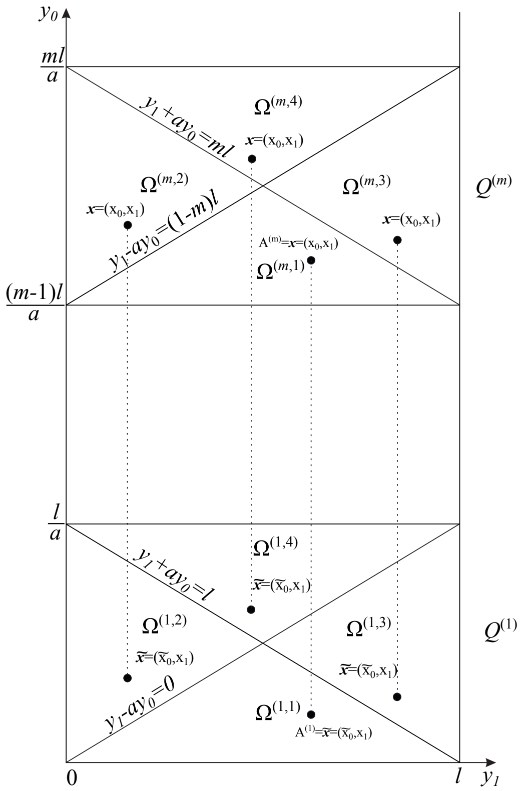

Figure 8: . $\widetilde{x_0} = x_0 - \frac{m-1}{a}l, m-$ is an odd number

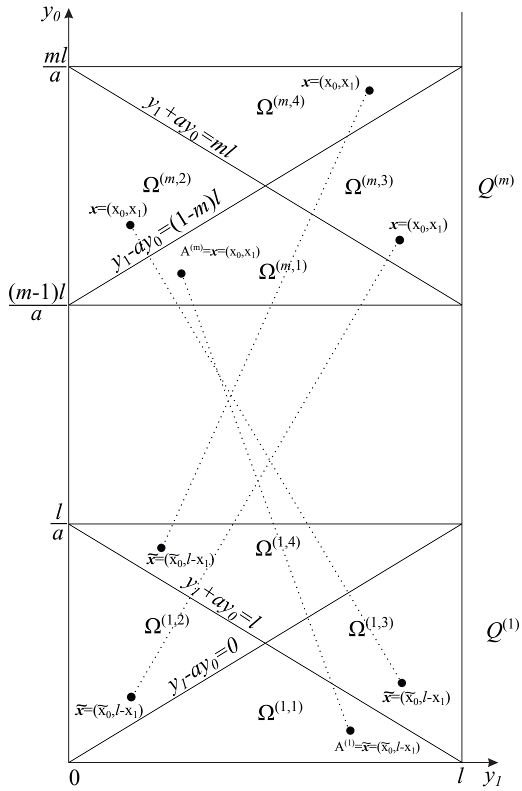

Figure 9: . $\widetilde{x_0} = x_0 - \frac{m-1}{a}l, m-$ is an even number

In the formula (49) (m is the odd number) there is the full correspondence, namely: if $\boldsymbol{x} \in \bar{\Omega}^{(m,j)}$, then the point $A^{(1)}$ with the coordinates $A^{(1)} = \widetilde{\boldsymbol{x}} = (\widetilde{x_0}, x_1) = (x_0 - \frac{m-1}{a}l, x_1)$ is to the set $\bar{\Omega}^{(1,j)}$ for the same j as $\boldsymbol{x} \in \bar{\Omega}^{(m,j)}$, $j \in \{1,2,3,4\}$.(see fig. 9)

References

[1] *Korzyuk V.I., Kozlovskaya I.S.* Klassicheskie resheniya zadach dlya giperbolicheskikh uravnenii (Classical Solutions of Problems for Hyperbolic Equations), A Lecture Course for Students of Mathematical and Physical-Mathematical Specialities, in 10 parts, Minsk: Belarus. Gos. Univ., (2017), Pt. 2.

[2] *Korzyuk V.I., Kozlovskaya I.S.* On compatibility conditions in boundary value problems for hyperbolic equations, Dokl. Nats. Akad. Nauk Belarusi (2013) 37-42

[3] *Korzyuk V.I., Kozlovskaya I.S., Naumavets S.N.* Classical solutions in the theory of partial differential equations, Studia i materialy Europejska Uczelnia Informatyczno-Ekonomiczna w Warszawie (2015) 55-78

[4] *Korzyuk V.I.* The characteristic parallelogram method on the example of the first mixed problem for a one-dimensional wave equation, Dokl. Nats. Akad. Nauk Belarusi (2017) 7-13

[5] *Korzyuk V.I., Kozlovskaya I.S., Naumavets S.N.* Classical problem solutions for hyperbolic equations, Studia i materialy Europejska Uczelnia Informatyczno-Ekonomiczna w Warszawie (2017) 55-78

[6] *Korzyuk V.I., Naumavets S.N., Sevastyuk V.A.* On the classical solution of the second mixed problem for the one-dimensional wave equation, Proceedings of the Institute of Mathematics (2018) 35-42

[7] *Korzyuk V.I., Naumavets S.N., Serikov V.P.* The characteristic parallelogram method for solving the second mixed problem for a one-dimensional wave equation, Proceedings of the Institute of Mathematics (2018) 43-53

[8] *Korzyuk V.I., Stoliarchuk I.I.* Mixed problem for Klein-Gordon-Fock eqaution with curve derivatives in boundary conditions, Computer Algebra Systems in Teaching and Research. Sedlice University of Natural Science and Humanities (2015) 166-181

[9] *Korzyuk V.I., Vinh N. V.* Classical solutions of mixed problems for onedimensional biwave equation, Vestsi Akad. Navuk Belarusi Ser. Fiz.-Mat. Navuk (2016) 69-79