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«BREST STATE TECHNICAL UNIVERSITY»

APPLIED MECHANICS

Structural Mechanics

Part 2: STATICALLY INDETERMINATE FRAMES

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The manual outlines the basic mechanics of plane frames, including the basic concepts and principles of structural mechanics, the determination of internal forces, and displacements in statically indeterminate frames. Calculation of statically indeterminate frames by an area-moment method (force method) and slope and deflection method (displacement method) are represented. Examples of calculations and problems (with keys) to independent solutions are given.

The manual is intended for international engineering students.

Compiled: **Ignatyuk V.** PhD, Associate Professor,
Applied Mechanic Department, BSTU, Brest, Belarus.
Zheltkovich A. PhD, Associate Professor,
Applied Mechanic Department, BSTU, Brest, Belarus.
Zhao Leiyang (赵磊阳), Senior Lecturer of Normal Education Department, Hohhot Vocational College, Hohhot, China.
Gao Jian (老师), English tutor, Jiamusi, China.

Reviewers: **Zhdanov D.** Director of Projekt Ltd, PhD, Associate Professor.
Semenyuk O. Associate Professor of the Department "Technology of Building Production" at Brest State Technical University, PhD, Associate Professor.

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4. Calculation of Statically Indeterminate Frames by an Area-Moment Method and Displacement Method

4.1. Idea of the Area-Moment Method

In the area-moment method, the calculation of statically indeterminate systems is reduced to well-known methods of calculating statically determinate systems.

Statically determinate system used for statically redundant system is obtained by discarding so called redundant constraints (with their replacement by reactions that may arise in these constraints) and is called the primary system or principal system (P.S.) of the area-moment method. This system should also work as a statically redundant system.

To comply with this, the following conditions must be met for the primary system:

1) In the primary system of the area-moment method, instead of redundant constraints, forces which correspond to the reactions in these constraints must be applied. In a statically indeterminate system, these constraints will experience reactive forces; these forces will be unknown of the area-moment method. As a result, the primary system will be under a set of loads – P and under unknowns – X_i ($i = 1 \ldots r$, where number of redundant constraints in the system – r);

2) Deflections of points (sections) in the direction of discarded (redundant) constraints in the primary system should be zero, because in a statically indeterminate system in these directions are constraints. Thus for the primary system, which is under a given loads (applied) – P_i and the unknowns of force method – X_i can be recorded analytically in the form of a system of equations

$$\begin{cases} \Delta_1(X_1,\ X_2, \ldots, X_r, P) = 0; \\ \Delta_2(X_1,\ X_2, \ldots, X_r, P) = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_r(X_1,\ X_2, \ldots, X_r, P) = 0. \end{cases} \tag{4.1}$$

The solution to this system of equations allows us to identify the primary unknowns of the area-moment method.

By applying together, the values of forces which have been found by use of equations of a system 4.1, with the applied loads to the primary system, the diagrams of internal forces – M, Q, N will be able to be plotted by using of usual way (by way of calculating statically non-redundant structures). These diagrams will be diagrams of internal forces in a given statically redundant system.

The following principles for calculating redundant systems are considered below. The calculation of redundant frames is represented in more detail and consistently with the allocation of all stages of calculation.

4.2. The Degree of Static Redundancy of the System

The degree of static redundancy of the system is the number of redundant constraints, the removal of which will turn the system into a statically determinate system. The number of redundant constraints equals the degree of freedom of the system with the reverse sign: $r = -W$. Formulas for determining the degree of freedom of the system are given and discussed in the section 2 "Kinematic analysis of structures" [1].

The degree of static redundancy of frames can be determined by formulas:

$$r = 3L - h, \tag{4.2}$$

where: r – the number of redundancy; L – the number of closed loops (contours) which form the structures; h – the number of ordinary hinges:

$$r = -(3D - 2H - C_o), \tag{4.3}$$

where: D – the number of disks in the system; C_0 – the number of kinematic restrictions (*reactions at the supports*) of the system.

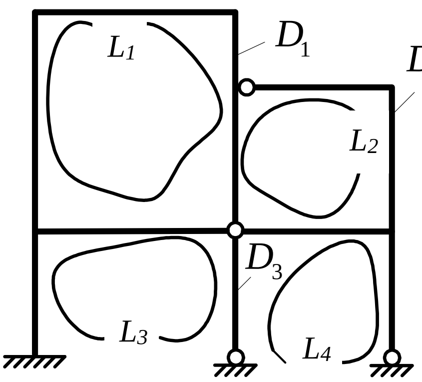


Figure 4.1

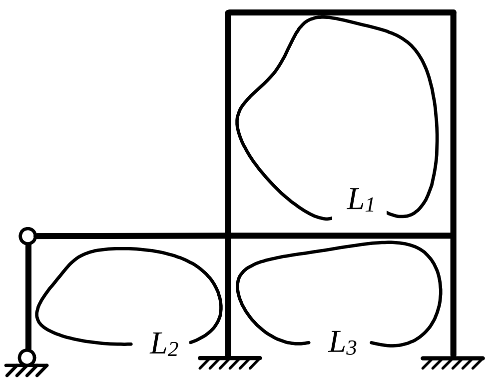


Figure 4.2

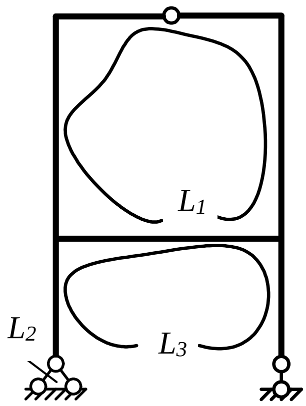


Figure 4.3

Let's calculate the number of redundant constraints for the frames presented in the figure 2÷4:

a) for the frame on the figure 4.1:

$$r = -(3D - 2H - C_o) = -(3\cdot 3 - 2\cdot 4 - 7) = 6;$$

b) For the frame on the figure 4.2:

$$r = 3L - h = 3\cdot 3 - 2 = 7;$$

The frame contains completely closed loop redundant constraints can be found according to the formula:

$$r = -(3D - 2H - C_o);$$

c) for the frame on the figure 4.3:

$$r = 3L - h = 3\cdot 3 - 7 = 2;$$

$$r = -(3D - 2H - C_o) = -(3\cdot 1 - 2\cdot 1 - 3) = 2.$$

4.3. Choosing the Primary System of the Area-Moment Method

The primary system (P.S.) of the force method is called a statically determinate, geometrically stable system (perfect frame). A geometrically stable system from a given redundant system must be taken by discarding redundant constraints and replacing them with unknown forces (that can arise in these constraints).

Stability of geometrical shape is determined by kinematic analysis of the primary system more precisely, by geometric analysis of the structure of the system (see «Kinematic Analysis of Structures»). Let's take a look at a few examples of selecting the principal systems of force method.

Example 4.1 The frame presented in the figure 4.4 *a*, has two redundant constraints:

$$r = 3L - h = 3\cdot 1 - 1 = 2,$$

or: $$r = -(3D - 2H - C_O) = -(3\cdot 2 - 2\cdot 1 - 6) = 2,$$

Some of primary systems shown in the figures 4.4*b,e* can be selected for it. The system depicted in the figure 4.4*f* cannot be accepted, as it is instantaneously variable system (at the top part of the frame) by the first sign of instant variability – three discs are connected by three hinges lying on one straight line.

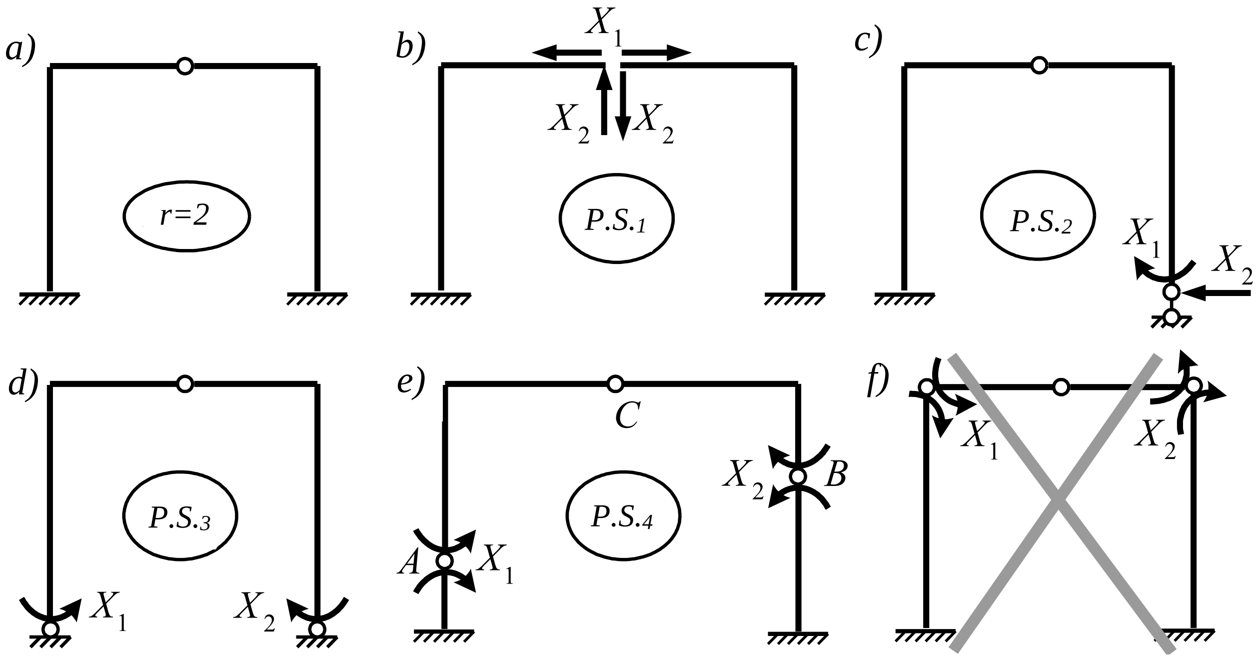


Figure 4.4

Example 4.2 The frame presented in the figure 4.4 *a*, has three redundant constraints:

$$r = 3L - h = 3\cdot 3 - 6 = 3,$$

or: $$r = -(3D - 2H - C_O) = -(3\cdot 4 - 2\cdot 4 - 7) = 3,$$

and for it, variants of the primary system are presented in the figure 4.5 *b* – 4.5 *f*. The schemes depicted on figure 4.5 *g*, cannot be accepted as primary system (by the first sign of instant variability – hinges lying on one straight line). Primary system on figure 4.5 *h*, cannot be accepted either because the system is variable in the right part, (which can rotate with respect to the hinge), and the left part is statically redundant, which is a consequence of incorrect discarding of redundant constraints).

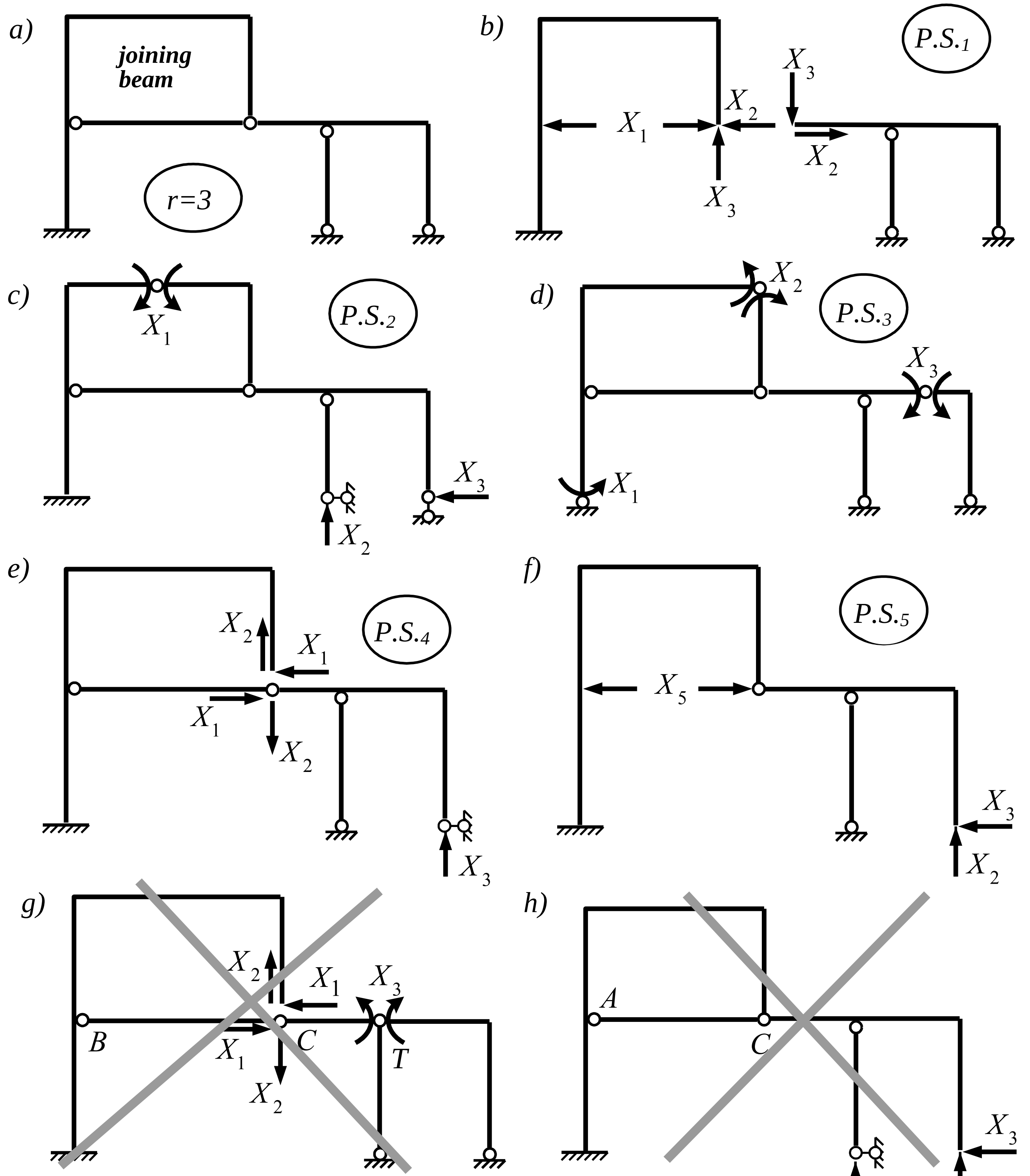


Figure 4.5

***Example* 4.3.** For the frame presented in the figure 4.6 *a*, the number of redundant constraints can be found:

$$r = 3L - h = 3 \cdot 1 - 0 = 3.$$

The possible variants of the primary system of the force method are shown in the figure 4.6 *b* – 4.6 *f*. As can be seen from the examples, primary system can be obtained by using the **following approaches in discarding redundant constraints:**

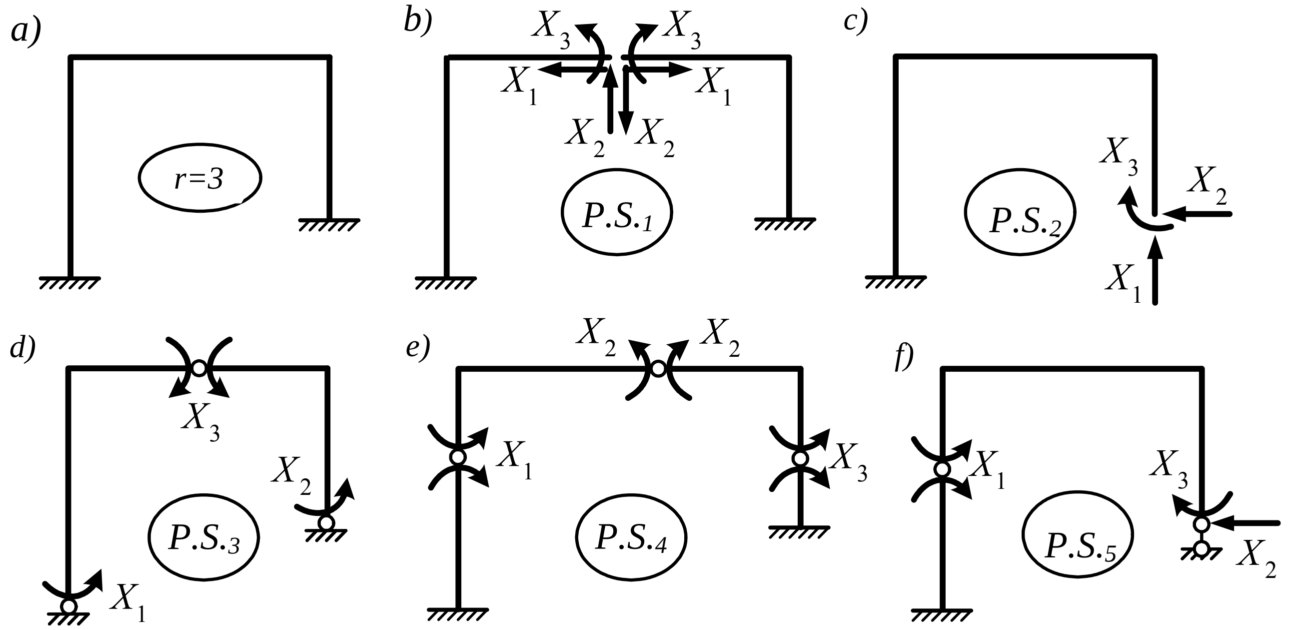


Figure 4.6

– discarding the supports. One constraint is removed when the hinged movable support is discarded, two constraints when the hinged immovable support is thrown away and three constraints when discarding the pinched support (embedding);

– Discarding separate support constraints (the number of unknowns equals the number of discarded support constraints);

– Cutting bracings. One connection acting along the bracing is removed;

– Setting-in hinged point (one constraint (angular) is removed);

– Sawing hinge (two constraints are removed when removed one ordinary hinge);

– Sawing the rods (three constraints are removed). Analysis of presenting primary systems leads to the following conclusion:

! ***For any statically redundant system, there are an infinite number of primary systems of the force method.***

To calculate by use of the force method, one primary system must be selected, which sometimes can be called the **design primary system** (D.P.S.)

Most rational primary system should be considered.

The rationality of the primary systems is determined by the following provisions:

1) In the design primary system, determining reactions at the supports and plotting of diagrams of internal forces should be as simple as possible;

2) The bending moment diagram should also be as simple as possible;

3) Symmetrical design primary systems should be chosen for symmetrical frames.

For the frame on the figure 4.4 – P.S.$_1$ can be taken as the most appropriate design scheme; for the frame on the figure 4.5 – P.S.$_1$; and for the frame on the figure 4.6 – P.S$_1$ or P.S$_2$.

4.4. System of Canonical Equations of the Force Method

The primary system of the force method adopted for calculation, as already stated (section 4.1), should be equivalent to a given statically redundant system, and this will be if these system is equally deformed and have the same deflections of all points. And accordingly, deflections (in the given P.S.) in the direction of discarded constraints should be zero (4.1), as in an actual statically redundant system.

Then is writing down the condition of equivalence of the primary system (loaded with unknowns of force method – $X_1,\ X_2,\ X_3$, ... X_r and an external applied force), a statically redundant system with ***n*** redundant constraints (4.1) in a deployed form, using the principle of independence of force. As a result, deflection in the direction of ***i***-discarded constraint will have the appearance of:

$$\Delta_i = \Delta_{i1} + \Delta_{i2} + \Delta_{i3} + \ldots + \Delta_{ik} + \ldots + \Delta_{ir} + \Delta_{iP} = 0,$$

where: Δ_{ik} – deflection in the direction of the *i* discarded constraint caused by the action of *k* unknown force (X_k); Δ_{iP} – deflection in the direction of *i* discarded constraint from the action of a given loading.

For linear-deformable systems, deflection caused by any force can be expressed in the form of a product of that force and the deflection in the same direction and the same sort (e.g. concentrated force causes linear deflection, point moment – angular deflection) from the action of the corresponding unit force:

$$\Delta_{ik} = \delta_{ik} \cdot X_k .$$

Expressing each of the deflections from the action of unknown forces through these forces and corresponding unit deflections, we get a system of canonical equations of the forces method in the form of:

$$\begin{cases} \delta_{11} X_1 + \delta_{12} X_2 + \delta_{13} X_3 + \ldots + \delta_{1r} X_r + \Delta_{1P} = 0; \\ \delta_{21} X_1 + \delta_{22} X_2 + \delta_{23} X_3 + \ldots + \delta_{2r} X_r + \Delta_{1P} = 0; \\ \delta_{31} X_1 + \delta_{32} X_2 + \delta_{33} X_3 + \ldots + \delta_{3r} X_r + \Delta_{3P} = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta_{r1} X_1 + \delta_{r2} X_2 + \delta_{r3} X_3 + \quad + \delta_{rr} X_r + \Delta_{rP} = 0, \end{cases} \tag{4.4}$$

where: δ_{ik} and Δ_{iP} – deflections, in the designations of which:

– First index identifies the points (sections) that displace in the directions of their deflections (they coincide respectively with the points (sections) in which the forces of X_i are attached, and with the directions of these forces);

– Second index points to the causes of these deflections, i.e. to the unit force of X_k if the second index is *k* (unit action, unit moment). Or if the second index is *P*, this means to the structure acts the actual loading.

Equation coefficients (4.4) with the same indices (δ_{ii}), will be termed hereafter principal deflection (coefficient), whereas the deflection such as (δ_{ik}) with different indices – secondary deflection (coefficient), and aforementioned deflection due to applied loads (Δ_{iP}) – absolute term.

The principal deflections here will always be positive and can't be zero. Secondary deflections can take any value, including zero, and for them on the basis of the Maxwell theorem of reciprocal deflections, equality will be always equal between themselves:

$$\delta_{ik} = \delta_{ki}. \tag{4.5}$$

Depending on the type of X_i force, deflections of δ_{ik} and Δ_{iP} on physical sense can be:

– ***linear deflection***, if X_i – concentrated (point) force;

– ***angular deflection*** if X_i – concentrated (point) moment;

– ***reciprocal linear deflection*** (convergence or divergence) of two points if X_i – two concentrated forces, applied at two points in a straight line, towards each other or from each other;

– ***the reciprocal angular deflection*** of the two-sections if X_i – two point moments attached in these sections turn towards each other or from each other.

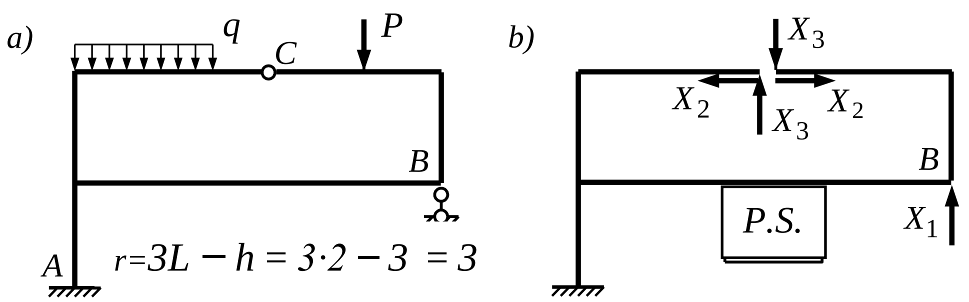


Figure 4.7

For example, for a frame depicted in a figure 4.7 *a*, when selecting the primary system of the force method in the view presented on the figure 4.7 *b*, will take place a system of three equations (4.4).

The physical meaning of this system's coefficients (on the example of several coefficients) will be as follows:

δ_{11} – vertical deflection of ***B*** point in the primary system from the action of the unit force X_1;

δ_{23} – mutual divergence of left and right sections in ***C*** point horizontally from the action of force of unit value – X_3;

Δ_{3P} – mutual convergence of left and right sections in ***C*** hinge vertically from the action of the assigned (external) loads.

<u>The physical meaning of the equations</u> as a whole will be:

<u>1st equation</u> – vertical deflection of ***B*** point from the action of X_1, X_2, X_3 forces and applied loads should be zero, because in a given statically redundant system (Figure 4.7 *a*) at ***B*** point there is a vertical restriction;

<u>The 2nd equation,</u> which should be zero, is a reciprocal divergence of left and right sections in ***C*** hinge horizontally from the action the forces of X_1, X_2, X_3 and applied loads, as these sections in actual construction are connected to each other by ***C*** hinge (Figure 4.7 *a*) and cannot diverge.

The physical meaning of <u>the 3rd equation</u> is similar to the meaning of the 2nd with a difference in the direction of mutual divergence of sections (vertically).

4.5. Calculating Deflections and Absolute Terms of Equations

Deflections and absolute terms of equations (4.4) are physical deflections and can be calculated by the Mohr's formula (3.2). In this case, for frames, as broken systems, in the Mohr's formula usually neglect the influence of shear force and longitudinal force, which for such systems are insignificant, omitting the corresponding components. As a result of the expression to determine the deflections and absolute terms of the systems of canonical equations of the force method will have in the form of:

$$\delta_{ii} = \sum_1^n \int_0^l \frac{\bar{M}_i^2\,dx}{EJ}\ ; \qquad \delta_{ik} = \sum_1^n \int_0^l \frac{\bar{M}_i\,\bar{M}_k\,dx}{EJ}\ ; \qquad \Delta_{iP} = \sum_1^n \int_0^l \frac{\bar{M}_i\,M_P\,dx}{EJ}\ , \tag{4.6}$$

where: $\bar{M}_i\,(\bar{M}_k)$ – law of change bending moment (diagram) in the primary system from action of $X_i = 1$ ($X_k = 1$); M_P – law of change bending moment in the primary system from the action of applied loads; EJ – flexural rigidity of the rod (part); ***n*** – number of integration sites; ***l*** – length of these sites.

Thus, in order to calculate the deflections and absolute terms of the canonical equations of the force method must be plotted in the primary system unit diagram of bending moment – $\bar{M}_i$ ($i = 1 \ldots r$) from the action of unit unknown ($X_i = 1$) and diagram of bending moment – $\boldsymbol{M_P}$ from the action of the applied loads. After that, will be gotten correspond opportunities to determine the sought quantities.

The principles of calculating of Morh's integrals (4.6) are set out in section 3.10. (Part I) [1]. After calculating the coefficients and absolute terms of the canonical equations by formula (4.6) it is necessary to check the correctness of the calculations, which can be used:

a) **Universal check** of the correct calculation of <u>secondary deflections</u>:

$$\sum_1^n \int_0^l \frac{\bar{M}_S^2\,dx}{EJ} = \sum_{i=1}^r \ \sum_{k=1}^r \delta_{ik}\ ; \tag{4.7}$$

Where the right part of the expression represents the sum of all coefficients of the system (4.4), and $\bar{M}_S = \bar{M}_1 + \bar{M}_2 + \ldots + \bar{M}_n$ – total unit diagrams.

b) To determine which one equation (which row) has incorrectly deducted coefficients, if a universal check is not performed, we can make ***<u>line checks</u>*** recorded as a form of equation:

$$\sum_{1}^{N}\int_{0}^{l}\frac{\overline{M}_i\,\overline{M}_S dx}{EJ}=\sum_{k=1}^{n}\delta_{ik}\quad (k=1\ \dots\ n); \qquad (4.8)$$

Where the right part represents the sum of all the coefficients in the ***i***-system equation (4.4).

Analysis of the performance or failure of separate *line checks* allows determining (at least approximately) which of the coefficient $-\delta_{ik}$ perhaps have been miscalculated.

As it is easy to see, all (***L***) *line checks* replace the *universal check*, and vice versa.

c) **A column check** of the correctness of calculating absolute terms (deflections due to applied loads) of the system of equations - is recorded as:

$$\sum_{1}^{n}\int_{0}^{l}\frac{\overline{M}_S M_P dx}{EJ}=\sum_{i=1}^{L}\Delta_{iP}\,; \qquad (4.9)$$

Where the right side of the expression (4.9) is the sum of all absolute terms of the system of equations (4.4).

4.6. Plotting the Final Diagram and Verifying Them

The calculated deflection and absolute terms of the system of canonical equations (4.4) is a heterogeneous system of linear algebraic equations and can be solved, for example, by the method of substitution, the way Gauss and other known ways.

Note that after finding an unknown of the force method, it is **necessary to check** the correctness of the solution of the equation of system by substituting the found values of X_i *(i = 1 ... n)* into all equations of the system. *If don't do this, may be turn out that all further computations and analysis will be a waste of time.*

After determining the unknowns of force method – X_i (i = 1 … n), plotting of the final diagrams of internal forces in a designed statically determinate system can be done in two ways:

1. Applied loads and all found unknowns can be attached to the primary system and we can plot in this primary system the diagrams – *M*, *Q* and *N*, as in the usual statically determinate system, which are the diagrams of internal forces in a given statically redundant system.

2. Take into consideration that from the action of each of the unknowns – $\boldsymbol{X_i}$ of a unit magnitude and applied loads, the diagram of bending moment in the primary system have already been plotted (before calculating the coefficients of the system's equations – see section 4.5), they (diagrams) can be used. In this case, the final graphs of bending moment in a given redundant system can be constructed using the principle of independence of force, according to the formula:

$$M=\overline{M}_1\cdot X_1+\overline{M}_2\cdot X_2+\overline{M}_3\cdot X_3+\ldots+\overline{M}_n\cdot X_n+M_P. \qquad (4.10)$$

Since they are used the results of diagrams and calculations of unknowns already performed earlier, this approach turns out simpler and faster leading to the goal, and therefore only this approach will be used.

The correctness of calculations and construction of a graph of bending moment is checked with ***deformation (kinematic) check***, which can be introduced in two variants:

1) ***Complete deformation check*** - recorded as:

$$\sum_{1}^{n}\int_{0}^{l}\frac{\bar{M}_S\, M\, dx}{EJ}=0\ ; \qquad (4.11)$$

And in a physical sense of it a total deflection in the directions of all unknowns of force method – X_i (i = 1 … n) from the action of these unknowns and applied loads, should be zero, because in a given statically determinate system in the direction of these unknowns (X_i) there are constraints (this physical meaning of equation coinciding with the physical meaning of all put together canonical equations of force method – see Section 4.4);

2) ***Line-up deformation (kinematic) checks*** have a form of:

$$\sum_{1}^{n}\int_{0}^{l}\frac{\bar{M}_i\, M\, dx}{EJ}=0\,, \qquad i = 1 \dots n; \qquad (4.12)$$

The physical meaning of each of these checks is equality to zero deflections to the all directions of each of the unknowns – X_i (i = 1 … n) from actions of these unknowns and applied loads. Deflections must be zero, as in an assigned statically redundant system in these directions there are constraints (and this corresponds to the physical meaning of the respective canonical equations of the force method – see. Section 4.4).

Altogether, *line-up deformation checks* (4.12) correspond to a complete *deformation check* (4.11), and if a full (*deformation*) check is carried out, it makes no sense to perform *line-up check*. This (*line-up check*) should be done if a full deformation check isn't performed (in order to determine in which direction we have to look for errors).

The diagram of the shear force of Q can be plotted on the diagram of ***M*** using the known differential dependence – $Q=\frac{dM}{dx}$, which ***for linear parts of the diagram*** of ***M*** can be presented as:

$$Q=\pm\left|\frac{M_{right}-M_{left}}{l}\right| \quad , \qquad (4.13)$$

where: $\boldsymbol{M_{left}}$, $\boldsymbol{M_{right}}$ – the magnitude of bending moment on the ends of the site (left and right); if the stretched fibers at these bending moments are on different sides of the rod, one of them is taken positive, and the other negative.

The sign before the absolute value in the formula (4.13) is accepted ***according to the rule (see figure 4.8)***:

If we have to combine the rod on which the diagram of M is plotted, with the tangent to this graph the rod must be rotated clockwise at the angle of the turn less than 90^0*, the sign " +" is accepted; if counterclockwise, the sign " _" is accepted.*

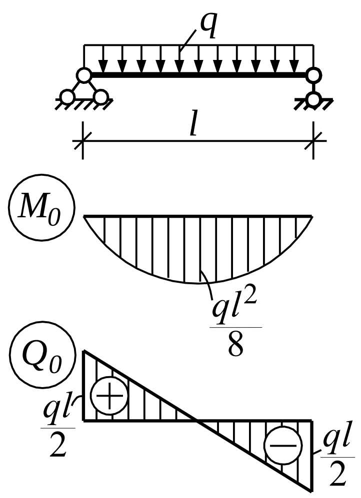


Figure 4.8

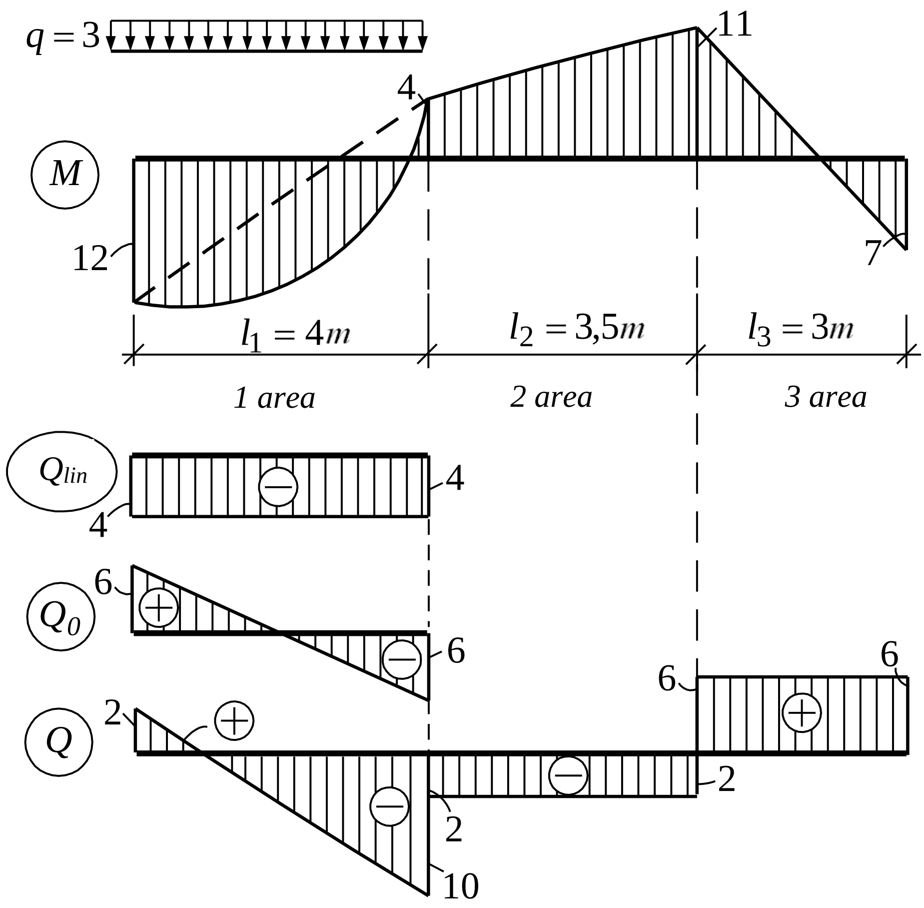


Figure 4.9

For curvilinear (parabolic) part of the diagram of M differential dependence – $Q = \dfrac{dM}{dx}$ can be represented in the following form:

$$Q = Q_0 \pm \left| \frac{M_{right} - M_{left}}{l} \right|, \tag{4.14}$$

where the second term is $\boldsymbol{Q_{lin}}$ (shear force) from the linear part of the ***M*** diagram, and the first summand – $\boldsymbol{Q_0}$ takes into account the curvilinear part of ***M*** diagram and represents itself the diagram (law of change) of the shear forces in the site (area) of frame considered as simple beam, from the action of a uniformly distributed load (see Figure 4.8).

For example, for the ***M*** on figure 4.9, represented by three areas with different laws of its change, the shear force in these areas (left to right) will be equal:

$$Q_1^{left,right} = \pm\frac{ql_1}{2} - \left|\frac{4-(-12)}{4}\right| = \pm\frac{3\cdot 4}{2} - 4 = \pm 6 - 4; \quad Q_1^{left} = +2; \quad Q_1^{right} = -10;$$

$$Q_2 = -\left|\frac{11-4}{3{,}5}\right| = -2; \qquad Q_3 = +\left|\frac{-7-11}{3}\right| = +6.$$

The Diagram of Longitudinal Force *N* are constructed on the diagram Q by the way of cutting out the nodes, i.e., cutting out the nodes of the frame, applying already known shear forces and unknown (as well as well-known) longitudinal forces (if applied concentrated forces are attached to the node, they are also must be taken into account) in area. The equilibrium equations are then drawn up: $\Sigma X = 0$; $\Sigma Y = 0$, from which unknown longitudinal forces are determined.

After plotting the diagrams of internal forces, cutting out the supporting nodes and considering their balance, reactions can be found in the supports of the frame. Then we need to perform a **static check** of the balance of the frame as a whole using, for example, equations:

$$\begin{aligned} &\Sigma X_{react.at\ the\ supports} + \Sigma X_{actual\ load} = 0; \\ &\Sigma Y_{react.at\ the\ supports} + \Sigma Y_{actual\ load} = 0; \\ &\Sigma M_{T\ react.at\ the\ supports} + \Sigma M_{T\ actual\ load} = 0. \end{aligned} \qquad (4.15)$$

4.7. Procedure for Calculation of Frames by Force Method

Thus, on the basis of calculation above, the following order of calculating the frames by force method is proposed:

1. Determine the degree of static redundancy of the frame (i.e. the number of redundant connections in the frame – ***n***), using, for example, formulas (4.2, 4.3).
2. Choose the design principal system of force method by presenting several possible variants of the primary system in advance.
3. Write down in general form the system of canonical equations of the force method (4.4) and find out the physical meaning of these equations and their constituents (values included in them).
4. Plot unit bending moment diagrams – $\overline{M}_1, \overline{M}_2, ..., \overline{M}_L$ and diagram of $\boldsymbol{M_R}$ due to the actual loading in the D.P.S. of the force method.
5. Calculate all coefficients – δ_{ik} and terms – Δ_{iP} of the system of canonical equations of the force method (4.6).
6. Check the correct calculation of coefficients (4.7 or 4.8) and terms of the equations of system (4.9).
7. Solve the system of canonical equations and find unknown – $\boldsymbol{X_1, X_2, ... X_n}$; Perform the check of correctness of the system's equations by substituting the unknowns found in all equations.
8. Plot the final diagram of bending moment – ***M*** in a given redundant frame (4.10).
9. Perform a deformation check of the ***M*** diagram (see. 4.12 or 4.11).
10. According to ***M*** graph plot the final diagram of the shear force – ***Q*** (using dependencies 4.13 or 4.14).
11. By cutting out the nodes on the diagram of ***Q***, taking into account the actual loads in the nodes, the diagram of longitudinal force – ***N*** can be plotted.
12. By cutting out the supporting joints, determine the reactions at the supports, and perform a static check of the balance of the frame as a whole (4.15).

4.8. Examples of Calculation

Here is an example of the calculation of the frame by the force method with one redundant constraint. Principles and approaches in frame calculations with more than one redundant constraints are no different from those which are presented in this example – the numbers will only be related to the difference, such as the number of equations in the system of equations, the number of design coefficients, the terms of the system's equations, the number of unit bending moment diagrams that will need to be built (by the parameters mentioned above calculations are performed).

Let's calculate the frame presented in the figure 4.10 *a*.

1. This frame has one redundant constraint:

$$r = 3L - h = 3 \cdot 3 - 8 = 1,$$

$$\text{or} \;\; r = -(3D - 2H - C_o) = -(3 \cdot 1 - 2 \cdot 0 - 4) = 1.$$

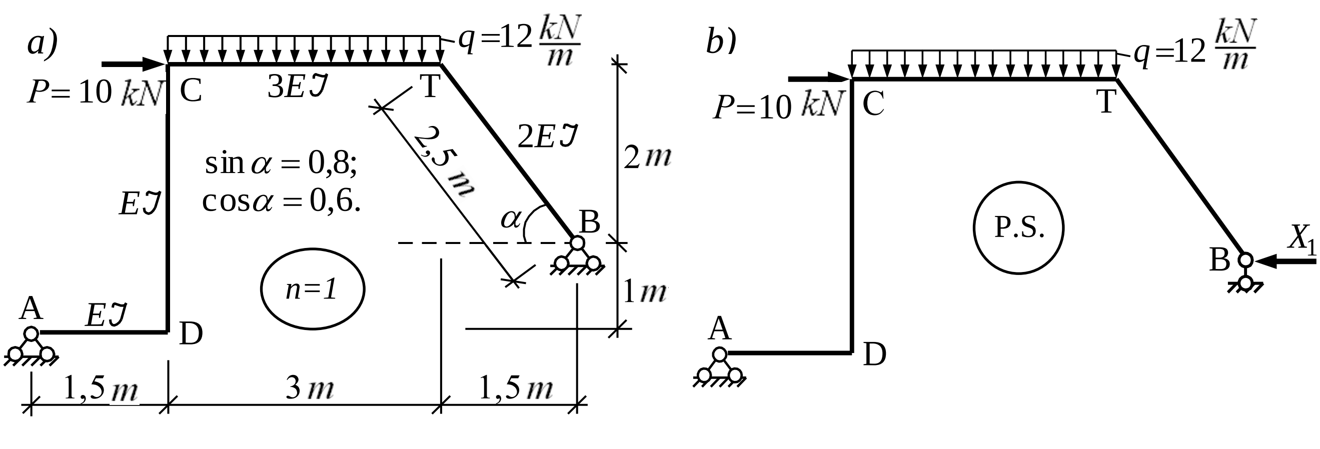
2. The accepted design diagram (scheme) of primary system of force method is shown in the figure 4.10 *b*. The choice of primary system here can also be made by discarding any other support restrictions, or by cutting the hinge in any section of the frame, except for the section lying at the intersection of the ***CD*** rod and the imaginary line of ***AB***, (e.g. this case the design diagram is an instantaneously variable system, which is formed by three discs connected by three hinges lying on one straight line the first sign of instantaneously variable system – see Part 1, p.13 [1]).

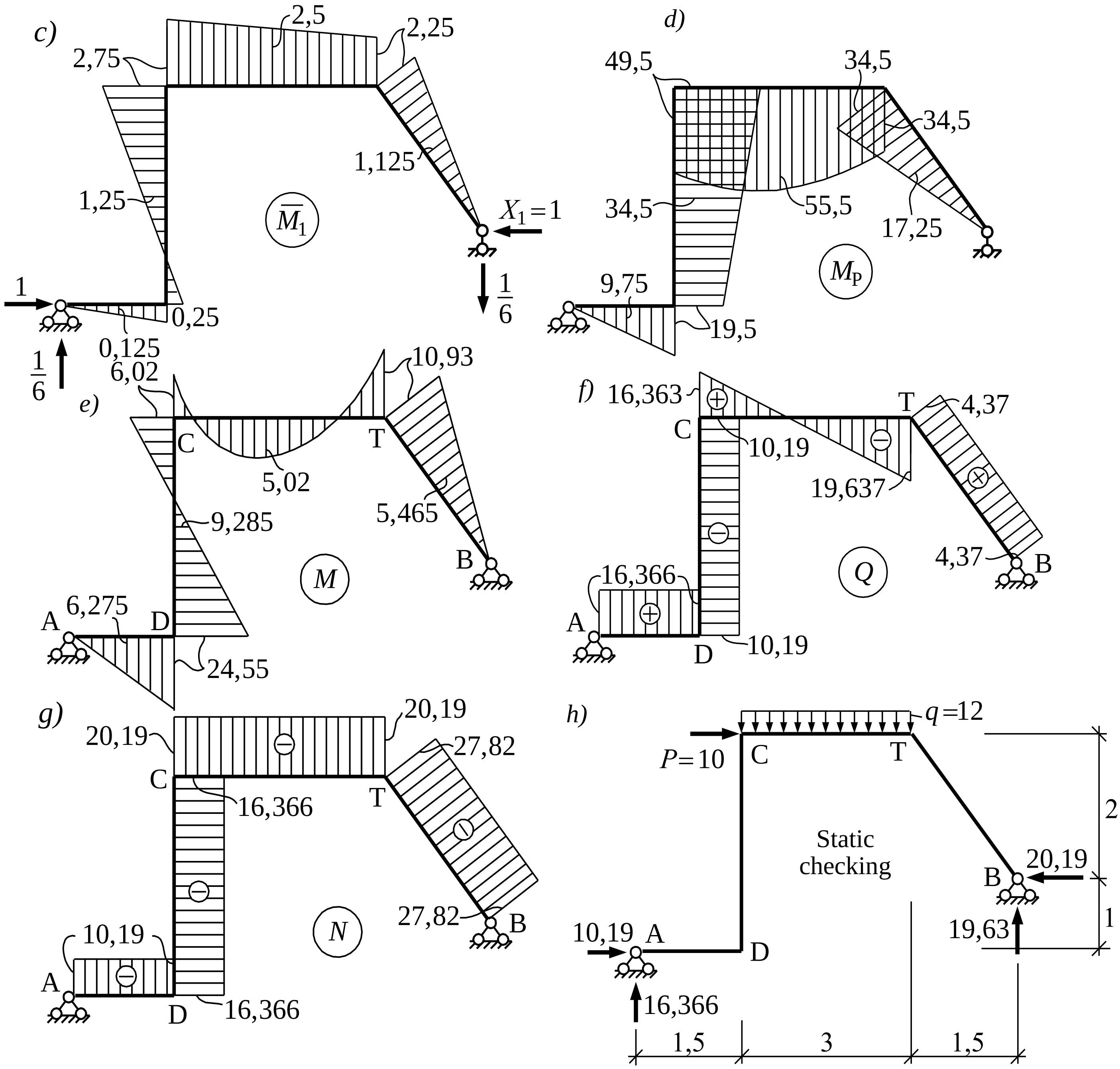
3. There will also be one canonical equation here representing in a physical sense the horizontal deflection of point ***B*** (in the primary system) from the action of X_1 force and actual applied load, which should be zero, as in the original system (Figure 4.10 *a*) at ***B*** point this is a horizontal constraint (here is a hinged immovable support):

$$\delta_{11} X_1 + \Delta_{1P} = 0, \tag{4.16}$$

4. In the primary system of the force method, as in an ordinary statically determinate system we need to plot a unit diagram of bending moment $\overline{M}_1$ (Figure 4.10 *c*) from the action of the X_1 force of a unit magnitude $(X_1 = 1)$ and a diagram M_P due to the actual loading (Figure 4.10 *d*).

5. Calculate the coefficient of δ_{11} and the term of Δ_{1P}. Let's show here the calculation of these values in different ways:





c)
2,75
2,5
2,25
1,125
1,25
$\overline{M}_1$
$X_1 = 1$
1
0,25
0,125
$\frac{1}{6}$
$\frac{1}{6}$
d)
49,5
34,5
34,5
34,5
55,5
17,25
M_P
9,75
19,5
6,02
10,93
e)
C
T
5,02
5,465
9,285
B
M
6,275
A
D
24,55
f)
16,363
C
T
4,37
10,19
19,637
Q
4,37
B
16,366
A
D
10,19
g)
20,19
20,19
27,82
C
16,366
T
N
27,82
B
10,19
A
D
16,366
h)
$q = 12$
$P = 10$
C
T
2
Static checking
20,19
B
1
19,63
10,19
A
D
16,366
1,5
3
1,5

Figure 4.10

a) according to the Vereshchagin's rule:

$$\delta_{11}=\sum_1^n\int_0^l\frac{\overline{M}_1^2dx}{EJ}=\frac{1}{EJ}\left(\frac{0,25\cdot1,5}{2}\right)\cdot\frac{2}{3}0,25+\frac{1}{EJ}\left[-(0,25\cdot3)\cdot1,25+\frac{(2,75+0,25)}{2}\cdot3\times\right.$$

$$\left.\times\left(\frac{2}{3}3-0,25\right)\right]+\frac{1}{3EJ}\left[(2,25\cdot3)\cdot2,5+\frac{(2,75-2,25)\cdot3}{2}\cdot\left(\frac{2}{3}0,5+2,25\right)\right]+$$

$$+\frac{1}{2EJ}\left(\frac{2,25\cdot2,5}{2}\right)\cdot\frac{2}{3}2,25=\frac{1}{EJ}(0,031+6,938+6,271+2,109)=\frac{15,35}{EJ};$$

$$\Delta_{1P}=\sum_1^n\int_0^l\frac{\overline{M}_1\,M_P\,dx}{EJ}=\frac{1}{EJ}\left(\frac{19,5\cdot1,5}{2}\right)\frac{2}{3}0,25+\frac{1}{EJ}\left[-(19,5\cdot3)\cdot1,25-\frac{(49,5-19,5)}{2}\cdot3\times\right.$$

$$\left.\times\left(\frac{2}{3}3-0,25\right)\right]+\frac{1}{3EJ}\left[-(34,5\cdot3)\cdot2,5-\frac{(49,5-34,5)}{2}\cdot3\left(2,25+\frac{2}{3}0,5\right)-\right.$$

$$\left.-\left(\frac{2}{3}\cdot\frac{12\cdot3^2}{8}\cdot3\right)\cdot2,5\right]-\frac{1}{2EJ}\left(\frac{1}{2}34,5\cdot2,5\right)\frac{2}{3}2,25=-\frac{309,906}{EJ};$$

b) according to the Simpson's formula:

$$\delta_{11}=\sum_1^n\int_0^l\frac{\overline{M}_1^2dx}{EJ}=\frac{1,5}{6EJ}\left(0^2+4\cdot0,125^2+0,25^2\right)+\frac{3}{6EJ}\left(0,25^2+4\cdot1,25^2+2,75^2\right)+$$

$$+\frac{3}{6\cdot3EJ}\left(2,75^2+4\cdot2,5^2+2,25^2\right)+\frac{2,5}{6\cdot2EJ}\left(2,25^2+4\cdot1,125^2+0^2\right)=\frac{15,35}{EJ};$$

$$\Delta_{1P}=\sum_1^n\int_0^l\frac{\overline{M}_1M_P\,dx}{EJ}=\frac{1,5}{6EJ}\left(0^2+4\cdot0,125\cdot9,75+0,25\cdot19,5\right)+\frac{3}{6EJ}(0,25\cdot19,5-$$

$$-4\cdot1,25\cdot34,5-2,75\cdot49,5)-\frac{3}{6\cdot3EJ}(-2,75\cdot49,5-4\cdot2,5\cdot55,5-2,25\cdot34,5)+$$

$$+\frac{2,5}{6\cdot2EJ}(2,25\cdot34,5+4\cdot1,125\cdot17,25+0\cdot0)=-\frac{309,906}{EJ}.$$

(!) Note that with such calculations of Mohr's integrals it is possible to perform calculations in different ways at different areas (according to the Vereshchagin's rule, according to the trapezoid formula, according to the Simpson's formula), combining them in terms of the convenience of computations.

7. Solve the canonical equation of the force method (4.16):

$$\frac{15,35}{EJ}X_1-\frac{309,906}{EJ}=0;\qquad X_1=20,19\ kN.$$

8. Plot the final diagram of bending moment according to the formula (4.10):

$$M=\overline{M}_1\cdot X_1+M_P\,,$$

multiplying all the characteristic ordinates of the $\overline{M}_1$ diagram by 20.19 *kN*, and adding up the results with the corresponding ordinates of the M_P diagram. The final diagram ***M*** is presented in the figure 4.10 *e*.

9. Deformation (kinematic) check of the final diagram of *M*:

$$\sum_1^n \int_0^l \frac{\overline{M}_1 M\,dx}{EJ} = 0; \quad \frac{1}{EJ}\left(\frac{24{,}55\cdot 1{,}5}{2}\right)\cdot\frac{2}{3}0{,}25 + \frac{3}{6EJ}\left(0{,}25\cdot 24{,}55 - 4\cdot 1{,}25\cdot 9{,}285 +\right.$$

$$\left. + 2{,}75\cdot 6{,}02\right) + \frac{3}{6\cdot 3EJ}\left(2{,}75\cdot 6{,}02 - 4\cdot 2{,}5\cdot 5{,}02 + 2{,}25\cdot 10{,}93\right) + \frac{1}{2EJ}\left(\frac{10{,}93\cdot 2{,}5}{2}\right)\frac{2}{3}2{,}25 =$$

$$= \frac{1}{EJ}\left(3{,}069 - 11{,}866 - 1{,}509 + 10{,}247\right) = \frac{1}{EJ}\left(13{,}316 - 13{,}375\right) = 0{,}059 \approx 0;$$

Discrepancy $\left|\dfrac{13{,}316 - 13{,}375}{13{,}375}\right|\cdot 100\% = 0{,}443\% < 1\%$ is insignificant.

10. Build the diagram of shear force in a given statically redundant system using formulas (4.13), (4.14).

a) at the parts of *AD* and *DC* according to the formula (4.13) we will receive:

$$Q_{AD} = +\left|\frac{24{,}55 - 0}{1{,}5}\right| = +16{,}366\ kN; \qquad Q_{DC} = -\left|\frac{24{,}55 - (-6{,}02)}{3}\right| = -10{,}19\ kN;$$

b) The formula should be used on the *CT* part (4.14):

$$Q_{CT} = \pm\frac{12\cdot 3}{2} - \left|\frac{10{,}93 - 6{,}02}{3}\right| = \pm 18 - 1{,}637\ kN;$$

$$Q_{CD}^{left} = +18 - 1{,}637 = 16{,}363\ kN; \qquad Q_{CD}^{right} = -18 - 1{,}637 = -19{,}637\ kN;$$

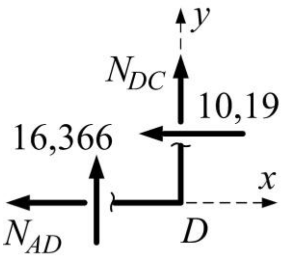
c) On the *TB* part, get shear force according to the formula (4.13):

$$Q_{TB} = +\left|\frac{10{,}93 - 0}{2{,}5}\right| = +4{,}37\ kN.$$

The diagram of the shear forces is depicted in the figure 4.10 *e*.

11. The diagram of longitudinal forces of N is plotted by the method of cutting out of nodes on the *Q* diagram:

a) Node D



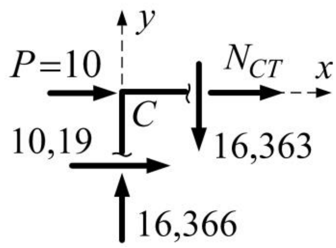
$\sum X = 0;$

$N_{AD} = -10{,}19;$

$\sum Y = 0;$

$N_{DC} = -16{,}366;$

b) Node C



$\sum X = 0;$

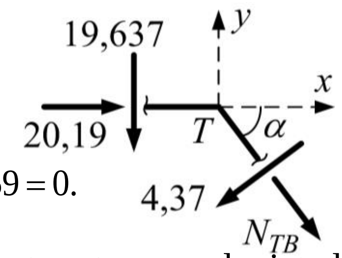
$10 + 10{,}19 - N_{CT} = 0;$

$N_{CT} = -20{,}19;$

$\sum Y = 0;$

$16{,}366 - 16{,}363 \approx 0.$

c) <u>*Node T*</u>

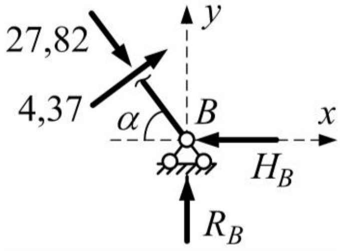


$\sum X = 0; \quad 20{,}19 - 4{,}37 \cdot 0{,}8 + N_{TB} \cdot 0{,}6 = 0; \quad N_{TB} = -27{,}823;$

$\sum Y = 0; \quad -19{,}637 - 4{,}37 \cdot 0{,}6 + 27{,}823 \cdot 0{,}8 = 0; \quad -22{,}259 + 22{,}259 = 0.$

Diagram of longitudinal forces in a given statically redundant system are depicted in figure 4.10 *g*.

d) <u>*Node B*</u>

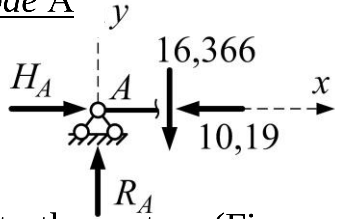


12. Cutting out the supporting nodes, can be gotten reactions at the supports:

$$\sum X = 0; \quad H_A = 10{,}19;$$

$$\sum Y = 0; \quad R_A = 16{,}366.$$

e) <u>*Node* A</u>



$\sum X = 0; \quad 27{,}82 \cdot 0{,}6 + 4{,}37 \cdot 0{,}8 - H_B = 0; \quad H_B = 20{,}19;$

$\sum Y = 0; \quad -27{,}82 \cdot 0{,}8 + 4{,}37 \cdot 0{,}6 + R_B = 0; \quad R_B = 19{,}63.$

Applying the external loads and reactions at the supports to the system (Figure 4.10 *h*), we can perform a static check of the balance of the frame:

$$\sum X = 0; \qquad 10{,}19 + 10 - 20{,}19 = 0; \qquad 20{,}19 - 20{,}19 = 0;$$

$$\sum Y = 0; \qquad 16{,}366 - 12 \cdot 3 + 19{,}63 = 0; \qquad 35{,}999 - 36 \approx 0; \qquad \sum M_C = 0;$$

$$12 \cdot 3 \cdot 1{,}5 - 10{,}19 \cdot 3 + 16{,}366 \cdot 1{,}5 - 19{,}63 \cdot 4{,}5 + 20{,}19 \cdot 2 = 0; \quad 118{,}929 - 118{,}905 \approx 0;$$

$$\text{Discrepancy } \left| \frac{118{,}929 - 118{,}905}{118{,}905} \right| \cdot 100\% = 0{,}02\% \text{ is insignificant.}$$

<u>**Example 2**</u> Consider the more complex frame shown in the figure 4.12 *a*.

1. Frame has two redundant connections:

$$r = 3C - H = 3 \cdot 2 - 4 = 2,$$

$$\text{or } \; r = -(3D - 2H - C_o) = -(3 \cdot 3 - 2 \cdot 3 - 5) = 2.$$

2. The accepted design primary system of force method is shown in the figure 4.12 *b*. P.S. possible variants here may also be frames presented in figure 4.11.

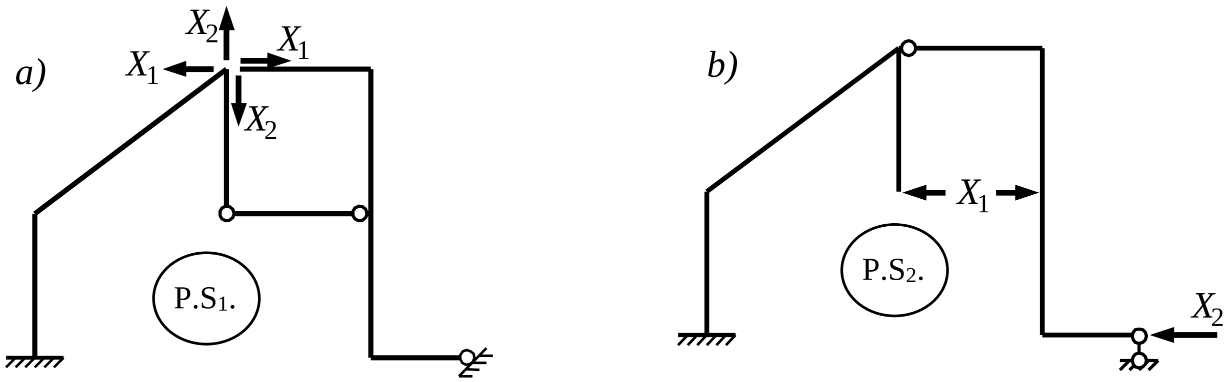
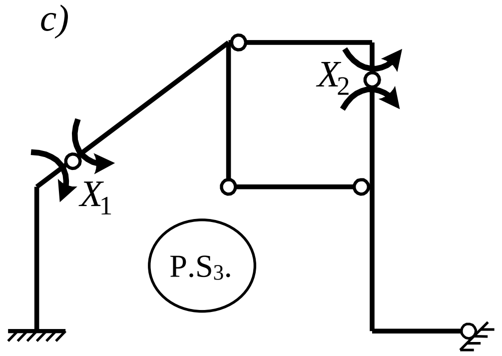


Figure 4.11



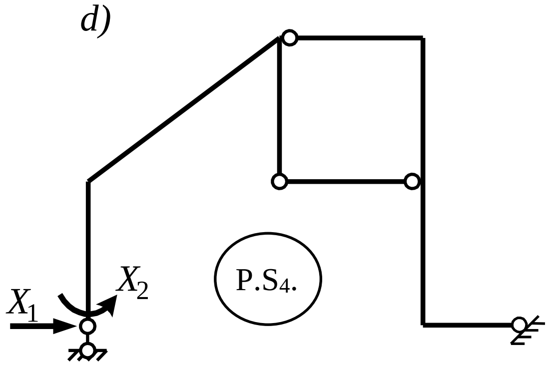


Figure 4.11 (continued)

3. The system of canonical equations of the force method here will have a view:

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P} = 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P} = 0. \end{cases}$$

<u>The physical meaning of equations:</u>

<u>1st equation</u> – is the angle of section rotation (angular displacement) in the rod above of *A* support (Figure 4.12*b*) from the forces X_1, X_2, and the given external load, which should be zero, as it is a cross-section in a given redundant frame rigidly attached to the base (the support at *A* point is a pinched);

<u>2nd equation</u> – represents a mutual divergence of D and K points from the unit forces X_1, X_2, and the external (applied) load, which should be zero, as these points are connected by a rod (by bracing), which is considered non-stretch (in the calculation neglect longitudinal deformations of bracing).

<u>The physical meaning of coefficients and terms:</u>

δ_{12} – angle of rotation (angular displacement) in the primary system of the cross-section of force method over the *A* support from the action of the X_2 unit force;

δ_{22} – mutual divergence of *D* and *K* points in the primary system from the action of X_2 unit force;

Δ_{2P} – reciprocal divergence of *D* and *K* points in the primary system from external loads.

4. Plot in the primary system the unit bending moment diagrams $\overline{M}_1$ and $\overline{M}_2$ from the action of unit unknowns X_1 and X_2. Plot of M_P diagram from the action of the applied external loads.

Note that the frame in the primary system is a three-hinged frame with supports in one level and the definition of reactions at the supports in it from any of the loads can be made, for example, from equations:

$$\sum M_A = 0; \qquad \sum M_B = 0; \qquad \sum M_C^{left} = 0; \qquad \sum M_C^{right} = 0,$$

and we can use equations to check them: $\sum X = 0;$ $\sum Y = 0.$

The $\overline{M}_1$, $\overline{M}_2$ and M_P diagrams are showed in figure 4.12 *c*÷4.12 *e*.

5. Calculating the coefficients of canonical equations:

a) ***Unit coefficients:***

$$\delta_{11} = \sum_1^N \int_0^l \frac{\overline{M}_1^2 dx}{EJ} = \frac{3}{6EJ}\left(1^2 + 4\cdot 0{,}875^2 + 0{,}75^2\right) + \frac{1}{EJ}\left(\frac{0{,}75\cdot 5}{2}\right)\cdot\frac{2}{3}0{,}75 +$$

$$+\frac{1}{EJ}\left(\frac{0{,}375\cdot 3}{2}\right)\cdot\frac{2}{3}0{,}375 + \frac{6}{6\cdot 2EJ}\left(0{,}375^2 + 4\cdot 0{,}125^2 + 0{,}125^2\right) +$$

$$+\frac{1}{EJ}\left(\frac{0{,}125\cdot 1}{2}\right)\cdot\frac{2}{3}0{,}125 = \frac{3{,}505}{EJ};$$

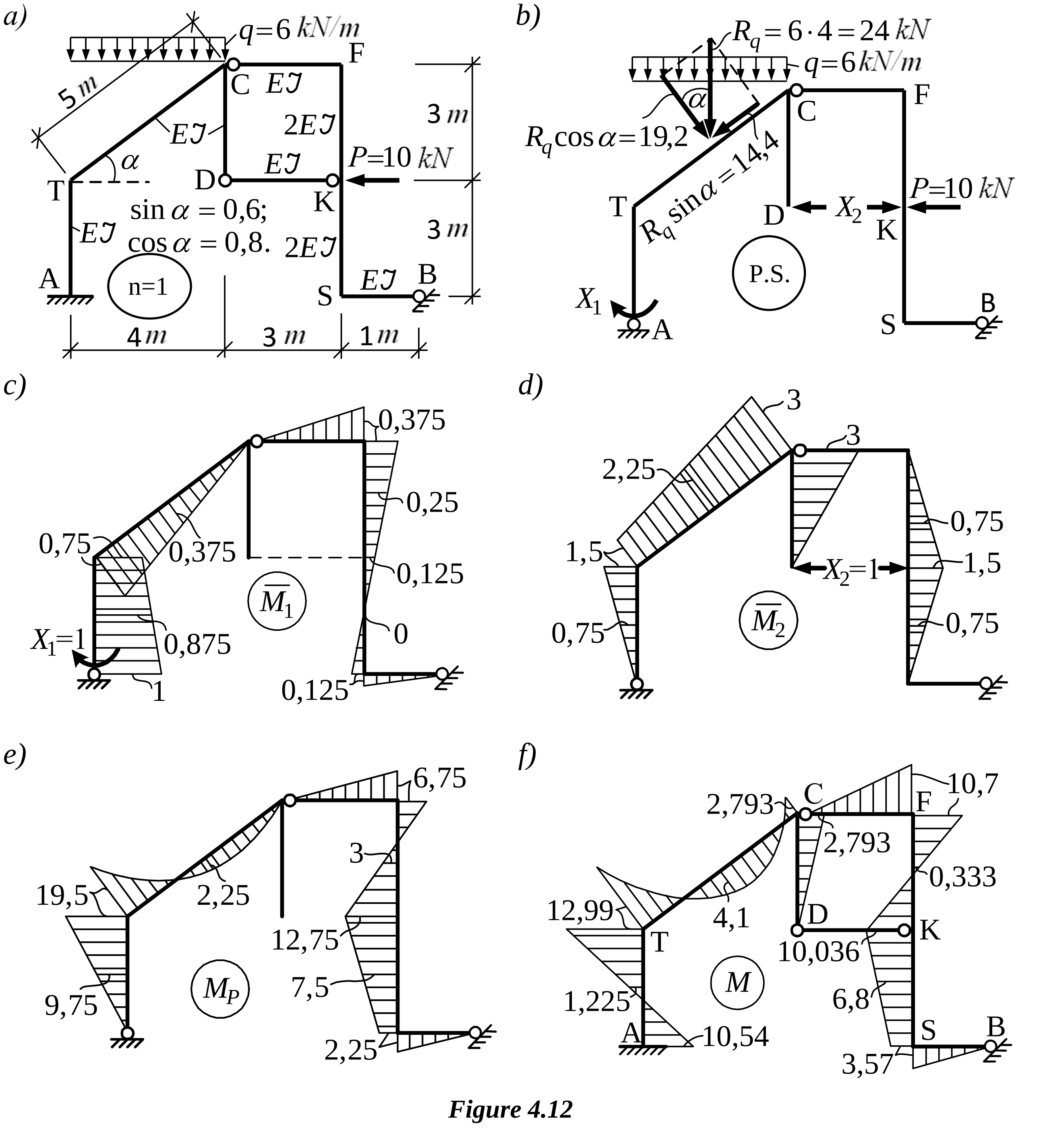


Figure 4.12

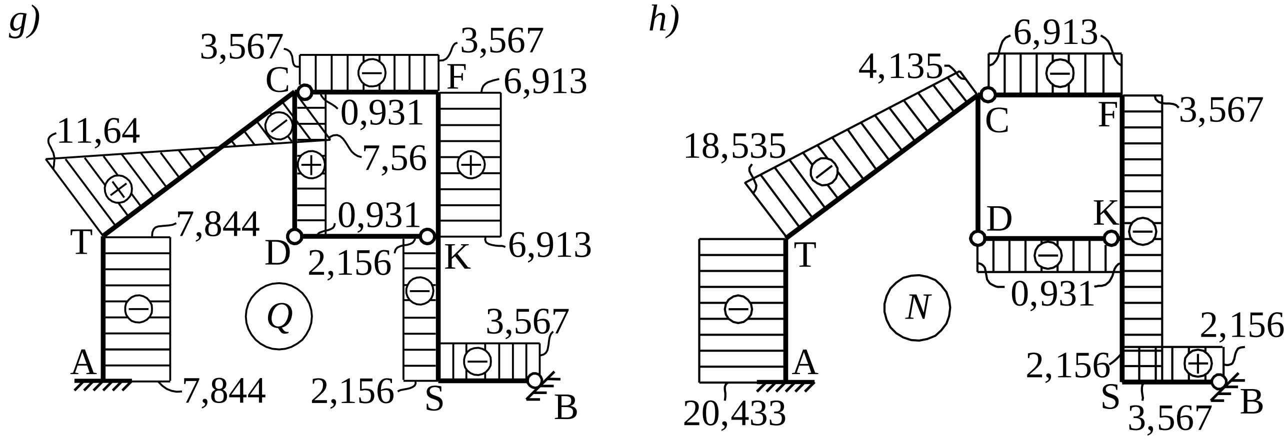


Figure 4.12 (continued)

$$\delta_{12}=\delta_{21}=\sum_{1}^{n}\int_{0}^{l}\frac{\overline{M}_1\overline{M}_2\,dx}{EJ}=\frac{3}{6EJ}\left(-1\cdot 0-4\cdot 0{,}875\cdot 0{,}75-0{,}75\cdot 1{,}5\right)+$$

$$+\frac{5}{6EJ}\left(-0{,}75\cdot 1{,}5-4\cdot 0{,}375\cdot 2{,}25-0\cdot 3\right)+\frac{3}{6\cdot 2EJ}\left(0{,}375\cdot 0+4\cdot 0{,}25\cdot 0{,}75+0{,}125\cdot 1{,}5\right)+$$

$$+\frac{3}{6\cdot 2EJ}\left(0{,}125\cdot 1{,}5+4\cdot 0\cdot 0{,}75+0{,}125\cdot 0\right)=-\frac{5{,}344}{EJ};$$

$$\delta_{22}=\sum_{1}^{n}\int_{0}^{l}\frac{\overline{M}_2^{\,2}dx}{EJ}=\frac{1}{EJ}\left(\frac{1{,}5\cdot 3}{2}\right)\cdot\frac{2}{3}1{,}5+\frac{5}{6EJ}\left(1{,}5^2+4\cdot 2{,}25^2+3^2\right)+\frac{1}{EJ}\left(\frac{3\cdot 3}{2}\right)\cdot\frac{2}{3}3+$$

$$+\frac{1}{2EJ}\left(\frac{1{,}5\cdot 3}{2}\right)\cdot\frac{2}{3}1{,}5\cdot 2=\frac{39{,}75}{EJ};$$

b) ***Absolute terms:***

$$\Delta_{1P}=\sum_{1}^{N}\int_{0}^{l}\frac{\overline{M}_1\,M_P\,dx}{EJ}=\frac{3}{6EJ}\left(1\cdot 0-4\cdot 0{,}875\cdot 9{,}75-0{,}75\cdot 19{,}5\right)+$$

$$+\frac{5}{6EJ}\left(-0{,}75\cdot 19{,}5+4\cdot 0{,}375\cdot 2{,}25+0\cdot 0\right)+\frac{1}{EJ}\left(\frac{0{,}375\cdot 3}{2}\right)\cdot\frac{2}{3}6{,}75+$$

$$+\frac{3}{6\cdot 2EJ}\left(0{,}375\cdot 6{,}75-4\cdot 0{,}25\cdot 3-0{,}125\cdot 12{,}75\right)+$$

$$+\frac{3}{6\cdot 2EJ}\left(-0{,}125\cdot 12{,}75+4\cdot 0\cdot 7{,}5+0{,}125\cdot 2{,}25\right)+\frac{1}{EJ}\left(\frac{0{,}125\cdot 1}{2}\right)\cdot\frac{2}{3}2{,}25=-\frac{31{,}972}{EJ};$$

$$\Delta_{2P}=\sum_{1}^{N}\int_{0}^{l}\frac{\overline{M}_2\,M_P\,dx}{EJ}=\frac{1}{EJ}\left(\frac{1{,}5\cdot 3}{2}\right)\cdot\frac{2}{3}19{,}5+\frac{5}{6EJ}\left(19{,}5\cdot 1{,}5-4\cdot 2{,}25\cdot 2{,}25+3\cdot 0\right)+$$

$$+\frac{3}{6\cdot 2EJ}\left(0\cdot 6{,}75-4\cdot 0{,}75\cdot 3-1{,}5\cdot 12{,}75\right)-\frac{3}{6\cdot 2EJ}\left(1{,}5\cdot 12{,}75+4\cdot 0{,}75\cdot 7{,}5+0\cdot 2{,}25\right)=$$

$$=\frac{19{,}313}{EJ}.$$

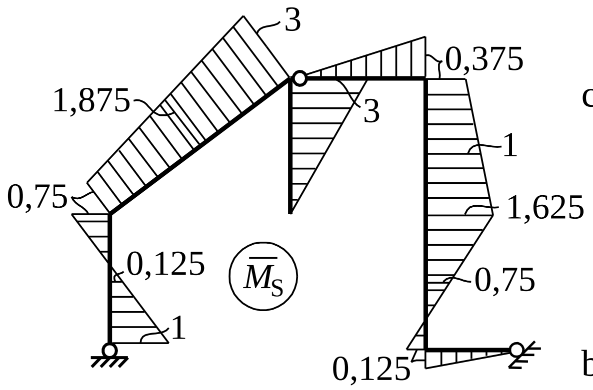


Figure 4.13

6. Checks correctly calculate unit coefficients and absolute terms:

a) ***universal verification:***

$$\sum_1^N \int_0^l \frac{\overline{M}_S^2 dx}{EJ} = \sum_{i=1}^2 \sum_{k=1}^2 \delta_{ik},$$

where: $\overline{M}_S = \overline{M}_1 + \overline{M}_2$ – a total unit bending moment diagram, having the view shown in the figure 4.13;

$$\sum_1^n \int_0^l \frac{\overline{M}_S^2 dx}{EJ} = \frac{3}{6EJ}\left(1^2 + 4\cdot 0{,}125^2 + 0{,}75^2\right) + \frac{5}{6EJ}\left(0{,}75^2 + 4\cdot 1{,}875^2 + 3^2\right) +$$

$$+\frac{1}{EJ}\left(\frac{3\cdot 3}{2}\right)\cdot\frac{2}{3}3 + \frac{1}{EJ}\left(\frac{0{,}375\cdot 3}{2}\right)\cdot\frac{2}{3}0{,}375 + \frac{3}{6\cdot 2EJ}\left(0{,}375^2 + 4\cdot 1^2 + 1{,}625^2\right) +$$

$$+\frac{3}{6\cdot 2EJ}\left(1{,}625^2 + 4\cdot 0{,}75^2 + 0{,}125^2\right) + \frac{1}{EJ}\left(\frac{0{,}125\cdot 1}{2}\right)\cdot\frac{2}{3}0{,}125 = \frac{32{,}568}{EJ};$$

$$\sum_{i=1}^2 \sum_{k=1}^2 \delta_{ik} = \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22} = \frac{3{,}505}{EJ} - \frac{5{,}344}{EJ}\cdot 2 + \frac{39{,}75}{EJ} = \frac{32{,}567}{EJ};$$

Check is done;

b) ***Column check:***

$$\sum_1^N \int_0^l \frac{\overline{M}_S M_P\, dx}{EJ} = \sum_{i=1}^2 \Delta_{iP}; \qquad \sum_{i=1}^2 \Delta_{iP} = \Delta_{1P} + \Delta_{2P} = -\frac{31{,}972}{EJ} + \frac{19{,}313}{EJ} = -\frac{12{,}659}{EJ};$$

$$\sum_1^N \int_0^l \frac{\overline{M}_S M_P\, dx}{EJ} = \frac{3}{6EJ}\left(1\cdot 0 - 4\cdot 0{,}125\cdot 9{,}75 + 0{,}75\cdot 19{,}5\right) +$$

$$+\frac{5}{6EJ}\left(0{,}75\cdot 19{,}5 - 4\cdot 1{,}875\cdot 2{,}25 + 3\cdot 0\right) + \frac{1}{EJ}\left(\frac{0{,}375\cdot 3}{2}\right)\cdot\frac{2}{3}6{,}75 +$$

$$+\frac{3}{6\cdot 2EJ}\left(0{,}375\cdot 6{,}75 - 4\cdot 1\cdot 3 - 1{,}625\cdot 12{,}75\right) +$$

$$+\frac{3}{6\cdot 2EJ}\left(-1{,}625\cdot 12{,}75 - 4\cdot 0{,}75\cdot 7{,}5 + 0{,}125\cdot 2{,}25\right) + \frac{1}{EJ}\left(\frac{0{,}125\cdot 1}{2}\right)\cdot\frac{2}{3}2{,}25 = -\frac{12{,}66}{EJ};$$

Check is done.

7. We solve the system of canonical equations:

$$\begin{cases} \dfrac{3{,}505}{EJ}X_1-\dfrac{5{,}344}{EJ}X_2-\dfrac{31{,}982}{EJ}=0; \\ -\dfrac{5{,}344}{EJ}X_1+\dfrac{39{,}75}{EJ}X_2+\dfrac{19{,}313}{EJ}=0. \end{cases}$$

Find: $X_1=10{,}54\ kNm;\quad X_2=0{,}931\ kN.$

We check the solution by substituting these values into equations:

$$\begin{cases} 3{,}505\cdot10{,}54-5{,}344\cdot0{,}931-31{,}972=0; \qquad 36{,}943-36{,}947\approx0; \\ -5{,}344\cdot10{,}54+39{,}75\cdot0{,}931+19{,}313=0; \qquad 56{,}325-56{,}321\approx0. \end{cases}$$

Discrepancies are insignificant

8. Plot the final diagram of bending moment in a given statically redundant system according to the formula:

$$M=\overline{M}_1\cdot X_1+\overline{M}_2\cdot X_2+M_P\,.$$

For the convenience of calculations, can be plotted intermediate diagrams separately $\overline{M}_1\cdot X_1$ and $\overline{M}_2\cdot X_2$ (see figure 4.14):

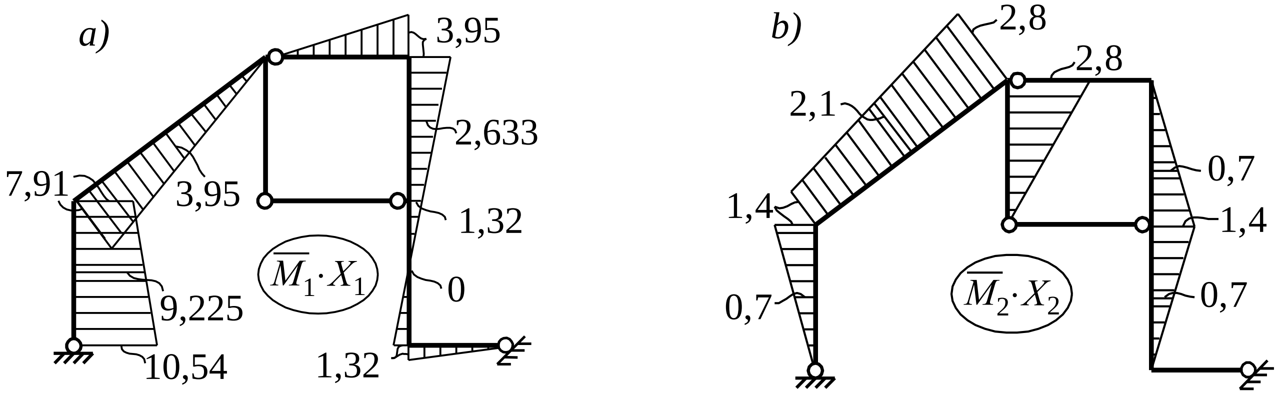


Figure 4.14

Adding together the ordinates of diagrams $\overline{M}_1\cdot X_1$, $\overline{M}_2\cdot X_2$ (figure 4.14) and M_P (figure 4.12 *e*), can be gotten the final diagram of bending moment in a given statically redundant system in the form depicted in the figure 4.12 *f*.

9. Deformation (kinematic) check of the diagram *M*:

$$\sum_1^N\int_0^l\frac{M\,\overline{M}_S\,dx}{EJ}=0;\qquad \frac{3}{6EJ}\big(1\cdot10{,}54-4\cdot1{,}225\cdot0{,}125+12{,}99\cdot0{,}75\big)+$$

$$+\frac{5}{6EJ}\big(12{,}99\cdot0{,}75-4\cdot4{,}10\cdot1{,}875+2{,}793\cdot3\big)+\frac{1}{EJ}\left(\frac{2{,}793\cdot3}{2}\right)\cdot\frac{2}{3}3+$$

$$+\frac{1}{EJ}\left(\frac{10{,}7\cdot3}{2}\right)\cdot\frac{2}{3}0{,}375+\frac{3}{6\cdot2EJ}\big(10{,}7\cdot0{,}375+4\cdot1\cdot0{,}333-10{,}036\cdot1{,}625\big)+$$

$$+\frac{3}{6\cdot 2EJ}\left(-10{,}036\cdot 1{,}625-4\cdot 6{,}8\cdot 0{,}75-3{,}57\cdot 0{,}125\right)+\frac{1}{EJ}\left(\frac{3{,}57\cdot 1}{2}\right)\cdot\frac{2}{3}0{,}125=$$

$$=\frac{1}{EJ}\left(9{,}835-10{,}524+8{,}379+4{,}013-2{,}741-9{,}066+0{,}149\right)=$$

$$=\frac{1}{EJ}\left(22{,}376-22{,}331\right)=\frac{0{,}045}{EJ}\approx 0;$$

Discrepancy $\frac{0{,}045}{22{,}331}\cdot 100\%=0{,}2\%<3\%$ is insignificant.

10. Diagram of shear force Q can be plotted on the diagram of bending moment by using formulas (4.13), (4.14).

At the same time, on the *TC* sloping section of the frame, the applied distributed load should be decomposed into components lengthways and perpendicularly to the rod. For this we must find a resulting force of applied distributed load R_q. At first we have to find $R_q=q\cdot l=6\cdot 4=24\,kN,$ which will then be divided into two components (see figure 4.12 *b*). If now the component, acting normally to the rod, divide by the length of the sloping rod (5m), can be gotten the intensity of uniformly distributed load $q_o\left(q_o=19{,}2/5=3{,}84kN/m\right)$ to the *TC* section, which acts perpendicularly to this section of the frame. After that may be able to calculate the ordinates of the Q_o diagram, included in the formula (4.13) – see Figure 4.15.

The values of the ordinates of Q graph at the edges of the TC section will then be equal:

$$Q_{TC}=Q_o+\left|\frac{M_{right}-M_{left}}{l}\right|=\pm\frac{3{,}84\cdot 5}{2}+\left|\frac{2{,}793-12{,}99}{5}\right|=\pm 9{,}6+2{,}04\ \ kN;$$

$$Q_{TC}^{left}=+9{,}6+2{,}04=11{,}64\ \ kN;\qquad Q_{TC}^{right}=-9{,}6+2{,}04=-7{,}56\ \ kN.$$

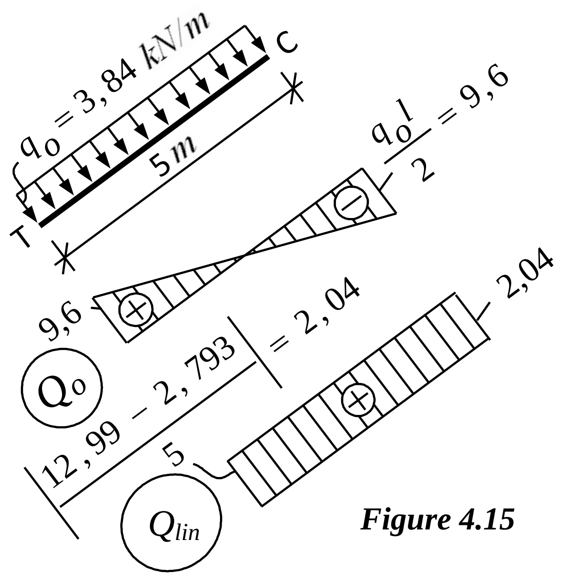


Figure 4.15

At the parts of linear change of the ***M*** diagram we get:

$$Q_{AT}=-\left|\frac{10{,}54-\left(-12{,}99\right)}{3}\right|=7{,}844\ kN;$$

$$Q_{CD}=+\left|\frac{0-2{,}793}{3}\right|=+0{,}931\,kN;$$

$$Q_{CF}=-\left|\frac{10{,}7-0}{3}\right|=-3{,}567\ kN;$$

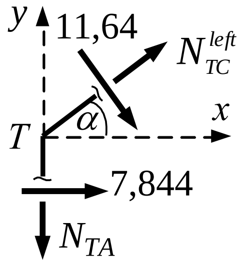
$$Q_{FK}=+\left|\frac{10{,}036-\left(-10{,}7\right)}{3}\right|=6{,}913\,kN;$$

$$Q_{KS} = -\left|\frac{3,57-10,036}{3}\right| = -2,156\ kN; \qquad Q_{SB} = -\left|\frac{0-3,57}{1}\right| = -3,567\ kN;$$

The diagram of the shear force in a given redundant frame is represented on the figure 4.12*g*.

11. Diagram of longitudinal force can be plotted a way of cutting out of nodes:

a) *Node T*



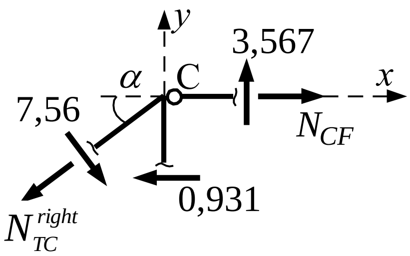
$$\sum X = 0; \qquad 7,844 + 11,64 \cdot 0,6 + N_{TC}^{left} \cdot 0,8 = 0;$$

$$N_{TC}^{left} = -18,535\,kN;$$

$$\sum Y = 0; \quad -N_{TA} - 11,64 \cdot 0,8 - (-18,535) \cdot 0,6 = 0;$$

$$N_{TA} = -20,433\ kN;$$

b) *Node C*



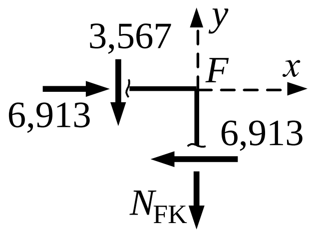
$$\sum Y = 0; \qquad -N_{TC}^{right} \cdot 0,6 - 7,56 \cdot 0,8 + 3,567 = 0;$$

$$N_{TC}^{right} = -4,135\,kN;$$

$$\sum X = 0; \; -(-4,135) \cdot 0,8 + 7,56 \cdot 0,6 - 0,931 + N_{CF} = 0;$$

$$N_{CF} = -6,913\,kN;$$

c) *Node F*



$$\sum X = 0; \qquad 6,913 - 6,913 = 0;$$

$$\sum Y = 0; \qquad -3,567 - N_{FK} = 0;$$

$$N_{FK} = 3,567\ kN.$$

Similarly are cut out of K and S nodes. The final diagram of longitudinal force in a given redundant frame is presented in figure 4.12*h*.

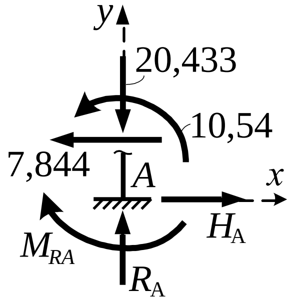
12. Cutting out the support nodes now and taking into account all kinds of forces in the support sections (*M*, *Q*, *N*), we can easily find the reactions at the *A* and *B* supports in nodes:

d) *Node A*

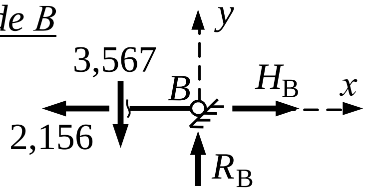
$$\sum X = 0; \quad H_A = 7,844\ kN;$$

$$\sum Y = 0; \quad R_A = 20,433\,kN;$$

$$\sum M_A = 0; \quad M_{RA} = 10,54\ kNm;$$



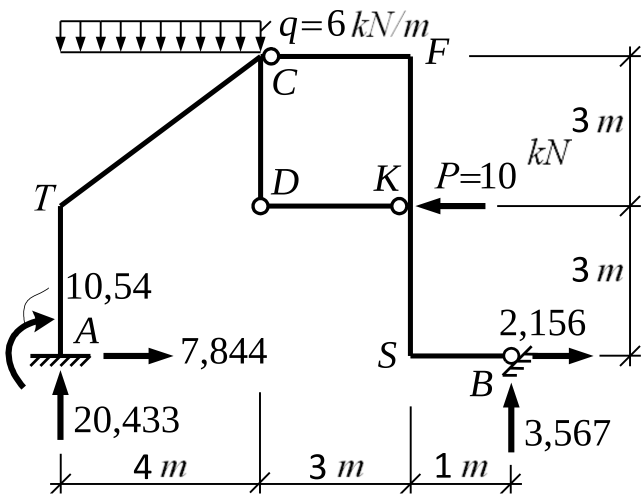
e) *Node B*



$$\sum X = 0; \quad H_B = 2,156\,kN;$$

$$\sum Y = 0; \quad R_B = 3,567\,kN.$$

13. After that, a static check of the balance of the frame has to be done:



$$\sum X = 0;$$
$$7{,}844 + 2{,}156 - 10 = 0;\ 10 - 10 = 0;$$
$$\sum Y = 0;$$
$$20{,}433 + 3{,}567 - 6 \cdot 4 = 0;\ 24 - 24 = 0;$$
$$\sum M_C = 0;$$
$$10{,}54 + 20{,}433 \cdot 4 - 7{,}844 \cdot 6 -$$
$$-(6 \cdot 4) \cdot 2 + 10 \cdot 3 - 2{,}156 \cdot 6 -$$
$$-3{,}567 \cdot 4 = 0; \quad 122{,}27 - 122{,}27 = 0.$$

4.9. Simplification in Calculations by the Force Method of Symmetrical Frames

4.9.1. General Concepts and Definitions

Symmetrical are frames that have symmetry w.r.t. a certain axis in the configuration of the rods, in the location and action of the support connections and in the rigidities of the rods.

Symmetrical frames will be distinguished between three kinds of diagrams of forces:

– Arbitrary diagrams;
– Symmetrical diagrams;
– Asymmetric diagrams.

We will call them **symmetrical** diagrams, where the axes of symmetry of the frame have symmetry on the ordinates of internal forces and deflections (for the diagram ***M*** on stretched fibers).

It should be noted that the symmetrical diagram of the shear force ***Q*** will have opposite signs in symmetrical sections (the physical action of the shear forces will be symmetrical, which is easy to check).

Asymmetric are called diagrams, where the axes of symmetry of the frame have symmetry in the magnitude of the ordinates of internal forces, but opposite in deflections.

If diagrams on one side of the axis of symmetry change deflections to opposite ones, these diagrams will become symmetrical.

Note that the asymmetric diagram of ***Q*** in symmetrical sections will have the same signs.

Applied load, forces (including unknowns of method of force) and impacts (influences), from which the symmetrical graphs of forces are obtained, will be called **symmetrical loads, forces and influences**.

Accordingly, the applied load, forces and influences, from which the asymmetric diagrams of forces are obtained, will be called **asymmetric (**oblique symmetrical**) loads, forces and influences**.

In the calculations of symmetrical frames by force method when selecting a certain type of primary systems – symmetrical primary systems, depending on the type of actual loading, a number of significant simplifications of calculations are possible. They are presented below.

4.9.2. Dividing the System of Equation into Two Independent Groups

Consider the symmetrical frame depicted in the figure 4.16, having four redundant connections:

$$r = 3L - h = 3 \cdot 2 - 2 = 4,$$

$$\text{or} \quad n = -(3D - 2h - C_o) = -(3 \cdot 2 - 2 \cdot 2 - 6) = 4\,.$$

If we choose for frame the primary systems of force method shown in the figure 4.16, the system of canonical equations of the force method will have a form:

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \delta_{14}X_4 + \Delta_{1P} = 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \delta_{24}X_4 + \Delta_{2P} = 0; \\ \delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \delta_{34}X_4 + \Delta_{3P} = 0; \\ \delta_{41}X_1 + \delta_{42}X_2 + \delta_{43}X_3 + \delta_{44}X_4 + \Delta_{4P} = 0. \end{cases} \tag{4.17}$$

If we choose the primary system in the form presented in the figure 4.16 *c*, the system of equations (21) can be significantly simplified. In this case (in these primary systems) unit bending moment diagrams will be plotted (see Figure 4.16 *d* and figure 4.16 *g*) and calculated one of coefficient of equations in the system (21):

$$\delta_{14} = \delta_{41} = \sum_1^n \int_0^l \frac{\overline{M}_1 \overline{M}_4 \, dx}{EJ} = \frac{1}{EJ}(a \cdot h) \cdot \frac{h + 2h}{2} - \frac{1}{EJ}(a \cdot h) \cdot \frac{h + 2h}{2} = 0.$$

The zero result (multiplication of diagrams was done by Vereshchagin's rule) is due to the fact that the $\overline{M}_1$ diagram is symmetric, and diagram $\overline{M}_4$ is asymmetric

And this result will always take place in such cases, i.e.

(!) Deflections received "by multiplying" according to Mohr's formula of symmetrical diagrams will always be zero.

In our case, respectively, zero will be the following unit coefficients:

$$\delta_{13} = \delta_{31} = 0; \qquad \delta_{23} = \delta_{32} = 0; \qquad \delta_{24} = \delta_{42} = 0.$$

Since the products of all these zero unit coefficients on the unknowns in the system of equations (4.17) will also give zero, the whole number of components in these equations will fall out, and as a result the system of equations (4.17) will essentially be reduced into two independent groups:

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P} = 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P} = 0; \end{cases} \qquad \begin{cases} \delta_{33}X_3 + \delta_{34}X_4 + \Delta_{3P} = 0; \\ \delta_{43}X_3 + \delta_{44}X_4 + \Delta_{4P} = 0. \end{cases} \tag{4.18}$$

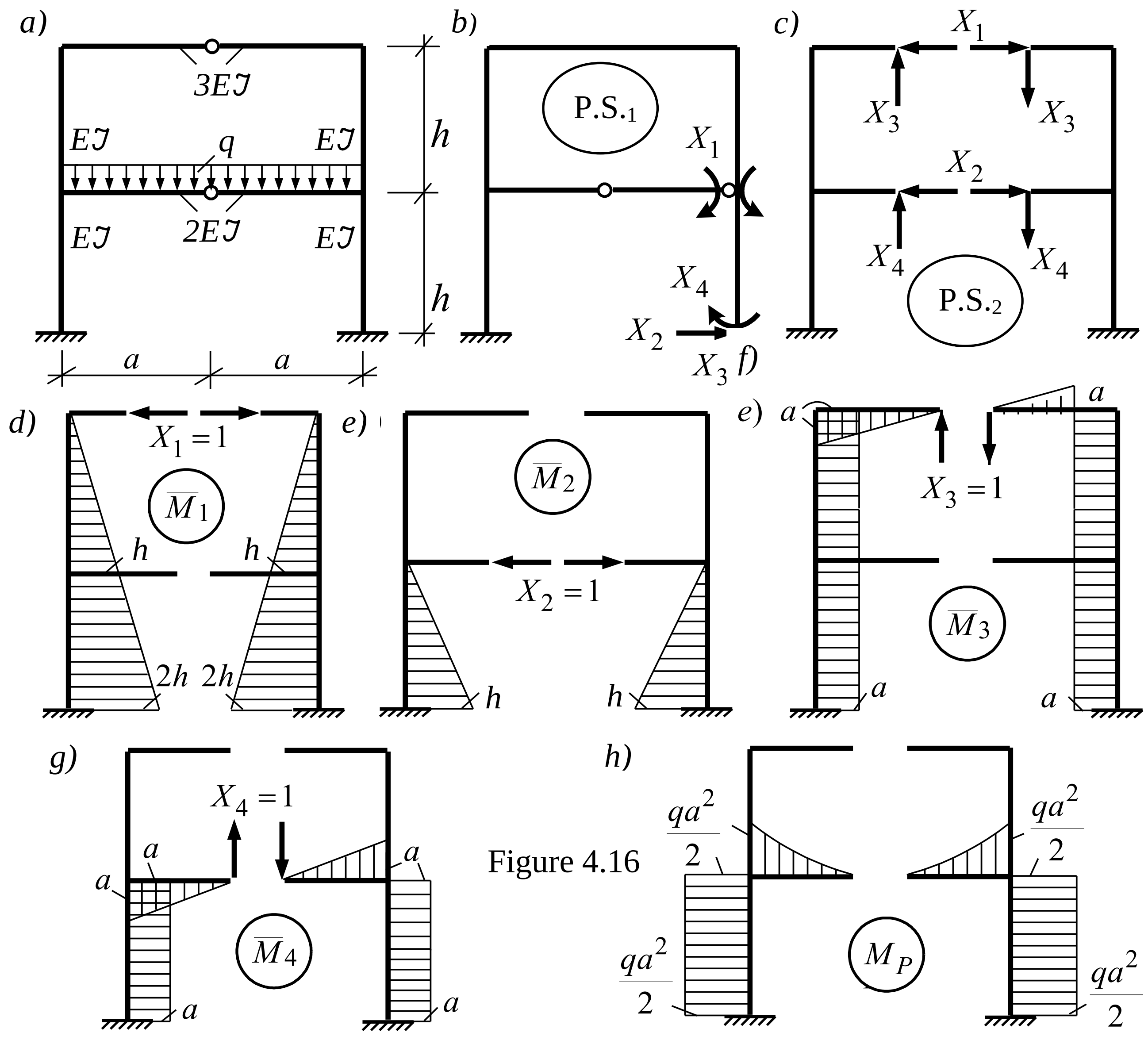


Figure 4.16

4.9.3. Simplifications of Symmetrical Frames with Symmetric (Asymmetric) Applied Loads

If the symmetrical redundant frame is loaded with a symmetric applied load, such as the frame on the figure 4.16 *a*, when choosing a symmetrical primary system (see, for example, Figure 4.16 *c*) the diagram from applied external loads $\boldsymbol{M_R}$ in the primary system will also be symmetrical (4.16 *c*).

This means (see the conclusions of the previous section) that absolute terms Δ_{3P} and Δ_{4P} will be zero, as received "by multiplying" of the symmetric diagram of $\boldsymbol{M_R}$ on the unit asymmetric ones $\overline{M}_3$ and $\overline{M}_4$. As a result, the second group (system) of equations in (4.18) becomes a homogeneous algebraic system of equations, and the solution to it will be zero values of unknowns ($X_3 = 0; X_4 = 0$). As a result, only symmetrical unknowns (in this case of X_1 and X_2) remain unknowns in the calculation.

Similar arguments can be held for the case of the loading of the symmetric frame with asymmetric applied load, and then the symmetric unknowns will turn to zero, and will remain only asymmetric unknowns.

<u>Conclusion</u>: When opting for a symmetrical redundant frame, the symmetrical primary system of the force method with symmetric and asymmetric unknowns and in the case of symmetric loading, all the asymmetric unknowns will be zero. In the case of an asymmetric loading, all symmetrical unknowns will be zero.

4.9.4. Grouping of unknowns

In some cases, when calculating symmetrical frames (e.g. frames with multiple spans) (see, for example, the frame depicted in figure 4.17 a), it is often difficult or even impossible to select a symmetrical P.S. in which the unknowns would immediately satisfy the symmetry conditions. Herewith primary system would be either symmetric or asymmetric. This can only be done when all redundant restrictions (connections) are discarded at points (sections) lying on the axis of symmetry of frame. In other cases, when choosing a symmetrical configuration of primary system, the unknowns of force method do not immediately satisfy the conditions of symmetry, these unknowns can be converted to symmetric and asymmetric. The basis for this transformation is that the received unknowns will act in symmetrical points (sections) and in symmetrical directions. This allows us separating such unknowns in a special way and the subsequent grouping of their parts to lead these unknowns to the satisfaction of the symmetry conditions. For example, for a frame on figure 4.17 *a*, (symmetrical w.r.t. vertical axis) having four redundant connections ($r = 3L - h = 3 \cdot 3 - 5 = 4$), the primary system of force method can be chosen in the form depicted in the figure 4.17 *b*, where the frame itself is symmetrical and unknowns – X_1', X_2' and X_3', X_4' are neither symmetric nor asymmetric. At the same time, these unknowns act in points (sections) and directions, symmetrical w.r.t. the axis of symmetry of the frame. Let's make a replace of such unknowns in accordance with dependencies which, mathematically, give a clear match of values (unambiguous correspondence) between the values of the left and the right of their parts of these dependencies, because the system of two equations has two unknowns.

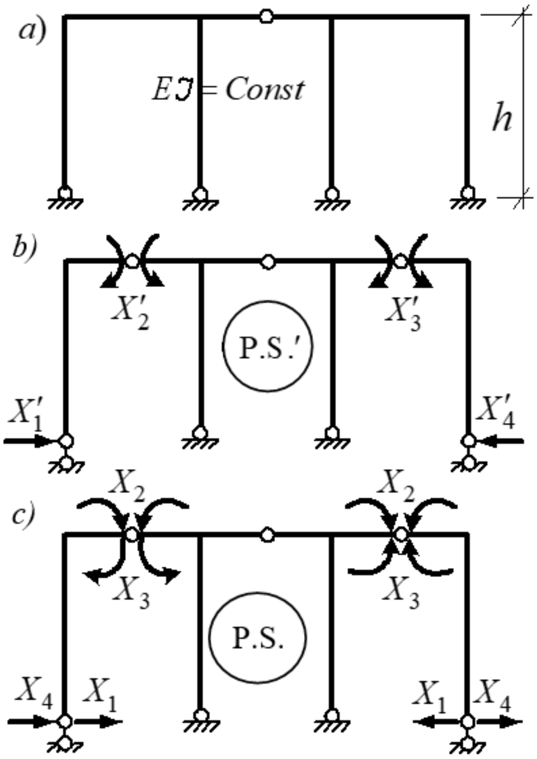


Figure 4.17

$$\begin{cases} X_1' = X_1 + X_4; \\ X_4' = X_1 - X_4; \end{cases} \quad \begin{cases} X_2' = X_2 + X_3; \\ X_3' = X_2 - X_3, \end{cases} \tag{4.19}$$

To grouping the eponymous unknowns on both sides of the axis of symmetry, we can get the primary system of force method, in which the unknowns will now be either symmetric or asymmetric.

In this case, the unknowns X_1 and X_2 are symmetrical, and the unknowns X_3 and X_4 are asymmetric. After such a transformation, called a **grouping of unknowns**, all of the above simplifications can be applied.

4.9.5. Decomposition of Applied Load on Symmetric and Asymmetric

Any applied load acting on a symmetrical system can be presented as a sum of symmetric and asymmetric loads. And it is done as follows:

1) a given applied load (see, for example, figure 4.18 a) are represented in the form of two identical halves (Figure 4.18 b);

2) At symmetrical points w.r.t. the axis of frame symmetry in relation to those in which the given applied load acts, we apply the same halves of loads (Figure 4.18 *b*), but only in different directions (Figure 4.18 *c*); the loads added in this way mutually annihilate
each other and thus they do not change the specified load;

3) After grouping these halves on one side and the other against the axis of symmetry of frame, we get the sum of symmetric loads (Figure 4.18 *d*) and asymmetric (Figure 4.18 *e*) loads.

Now for the considered frame (Figure 4.18 *a*), containing four redundant constraints if we choose a primary system of force method, (for example, in the form shown in the figure 4.18 *f*), then, in accordance with the stipulations above (simplifications of the calculation of symmetric frames) and the principle of independence of force, the calculation for the considered frame will be divided essentially in two calculations. Separately, the symmetric load can be calculated in which we will have a system of two types of equations.

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P}^{sym} = 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P}^{sym} = 0, \end{cases}$$

In which the bending moment diagram in a statically redundant system can be plotted by a formula:

$$M^{sym} = \overline{M}_1 X_1 + \overline{M}_2 X_2 + M_P^{sym}.$$

And the diagram of bending moment from asymmetric load can be plotted according to the formula:

$$M^{antisym} = \overline{M}_3 X_3 + \overline{M}_4 X_4 + M_P^{antisym}.$$

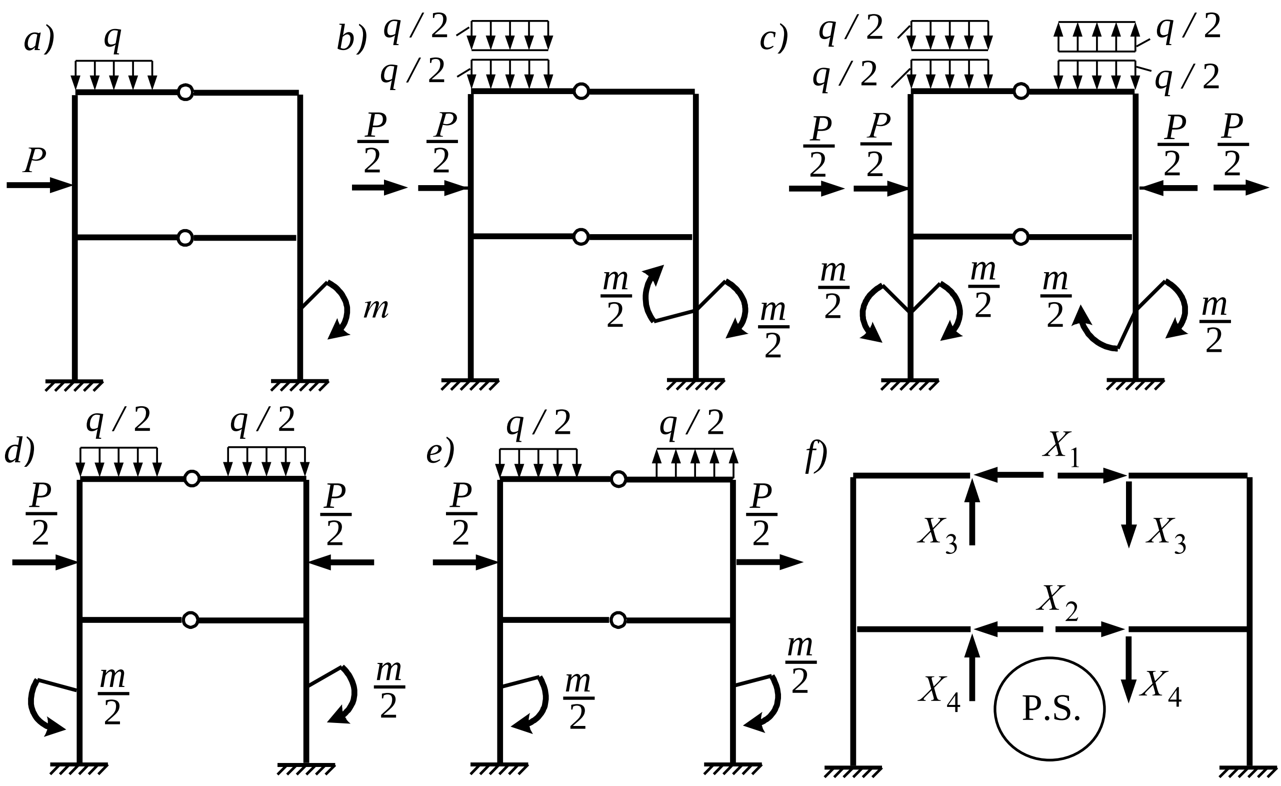


Figure 4.18

The diagram of bending moment corresponding to the original applied load, which is arbitrary in terms of symmetry, can eventually be obtained by the formula:

$$M = M^{sim} + M^{antisim}.$$

Analysis of the calculated procedure in comparison with the option without decomposing applied load shows that the simplification in the considered version associates with a reduction (approximately twice) in the volume of calculations of absolute terms of system of canonical equations of the force method. Considering that calculation of deflection due to applied load under the Morh's formula is usually the most laborious in comparison with the calculation of unit deflections (as far as plotting of $\boldsymbol{M_R}$ diagram from applied loads in most cases are much more difficult than unit diagram $\overline{M}_i$), the decomposition of applied load on symmetric and asymmetric often makes sense.

4.9.6. An Example of a Symmetrical Frame

Consider the frame presented in the figure 4.19 *a*. Despite the fact that at *A* point there is a hinged fixed supports, in the symmetrical frame w.r.t. the axis of symmetry of the *B* point is a hinged movable support. This frame in terms of considered classical form of force method can be named as symmetrical, so we do not take into account longitudinal deformations of rods. If we only have to take into consideration bending deformations, the frame will be symmetrical (point B, like point A, cannot move horizontally either). Meanwhile applied load on the frame is asymmetric.

1. The degree of static redundancy of the frame is equal to:

$$r = 3L - h = 3 \cdot 5 - 10 = 5,$$

i.e. the frame has five redundant connections.

2. The primary system of force method can be selected in the form shown in the figure 4.19 *b*, where unknowns X_2', X_3' and X_4' are fully satisfying the symmetry conditions, being either symmetric (X_3', X_4'), or asymmetric (X_2'), and the unknowns X_1' and X_5' do not satisfice the conditions of symmetry, but they act in symmetrical points and directions. Therefore, we can group by replacing them:

$$\begin{cases} X_1' = X_1 + X_5; \\ X_5' = X_1 - X_5. \end{cases}$$

To the given primary system, this replacement is taking into account, which is shown in the figure 4.19 *c*. Now the grouped unknown of X_1 will be symmetrical and the unknown of X_5 will be asymmetric. Given that the applied load is asymmetric, the symmetrical unknowns should be zero, i.e. $X_1 = 0$; $X_3 = 0$; $X_4 = 0$.

Given this simplification, for this frame only two unknowns X_2 and X_5 will remain in the primary system of force method (see figure 4.19 *d*).

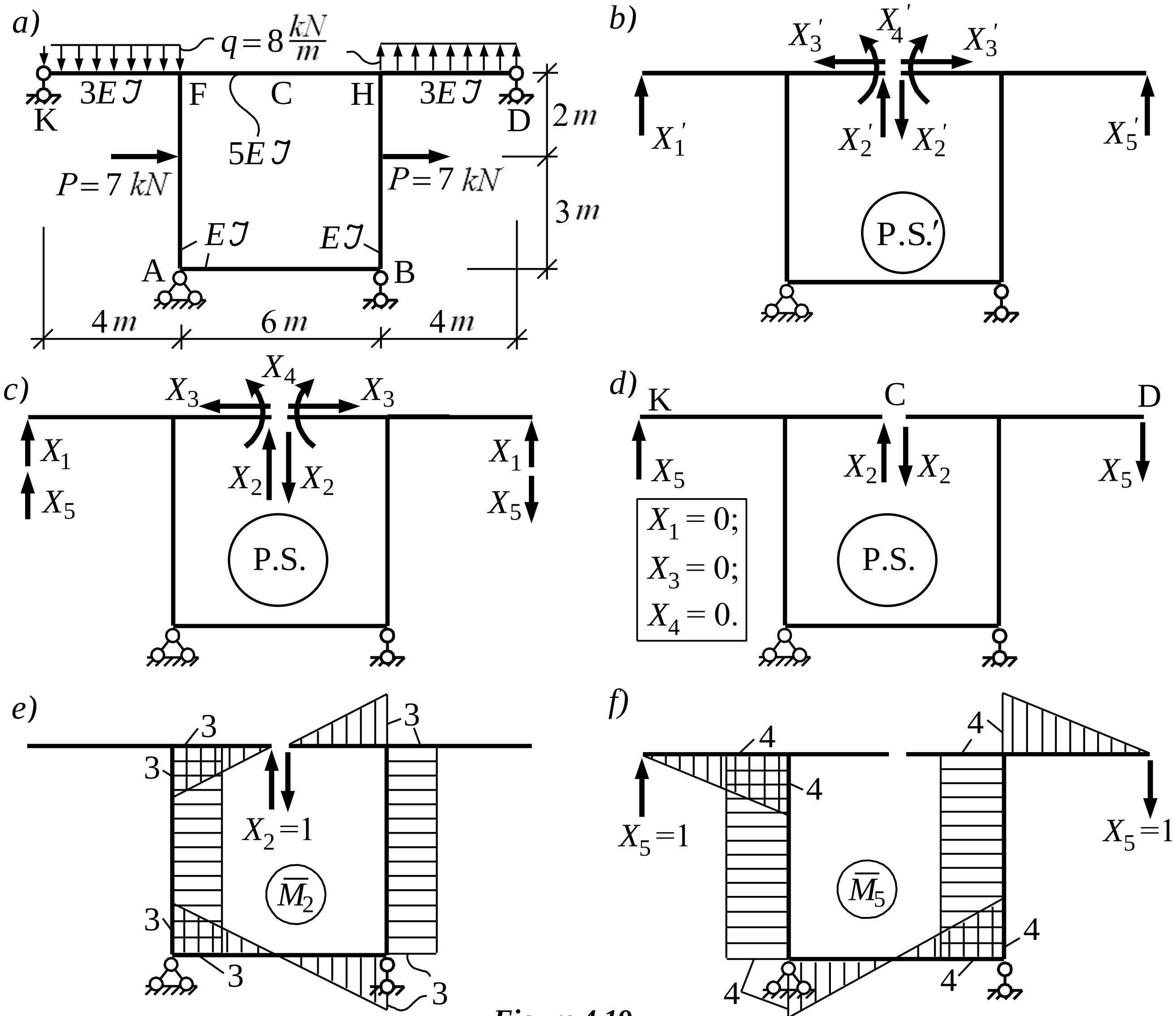


Figure 4.19

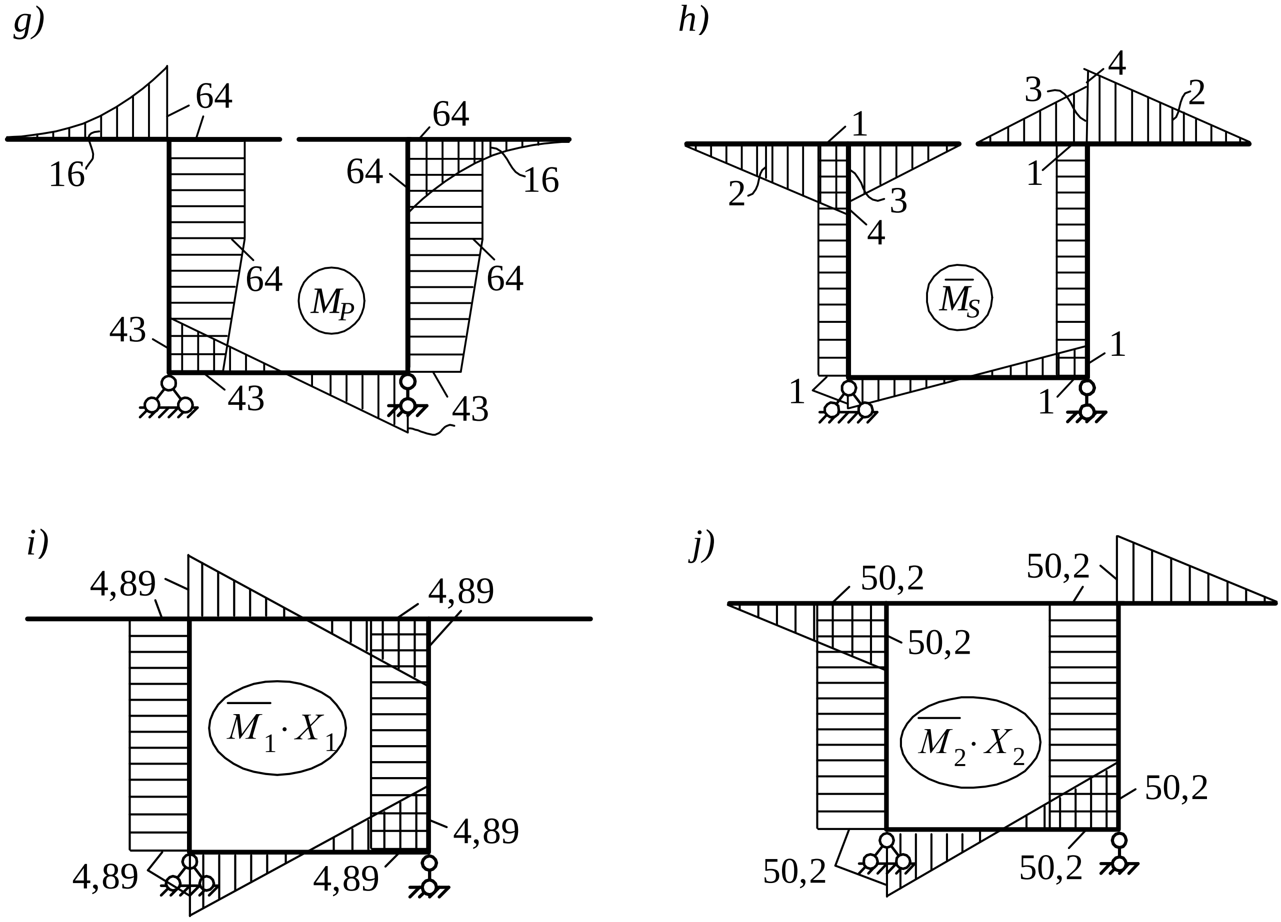


Figure 4.19 (continued)

3. And the system of canonical equations of the force method, based on the simplifications above, will have the appearance:

$$\begin{cases} \delta_{22} X_2 + \delta_{25} X_5 + \Delta_{2P} = 0; \\ \delta_{52} X_2 + \delta_{55} X_5 + \Delta_{5P} = 0. \end{cases} \tag{4.20}$$

<u>The physical meaning of equations:</u>

<u>1st equation</u> is a mutual divergence in the primary system left and right sections from the cut at *C* point (see Figure 4.19 *d*) vertically (from the unit forces X_2, X_5, and applied loads). Meanwhile the divergence should be zero, because in the original frame these sections are rigidly connected to each other and then will not be able to diverge;

<u>The 2nd equation</u> is a reciprocal divergence in primary system vertically in the *KD* line (*K* and *D* points, see Figure 4.19 *a*, *d*) from the action of unit forces X_2, X_5, and applied loads. Meanwhile the divergence should be zero, because at *K* and *D* points there are hinged fixed supports, fixing these points from vertical movements.

4. In the design primary system (Figure 4.19 *d*) we plot unit bending moment diagram from the action of unknowns of unit values $\overline{M}_2$, $\overline{M}_5$ and bending moment diagram from applied loads M_P, (this are presented respectively on figure 4.19 *e*, *f* and figure 4.19 *g*).

5. We calculate coefficients and absolute terms of equations:

a) ***coefficients (unit deflections):***

$$\delta_{22}=\sum_1^n\int_0^l\frac{\overline{M}_2^2\,dx}{EJ}=2\left[\frac{1}{5EJ}\left(\frac{3\cdot3}{2}\right)\cdot\frac{2}{3}3+\frac{1}{EJ}(3\cdot5)\cdot3+\frac{1}{EJ}\left(\frac{3\cdot3}{2}\right)\cdot\frac{2}{3}3\right]=\frac{111{,}6}{EJ};$$

$$\delta_{25}=\delta_{52}=\sum_1^n\int_0^l\frac{\overline{M}_2\overline{M}_5\,dx}{EJ}=2\left[-\frac{1}{EJ}(3\cdot5)\cdot4-\frac{1}{EJ}\left(\frac{3\cdot3}{2}\right)\cdot\frac{2}{3}4\right]=-\frac{144{,}0}{EJ};$$

$$\delta_{55}=\sum_1^n\int_0^l\frac{\overline{M}_5^2\,dx}{EJ}=2\left[\frac{1}{3EJ}\left(\frac{4\cdot4}{2}\right)\cdot\frac{2}{3}4+\frac{1}{EJ}(4\cdot5)\cdot4+\frac{1}{EJ}\left(\frac{4\cdot3}{2}\right)\cdot\frac{2}{3}4\right]=\frac{206{,}222}{EJ};$$

b) ***absolute terms (deflections from applied loads):***

$$\Delta_{2P}=\sum_1^n\int_0^l\frac{\overline{M}_2M_P\,dx}{EJ}=2\left[\frac{1}{EJ}(64\cdot2)\cdot3+\frac{1}{EJ}\left(\frac{64+43}{2}\cdot3\right)\cdot3+\frac{1}{EJ}\left(\frac{43\cdot3}{2}\right)\cdot\frac{2}{3}3\right]=\frac{1989{,}0}{EJ};$$

$$\Delta_{5P}=\sum_1^n\int_0^l\frac{\overline{M}_5M_P\,dx}{EJ}=2\left[\frac{4}{6\cdot3EJ}(-4\cdot16\cdot2-64\cdot4)-\frac{1}{EJ}(64\cdot2)\cdot4-\frac{1}{EJ}\left(\frac{64+43}{2}\cdot3\right)\cdot4-\right.$$

$$\left.-\frac{1}{EJ}\left(\frac{43\cdot3}{2}\right)\cdot\frac{2}{3}4\right]=-\frac{2822{,}667}{EJ}.$$

6. To perform the check of correctness of calculations of coefficients and absolute terms of the system's equations, need to plot a total unit bending moment diagram of $\overline{M}_S=\overline{M}_1+\overline{M}_2$ (see Figure 4.19 *h*), after which can be performed check:

a) universal verification: $\sum_1^n\int_0^l\frac{\overline{M}_S^2\,dx}{EJ}=\sum\sum\delta_{ik}$,

where: $\sum\sum\delta_{ik}=\delta_{22}+\delta_{25}+\delta_{52}+\delta_{55}=\frac{111{,}6}{EJ}-\frac{144}{EJ}\cdot2+\frac{206{,}222}{EJ}=\frac{29{,}822}{EJ};$

$$\sum_1^n\int_0^l\frac{\overline{M}_S^2\,dx}{EJ}=2\left[\frac{1}{3EJ}\left(\frac{4\cdot4}{2}\right)\cdot\frac{2}{3}4+\frac{1}{5EJ}\left(\frac{3\cdot3}{2}\right)\cdot\frac{2}{3}3+\frac{1}{EJ}(1\cdot5)\cdot1+\frac{1}{EJ}\left(\frac{1\cdot3}{2}\right)\cdot\frac{2}{3}1\right]=\frac{29{,}822}{EJ};$$

Check is done;

b) <u>column check:</u>

$$\sum_1^n \int_0^l \frac{\overline{M}_S M_P \, dx}{EJ} = \sum \Delta_{iP} \text{ , где: } \quad \sum \Delta_{iP} = \Delta_{2P} + \Delta_{5P} = \frac{1989{,}0}{EJ} - \frac{2822{,}667}{EJ} = -\frac{833{,}667}{EJ};$$

$$\sum_1^n \int_0^l \frac{\overline{M}_S M_P \, dx}{EJ} = 2\left[\frac{4}{6 \cdot 3EJ}(-4 \cdot 16 \cdot 2 - 64 \cdot 4) - \frac{1}{EJ}(64 \cdot 2) \cdot 1 - \frac{1}{EJ}\left(\frac{64+43}{2}\right) \cdot 3 \cdot 1 - \right.$$

$$\left. - \frac{1}{EJ}\left(\frac{43 \cdot 3}{2}\right) \cdot \frac{2}{3} 1\right] = -\frac{833{,}667}{EJ}; \text{ Check is done.}$$

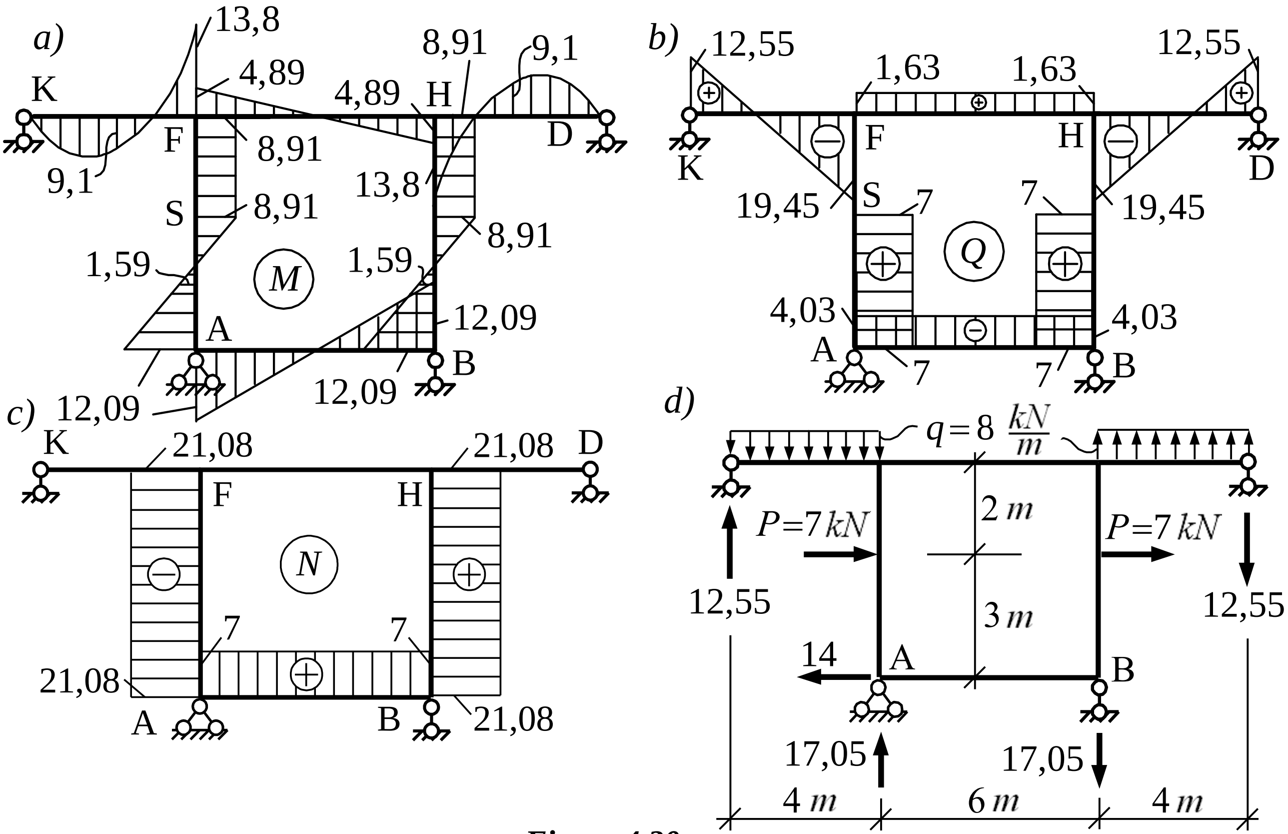


Figure 4.20

7. Solve the system of force method equations:

$$\begin{cases} \dfrac{111{,}6}{EJ} X_2 + \dfrac{144}{EJ} X_5 + \dfrac{1989}{EJ} = 0; \\ -\dfrac{144}{EJ} X_2 + \dfrac{206{,}222}{EJ} X_5 + \dfrac{2822{,}667}{EJ} = 0. \end{cases} \quad \text{Find:} \quad \begin{aligned} X_2 &= -1{,}63 \quad kN; \\ X_5 &= 12{,}55 \quad kN. \end{aligned}$$

Check the solution by substituting the found values into equations:

$$111{,}6 \cdot (-1{,}63) - 144 \cdot 12{,}55 + 1989{,}0 = 0; \qquad -1989{,}108 + 1989{,}0 \approx 0;$$

$$-144 \cdot (-1{,}63) + 206{,}222 \cdot 12{,}55 - 2822{,}667 = 0; \qquad 2822{,}806 - 2822{,}667 \approx 0.$$

Discrepancies are insignificant, check is carried out.

8. Plot the final graph of bending moment according to the formula:

$$M = \overline{M}_2 \cdot X_2 + \overline{M}_5 \cdot X_5 + M_P \ .$$

Separately, are showed intermediate (the multiplied) unit diagrams (see. Figure 4.19 *i, j*). The final diagram *M* is presented in the figure 4.20 *a*.

9. Perform a deformation test of the diagram of *M*:

$$\sum_1^N \int_0^l \frac{M \cdot \overline{M}_S \, dx}{EJ} = 0; \quad 2\left[\frac{4}{6 \cdot 3EJ}(4 \cdot 9{,}1 \cdot 2 - 13{,}8 \cdot 4) - \frac{1}{5EJ}\left(\frac{4{,}89 \cdot 3}{2}\right) \cdot \frac{2}{3} 3 - \frac{1}{EJ}(8{,}91 \cdot 2) \cdot 1 + \right.$$

$$\left. + \frac{1}{EJ}(1 \cdot 3) \cdot 1{,}59 + \frac{1}{EJ}\left(\frac{12{,}09 \cdot 3}{2}\right) \cdot \frac{2}{3} 1\right] = \frac{1}{EJ}(41{,}542 - 41{,}508) \approx 0;$$

Discrepancy $\left|\frac{41{,}542 - 41{,}508}{41{,}508}\right| \cdot 100\% = 0{,}08\% < 3\%$ is insignificant.

10. Diagram of *Q* shear force is plotted on the base of the *M* bending moment diagram using formulas (4.13) и (4.14):

$$Q_{KF} = Q_o - \left|\frac{M_{right} - M_{left}}{l}\right| = \pm \frac{8 \cdot 4}{2} - \left|\frac{-13{,}8 - 0}{4}\right| = \pm 16 - 3{,}45 \, kN;$$

$$Q_{KF}^{left} = +16 - 3{,}45 = +12{,}55 \, kN; \qquad Q_{KF}^{right} = -16 - 3{,}45 = -19{,}45 \, kN;$$

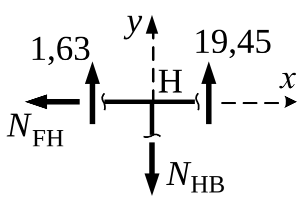
$$Q_{FH} = +\left|\frac{4{,}89 - (-4{,}89)}{6}\right| = 1{,}63 \ kN; \qquad Q_{FS} = \left|\frac{8{,}91 - 8{,}91}{2}\right| = 0 \, kN;$$

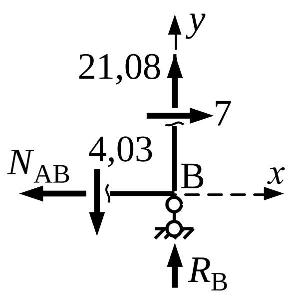
$$Q_{SA} = +\left|\frac{12{,}09 - (-8{,}91)}{3}\right| = +7 \ kN; \qquad Q_{AA} = -\left|\frac{-12{,}09 - 12{,}09}{6}\right| = -4{,}03 \, kN.$$

The shear force diagram is shown in the figure 4.20 *b*.

11. Diagram of longitudinal force in a given statically redundant system is plotted on the base of diagram – the way of cutting the node:

a) Node H *b) Node B*





$$\sum X = 0; \quad N_{FN} = 0; \qquad \sum X = 0; \quad N_{AB} = +7 \ kN;$$

$$\sum X = 0; \quad N_{HB} = +21{,}08 \ kN; \qquad \sum X = 0; \quad R_B = -17{,}05 \ kN.$$

F and *A* nodes are cut in the same way.

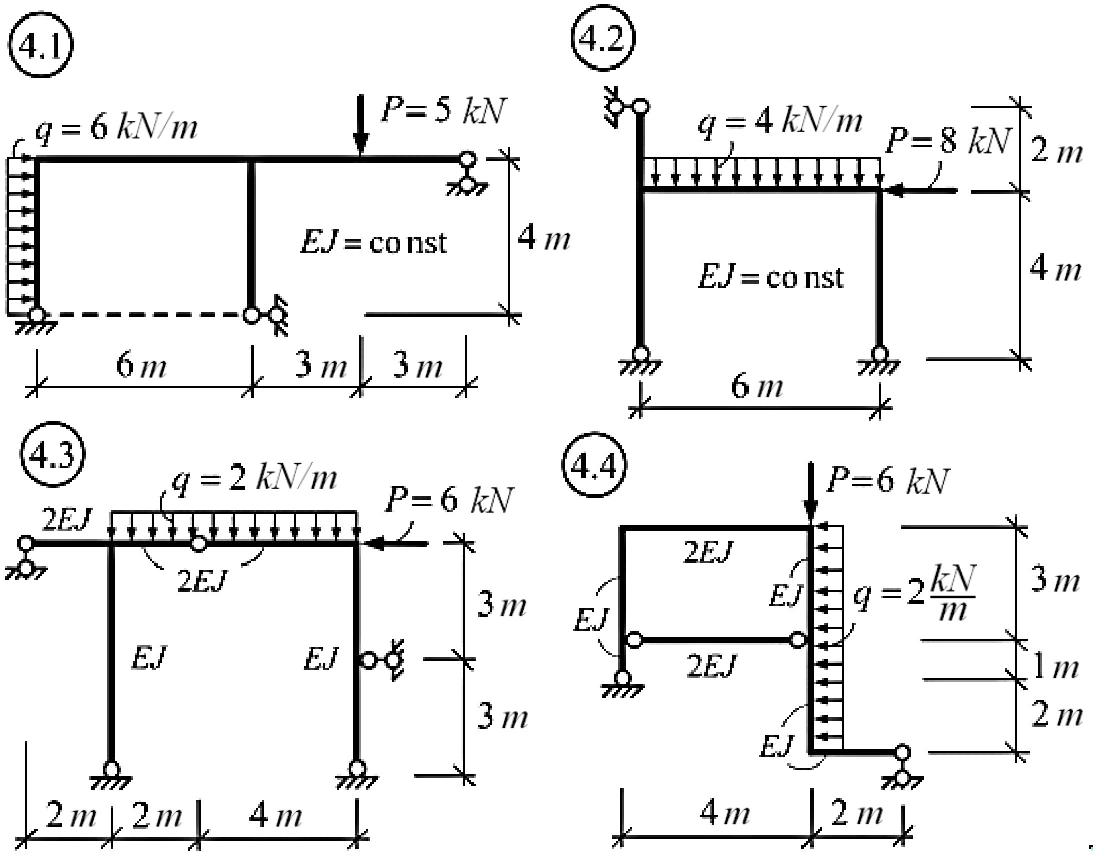
N diagram is shown in the figure 4.20 *b*.

12. Static frame balance check is in a Figure 4.20 *d*:

$$\sum X = 0;\quad 7+7-14=0;\quad 14-14=0;$$
$$\sum Y = 0;\quad 12{,}55+17{,}05-8\cdot 4-17{,}05+8\cdot 4-12{,}55=0;$$
$$\sum M_A = 0;\quad (7+7)\cdot 3+12{,}55\cdot 4+12{,}55\cdot 10+17{,}05\cdot 6-(8\cdot 4)\cdot 2-8\cdot 4(6+2)=0;$$
$$320-320=0.$$

4.10. Tasks to Solve Yourself

Plot in the below represented frames the diagrams of bending moment, shear force and longitudinal force, performing their calculation by force method.



Answers to these tasks are presented at the end of the manual in the "Key to Tasks for Self-Solution" section (p. 64-67).

5. Calculation of Redundant Frames by Slope Deflection Method

5.1. Approaches and Assumptions that Underlie the Slope Deflection Method

In calculating statically redundant systems by force method for the unknown forces are accepted forces in redundant restrictions, after which internal forces (M, Q, N) and the deflections of frame points can easily be found in the determined sections.

But the problem can be solved in the opposite direction. If first determine the deformed type of frame (deflections of sections of frame), then it is possible to establish the corresponding distribution of internal forces, which illustrates the well-known dependence:

$$M = EJ \cdot y''.$$

This is the approach used in the slope and deflection method (deflection method). At the same time, the analysis shows that the deformed view of the system is fully defined if the angles of twist (angular rotations) and linear displacements of its nodes are known. This is because the deflections of nodes are equal to the deflections of the ends of the rods connecting in these nodes, and the deformations of the rods are fully and unequivocally determined by the movements of the ends of the rods. The latter also applies to loaded rods, for which, however, the deformed form will depend additionally on the loads on them.

Consider, for example, the frame depicted in the pic. 5.1 *a*. The deformed state of this frame is determined by linear displacement and twist of nodes 1 and 2. The number of these deflections and twists or rotations depends from the assumptions used in the method. Thus, in general, the number of deflections and angles of twist in the frame (Figure 5.1 *b*) is five; in the case of neglecting of shear and longitudinal deformations and ignoring of changing in the length of rods when they bend (the effect of these values for the curved frame-rod systems due to their smallness is usually neglected – the classical formulation of a problem) the number of these unknowns equal only two (Figure 5.1 *c*) Δ and φ.

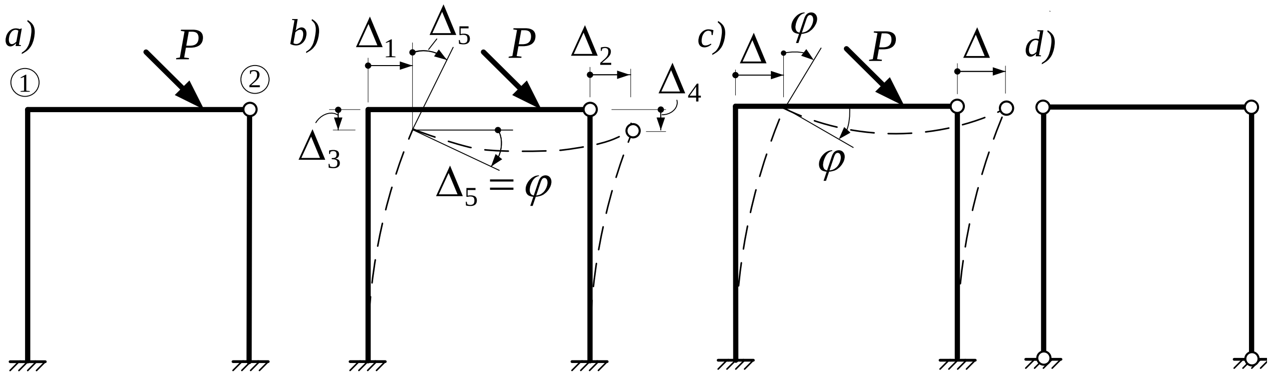


Figure 5.1

The number of independent angular and linear shifts of nodes completely and unequivocally defined by the deformed type of system is called ***the degree of its instability.***

All these independent nodes rotations and linear deflections are accepted as unknowns in the slope and deflection method. Hence the name of the method briefly is method of deflection.

Note that in the classical form of the deflection method, as well as in the force method, the following assumptions are used:

a) neglected shear and longitudinal deformations of elements of the system when it is deformed;

b) Is accepted that the projection of the curved (deformed) rod on its original direction is equal to the original length of the rod (Figure 5.2);

c) Is assumed that the value of angles in rigid node do not change during the system's deformation process;

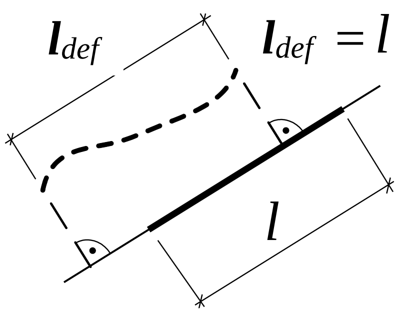


Figure 5.2

d) Angles of rotation of nodes and sections of rods at deformation of systems due to their small value are taken equal to tangents of these angles.

It should be noted that for a number of redundant frames, the degree of instability (kinematic instability) is lower than the degree of static redundancy, and that the deflection method with an equal number of unknowns is somewhat easier to calculate some redundant systems than using the method of force.

5.2. Determining the Degree of Instability of Frames

The degree of instability of the system, i.e. the number of unknown independent angular and linear shifts of the nodes, must be found in order to determine the *conjugate redundant system* (C.S.), in according with the formula:

$$n=n_a+n_l \tag{5.1}$$

Here: n_a – the number of independent angular twisting of nodes (unknowns of slope and deflection method), determined by the number of rigid nodes in the structure; at the same time, under the rigid nodes of the deflection method (then we will call them simply – rigid nodes) here we will understand those in which two conditions are met:

– in which two or more rods are rigidly connected at any angle.

– where happens change of bending moment, which cannot be determined from the usual equilibrium equations (statics).

For example, in a frame depicted in a figure 5.3 *a*, the rigid nodes where is met these conditions will be nodes: 1, 2, 3 ($n_a = 3$); for nodes: A, B, C, the second condition is not met, as they relate essentially to statically determinate parts of the structure, in which all internal forces can be determined by conventional method of calculating statically determinate systems;

n_l – the number of independent linear deflections of system's nodes (unknowns), which can be determined in two ways:

1. According to the number of possible independent linear displacements of rigid nodes (see definition of n_a), and hinged nodes of the structure based on the analysis of its possible elastic deformation taking into account some accepted assumptions (section 5.1).

For example, in a frame on figure 5.3 *a*, we need to analyze the possibility and independence of deflections of 1, 2, 3 rigid nodes and 4, 5 hinge nodes with arbitrary possible elastic deformation of this system:

1 node cannot move vertically, as it is fixed from vertical displacement with the help of the 1–D rod, which on the basis of ***a*** and ***b*** assumptions (section 5.1) not allow to diverge (or converge) of 1 and D points, and D pinched support;

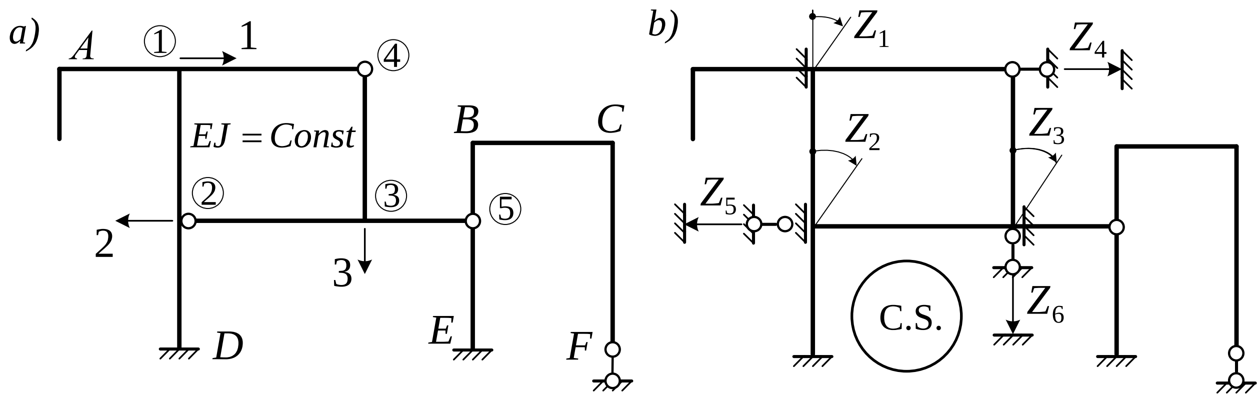


Figure 5.3

– horizontally, node of 1 can shift, as well as the rods of 1 to 2 and 4 to 3 can bend (horizontal displacement of 1 node is shown by an arrow with the number – 1);

–The 2 node cannot be shifted vertically based on the same reasoning as has been considered for node of 1;

–Horizontally, the same 2 node can shift, in respect that the rods: 1-2, 2-D, 4-3 and 5-E can bend (horizontal displacement of 2 node is marked by an arrow with the number – 2);

–The 3 node can move horizontally to the right or to the left (similar to 2 node). Moreover this shift will be the same as the horizontal displacement of 2 node, since the 2-3 rod connecting the nodes of 2 and 3, (based on ***a*** and ***b*** assumptions, section 5.1) does not allow change the distance between these nodes;

– On the vertical direction the 3 node, considering the possibility of elastic deformation of rods: 1-4, 2-3 and 3-5, can shift (shown by an arrow with the number – 3);

–The 4 hinge node in this case can move both horizontally rightward or leftward, the same as 1 node since the rod 1-4 does not allow for 1 and 4 sections to diverge (this is based on ***a*** and ***b*** assumptions (section 5.1)), and vertically but along with the node of 3 (due to the presence of 3-4 rod);

–The 5 hinge node can't move nor vertically, neither horizontally, but it will shift to the same direction as nodes of 2 and 3.

So for the frame on the figure 5.3 exists three independent deflections of nodes ($n_l = 3$), and in total we get six unknowns when calculating the frame by the deflection method:

$$n = n_a + n_l = 3 + 3 = 6.$$

2. A number of textbooks are proposing another formula to determine the $\boldsymbol{n_l}$:

$$n_l = W_{hinge\ frame\ scheme} = 3D - 2H - C_O, \tag{5.2}$$

According to which n_l equals the degree of freedom of the hinge scheme of the frame obtained by the introduction (cutting-in) of hinges in all rigid nodes of the structure, including pinched support.

For example, for a frame on figure 5.1 *a*, the hinge scheme of frame has the appearance presented on the figure 5.1 *g*, according to which we will receive:

$$n_l = W_{h.f.s.} = 3D - 2H - C_O = 3\cdot 3 - 2\cdot 2 - 4 = 1. \qquad (4.3)$$

For the frame of on figure 5.4 *a*, the hinge scheme of which is presented on the figure 5.4 *b*, we'll have:

$$n_l = W_{h.f.s.} = 3\cdot 4 - 2\cdot 3 - 6 = 0.$$

However, analysis of the possible frame deformation in the first way shows that 1 and 2 nodes can shift horizontally (figure 5.4*a*). Thus, application of formula (5.2) leads here to an incorrect result caused by instantaneous variation (variability) of the hinged scheme of the considered frame (figure 5.4*b*).

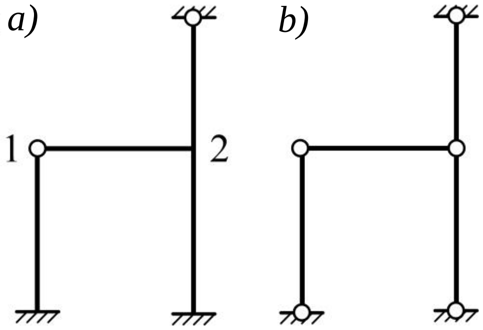


Figure 5.4

This situation can occur frequently, and the formula (5.2) will always give incorrect results in cases when hinge scheme creates an instantaneously variable system in the frame. Taken this into consideration, it is not recommended to use this version of calculation of n_l, but is suggested in all cases to use the first option of the definition of n_l, which is both simple and reliable.

5.3. Conjugate System of Redundant Structure of the Slope and Deflection Method

The conjugate system of the slope and deflection method can be gotten by introduction of additional (imaginary) restrictions, fixing the nodes from their possible angular twist and linear deflection, which were derived earlier (in computing the ***n*** degree of instability of the system). Thus, in all rigid nodes that can rotate (n_l), we install additional pinched supports, fixing them from the rotation, and in all nodes (hinged), which can linearly shift (n_l), we install additional supporting bars (which work as hinged movable supports) fixing from these shifts. These additional connections (together, of course, with nodes) are accepted as unknowns. Possible movements of these nodes (where were placed additional supports) are marked through Z_i (i = 1...n). It should be noted that the additional pinched support, unlike the actual pinched support, has only one restriction that secures the node from the rotation (from linear deflection it does not fix).

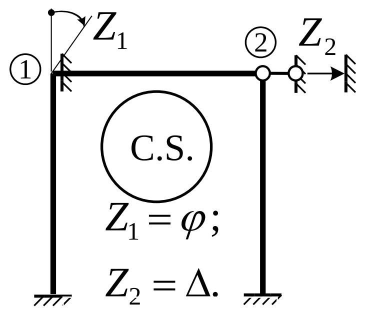


Figure 5.5

For the frame depicted in the figure 5.1 *a*, the conjugate system of the deflection method has the appearance presented on figure 5.5.

Based on accepted assumptions and established additional supports, the nodes of the conjugate system will be stationary. Given that in the classical form of the slope and deflection method can be neglected by shear and normal forces, in conjugate system further we will build only the diagram of bending moment. At the same time, the impact on the areas of the conjugate system (external applied loads, forced displacement of nodes) will cause bending moments only in those areas that are directly exposed to these impacts; namely, through the pinched support in rigid nodes and through the hinge nodes of the conjugate system bending impact will not be transmitted.

(!) Thus, the conjugate system of the deflection method will be a set of individual one-span beams independent of each other. These beams, depending on the conditions for attaching their ends to the nodes (support conditions) can be three types, the kinds of which are represented on the figure 5.6.

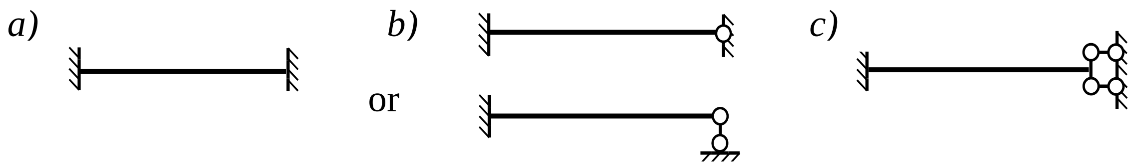


Figure 5.6

For the frame on the figure 5.3 *a* the conjugate system of the deflection method is represented by figure 5.3 *b*.

5.4. Canonical Equations of the Slope and Deflection Method

The calculation of frames by use of deflection method uses the conjugate system of this method. Extra supports are installed to nodes of frame that can shift; they need to restrict these shifts. At the same time, the conjugate system should work in the same way as the original system, in which there are no pointed out additional connections. As conditions equating the work of the conjugate system to the work of a given system, equality of zero reactions in additional supports is accepted, because they are not in the original system. For example, for a frame on figure 5.5 should be written down the reactions $R_1=0$, $R_2=0$. Given that reactions in the conjugate system arise from applied load and forced displacement of nodes, we will get:

$$R_1(Z_1, Z_2, P) = 0, \qquad R_2(Z_1, Z_2, P) = 0.$$

Taking advantage of the principle of independence of forces and external actions (displacement), these expressions can be presented as:

$$\begin{cases} R_1(Z_1) + R_1(Z_2) + R_1(P) = 0; \\ R_2(Z_1) + R_2(Z_2) + R_2(P) = 0, \end{cases} \quad \text{or} \quad \begin{cases} R_{1Z_1} + R_{1Z_2} + R_{1P} = 0; \\ R_{2Z_1} + R_{2Z_2} + R_{2P} = 0. \end{cases}$$

Since the Z_1 and Z_2 movements are not known, the reactions from their actions are expressed through appropriate single reactions $R_{iZ_k} = r_{ik} Z_k$.

As a result, we get the next system of equations:

$$\begin{cases} r_{11} Z_1 + r_{12} Z_2 + R_{1P} = 0; \\ r_{21} Z_1 + r_{22} Z_2 + R_{2P} = 0. \end{cases}$$

This form of recording of deflection method equations is called – canonical. In general, the system of canonical equations of the slope and deflection method has the form of:

$$\begin{cases} r_{11}Z_1 + r_{12}Z_2 + r_{13}Z_3 + ... + r_{1k}Z_k + ... + r_{1n}Z_n + R_{1P} = 0; \\ r_{21}Z_1 + r_{22}Z_2 + r_{23}Z_3 + ... + r_{2k}Z_k + ... + r_{2n}Z_n + R_{2P} = 0; \\ ------------------------- \\ r_{i1}Z_1 + r_{i2}Z_2 + r_{i3}Z_3 + ... + r_{ik}Z_k + ... + r_{in}Z_n + R_{iP} = 0; \\ ------------------------- \\ r_{n1}Z_1 + r_{n2}Z_2 + r_{n3}Z_3 + ... + r_{nk}Z_k + ... + r_{nn}Z_n + R_{nP} = 0; \end{cases} \qquad (5.3)$$

Here: n – the number of unknowns (angular twists and linear deflections) of shifts of nodes in the system, or the degree of instability of the system; Z_k $(k = 1...n)$ unknown values of displacements (angular and linear) of nodes of a structure; r_{ik} – reactive force (moment) in the ***i***-additional restriction (fixed support, hinged movable support) from movement (rotation or linear deflection) of ***k***-additional restriction (pinched support, hinged movable support) to a single amount of movement ($Z_k = 1$);

R_{iP} – reactive force (moment, force) in ***i***-additional restriction (pinched support, hinged movable support) from the action of applied loads.

The physical meaning of the equations (for the i-equation): reactive force (moment) in the ***i***-additional restriction (pinched support, hinged movable support) from the movements of all additional restrictions (rotations, linear deflections) $Z_1, Z_2, ..., Z_n$ and the given applied load is zero, as this ***i***-restriction in the original (design) system is not.

Here are examples of the physical meaning of coefficients and equations in general, for example, for conjugate system of the slope and deflection method presented on figure 5.3:

r_{11} – reactive moment in the 1-st extra pinched support from it same twist at a single angle;

r_{53} – reactive force in the 5-th additional hinged movable support from the turn of the 3-rd additional pinched support to the angle which, equal to the unit;

r_{26} – reactive moment in the 2-nd additional pinched support from the linear shift of the 6-th additional hinged movable support to the distance, equal to the unit;

R_{4P} – reactive force in the 4-th additional hinged movable support from the action of the applied loads;

3-rd equation: $r_{31}Z_1 + r_{32}Z_2 + r_{33}Z_3 + ... + r_{3n}Z_n + R_{3P} = 0$; a reactive moment in the 3-rd extra pinched support from moving all additional restrictions (supports) to the magnitude $Z_1,\ Z_2, ...,\ Z_6$ and from the effect of the external applied load is zero, because in a given system (Figure 5.3*a*) this (3-rd) pinched support does not exist.

Note that coefficients r_{ik} with the same indices (r_{11}, r_{22}), are called the principal deflections, and the rest are called secondary deflections (coefficients); the principal deflections cannot be negative and zero, and secondary deflections (factors) should satisfy the theorem of reciprocity of unit reactions ($r_{ik} = r_{ki}$).

5.5. Slope and Deflection Method Table Diagrams

To determine the r_{ik}, R_{iP} reactive forces it is necessary to be able to identify internal forces (plot their diagrams) in the conjugate system of the slope and deflection method from unite displacements of additional restrictions or supports (together, of course, with the respective nodes) and from the action of the given applied loads. Plotting these diagrams, due to the fact that the conjugate system of the slope and deflection method is a set of individual beams, completely independent of each other, is associated with the ability to calculate these beams (Figure 5.6). External actions (exposures) here will be turns of the pinched support, linear shifts of the hinge support and force factors (concentrated forces, concentrated moments, distributed loads). The calculation of such beams is usually done by force method, and the calculation results are tabulated (see table 5.1).

Table 5.1

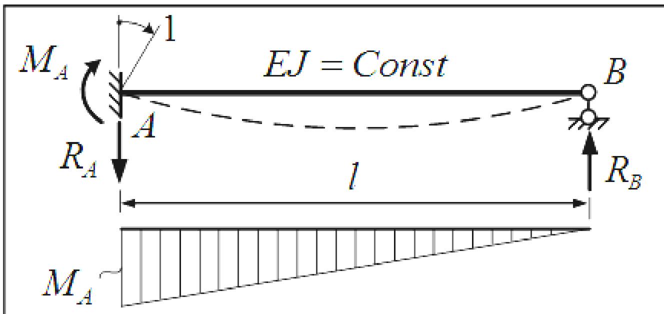
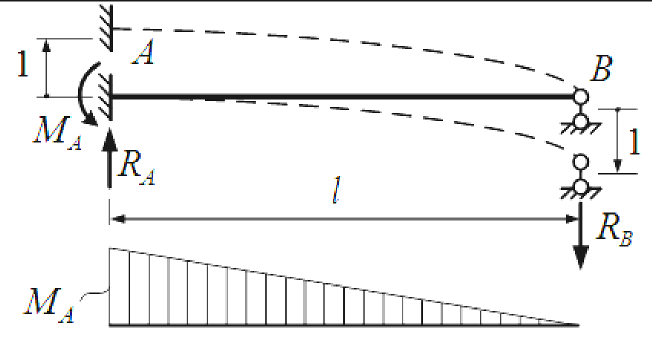
	$M_A = 3i$; where:	$R_A = R_B = \frac{3i}{l}$, $i = \frac{EJ}{l}$.
	$M_A = \frac{3i}{l}$; where:	$R_A = R_B = \frac{3i}{l^2}$, $i = \frac{EJ}{l}$.

Table 5.1 continuation

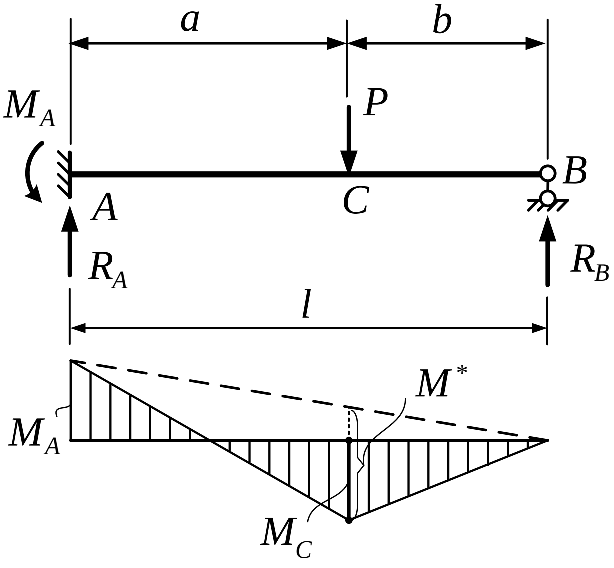
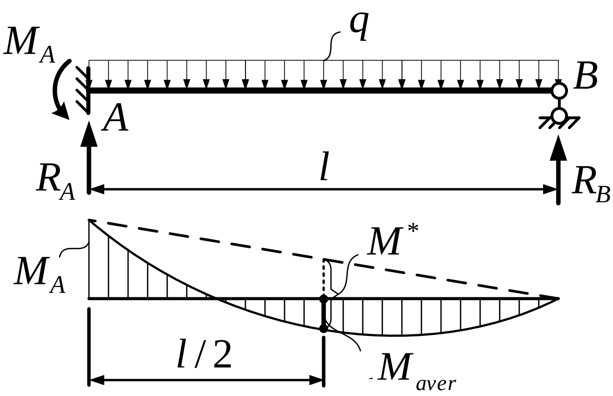
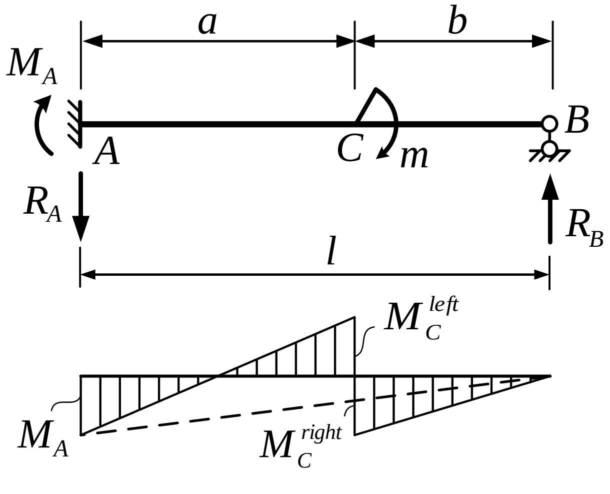
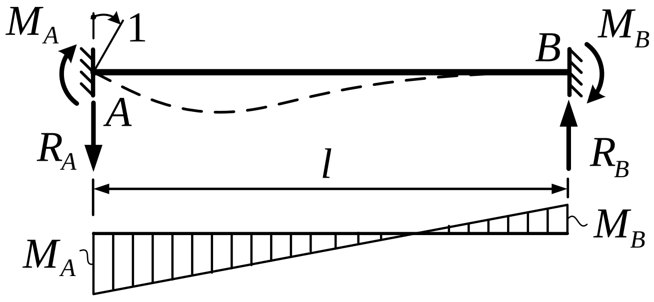
	$M_A=\dfrac{Pb(l^2-b^2)}{2l^2}$; $M^*=\dfrac{Pab}{l}$; $M_C=\dfrac{Pb}{2}\left[\dfrac{2a}{l}-\dfrac{b}{l^3}(l^2-b^2)\right]$; $R_A=\dfrac{Pb\,(3l^2-b^2)}{2l^3}$; $R_B=\dfrac{Pa^2(3l-a)}{2l^3}$; At: $\underline{a=b=\dfrac{l}{2}}$: $M_A=\dfrac{3}{16}Pl$; $M^*=\dfrac{Pl}{4}$; $M_C=\dfrac{5}{32}Pl$; $R_A=\dfrac{11}{16}P$; $R_B=\dfrac{5}{16}P$.
	$M_A=\dfrac{ql^2}{8}$; $R_A=\dfrac{5}{8}ql$; $M^*=\dfrac{ql^2}{8}$; $R_B=\dfrac{3}{8}ql$. $M_{aver}=\dfrac{ql^2}{16}$.
	$M_A=\dfrac{m\left(l^2-3b^2\right)}{2l^2}$; $M_C^{left}=\dfrac{ma}{l}-\dfrac{b}{l}M_A$; $M_C^{right}=\dfrac{mb}{l}+\dfrac{b}{l}M_A$; $R_A=R_B=\dfrac{3m\left(l^2-b^2\right)}{2l^3}$. At: $\underline{a=b=l/2}$: $M_A=\dfrac{m}{8}$; $R_A=R_B=\dfrac{9}{8}\dfrac{m}{l}$.

Table 5.1 continuation

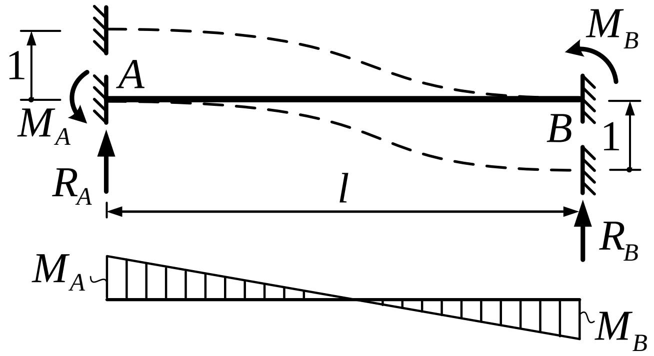


$$M_A = 4i;$$

$$M_B = 2i;$$

$$R_A = R_B = \frac{6i}{l},$$

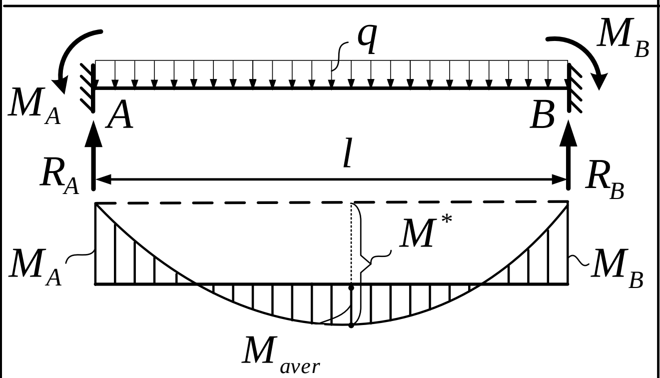
where: $i = \dfrac{EJ}{l}$.



$$M_A = M_B = \frac{6i}{l};$$

$$R_A = R_B = \frac{12i}{l^2},$$

where: $i = \dfrac{EJ}{l}$.

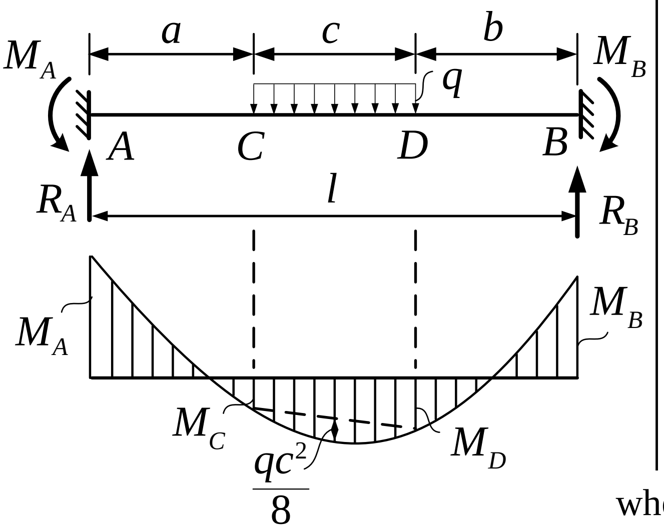


$$M_A = M_B = \frac{ql^2}{12};$$

$$M^* = \frac{ql^2}{8};$$

$$M_{aver} = \frac{ql^2}{24};$$

$$R_A = R_B = \frac{ql}{2}.$$



$$M_A = 2qcl\left[\frac{1}{2l}\left(a + \frac{c}{2}\right) + (\beta - \gamma)\right];$$

$$M_B = 2qcl(\gamma - 2\beta);$$

$$R_B = 2qc(2\gamma - 3\beta);$$

$$M_C = 2qc\left[(l - 2a)\gamma + (3a - l)\beta - \frac{c}{4}\right];$$

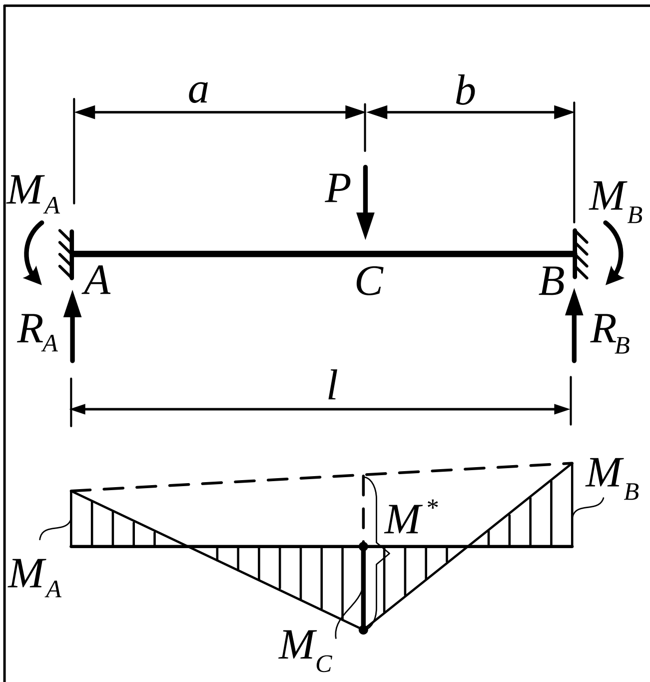
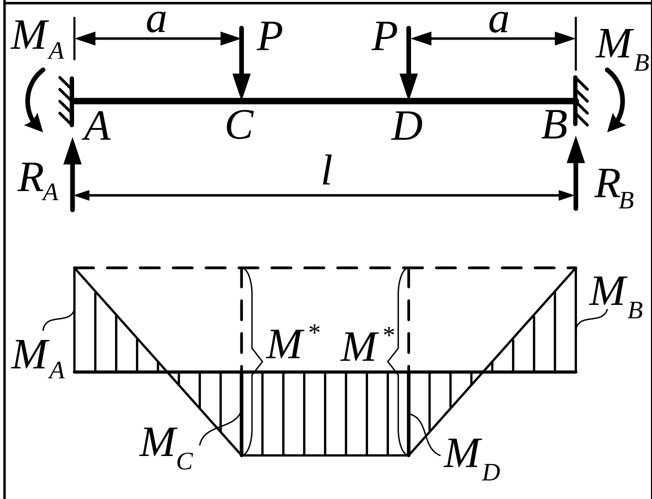
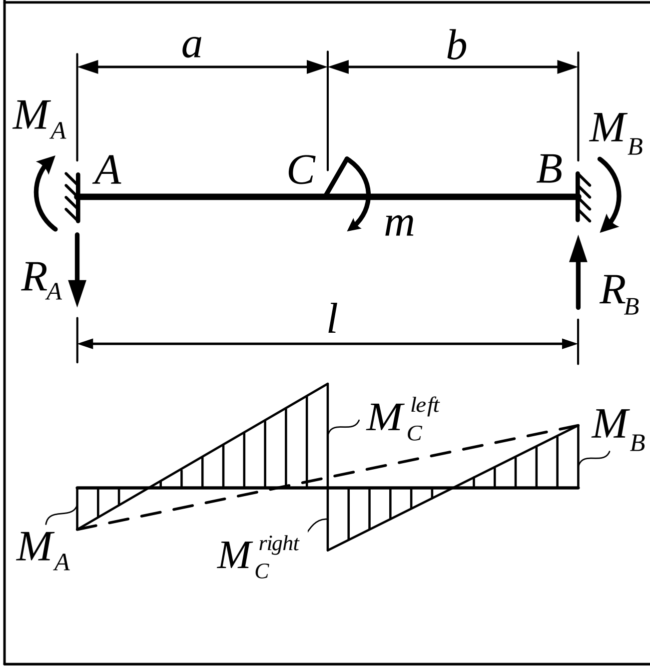
$$M_D = 2qc\left[(2b - l)\gamma + (2l - 3b)\beta\right];$$

$$R_A = qc - R_B = qc\left[1 - 2(2\gamma - 3\beta)\right],$$

where: $\beta = \dfrac{1}{2l^2}\left[\dfrac{c^2}{3} + \gamma(l - b)\right];$

$$\gamma = \frac{1}{2l^3}\left[c^2\left(b + \frac{3}{4}c\right) + la^2 + a(2l - a)\left(l + \frac{c}{2} - b\right)\right].$$

Table 5.1 continuation

	$M_A=\dfrac{Pab^2}{l^2}$; $M_B=\dfrac{Pa^2b}{l^2}$; $M^*=\dfrac{Pab}{l}$; $M_C=\dfrac{Pab}{l}\left[1-\dfrac{a}{l}-\dfrac{(b-a)b}{l^2}\right]$; $R_A=\dfrac{Pb^2}{l^2}\left(1+\dfrac{2a}{l}\right)$; $R_B=\dfrac{Pa^2}{l^2}\left(1+\dfrac{2b}{l}\right)$. At: $\underline{a=b=l/2}$: $R_A=R_B=\dfrac{P}{2}$; $M_A=M_B=M_C=\dfrac{Pl}{8}$; $M^*=\dfrac{Pl}{4}$.
	$M_A=M_B=Pa\left(1-\dfrac{a}{l}\right)$; $M^*=Pa$; $M_C=M_D=\dfrac{Pa^2}{l}$; $R_A=R_B=P$.
	$M_A=\dfrac{mb}{l^2}(3a-l)$; $M_B=\dfrac{ma}{l^2}(3b-l)$; $M_C^{left}=\dfrac{mb}{l^3}(l^2-3al+6a^2)$; $M_C^{right}=\dfrac{ma}{l^3}(l^2-3bl+6b^2)$; $R_A=R_B=\dfrac{6ab}{l^3}m$. At: $\underline{a=b=l/2}$: $M_A=M_B=\dfrac{m}{4}$; $M_C^{left}=M_C^{right}=\dfrac{m}{2}$; $R_A=R_B=\dfrac{3}{2}\dfrac{m}{l}$.

5.6. Plotting of a Bending Moment Diagram from Unit Load and Bending Moment Diagram from Applied Loading in Conjugate System

Plotting of an unit diagram of bending moment $\overline{M}_1$ from the action of $\boldsymbol{Z_i}$ shifting of unit magnitude ($\boldsymbol{Z_i}$ =1) and M_P diagram due to the actual applied loading to the conjugate system is carried out using ready-made (table) diagrams for individual sections of the conjugate system (see Table 5.1). Each part of a frame works regardless of others. Plotting of a diagram is performed as follows: depending on the actions directly on the structural element (concentrated force, concentrated moment, distributed load), the table moment diagram is selected, taking into account the specific parameters of this area (element), and then transferred to the design area (onto the conjugate scheme). At the same time, when plotting unit diagrams, it is expedient to present the deforming scheme of the conjugate system from the appropriate displacement of the node. This allows us to identify which elements of the conjugate system "work", how they "work" and with which side the fibers are stretched, and where the fibers are compressed (the bending moment diagram should be plotted from the stretched fibers side). Ordinates of unit diagrams, as can be seen from table diagrams, are expressed through the relative stiffness of the areas, which is the ratio of the real rigidity of the areas to their lengths $i_s = EJ_s / l_s$. If we set the stiffness in general, it can cause some difficulty in comparing with the ordinates of unit diagrams at different sites. To avoid these difficulties, the next way is suggested:

– we can choose one relative stiffness of the areas for the base (indicating it, for example, through $\boldsymbol{i}$) and then, through it, express the relative stiffness of the rest of the system; So, for the frame on figure 5.10 *a*, the rigidity of the areas is recorded as follows:

$$i_{01} = i_{23} = \frac{EJ}{h}; \qquad i_{12} = \frac{EJ_{12}}{l} = \frac{kEJ}{\alpha \cdot h} = \frac{k}{\alpha} \cdot \frac{EJ}{h},$$

and if we're noting $\frac{EJ}{h} = i$ or $EJ = i \cdot h$, we'll get: $i_{01} = i_{23} = i; \quad i_{12} = \frac{k}{\alpha} i$

(this manual uses further this approach of expressing the rigidity of the elements);

– We can express the stiffness of the areas through a certain amount of *EJ*, common to all areas $EJ_S = \beta_S EJ$; in this case for the frame on figure 5.10 *a*, we'll get:

$$i_{01} = i_{23} = \frac{EJ}{h}; \qquad i_{12} = \frac{k}{\alpha h} EJ;$$

– We can conveniently set for the parameters i, EJ some numerical values for further calculations, from our point of view. This can be done, because when the final diagrams of the forces are plotted, the values that are common to all areas are reduced. So their parameters i, EJ do not affect the results of the calculation (area stiffness ratios have significance); for the frame on the figure 5.10 *a*, it is convenient, for example, to take: $EJ=ih$.

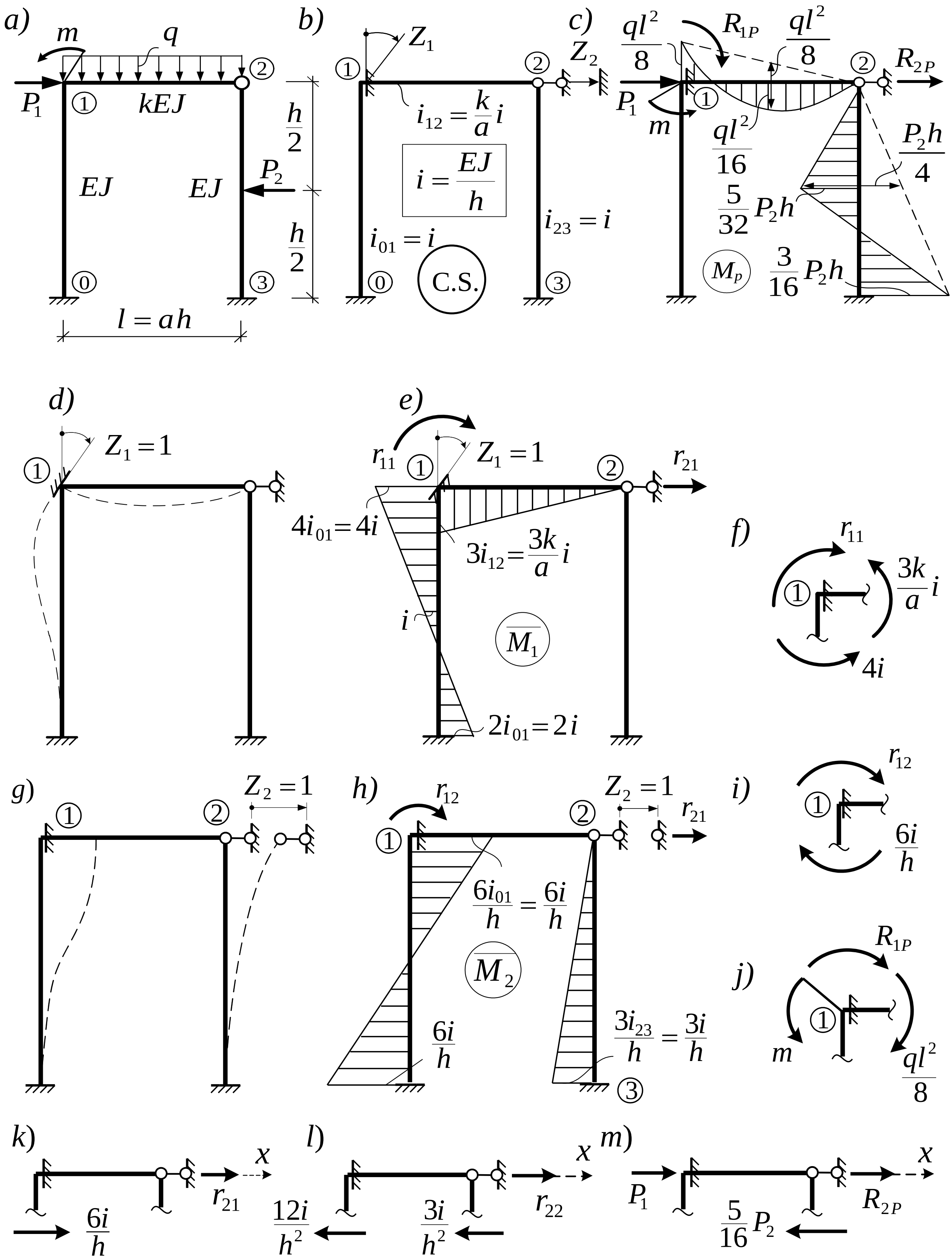

a)
m
q
P1
1
kEJ
2
EJ
EJ
P2
h/2
h/2
0
3
l = ah
b)
Z1
1
2
Z2
i12 = k/a i
i = EJ/h
i01 = i
i23 = i
C.S.
0
3
c)
ql²/8
R1P
ql²/8
R2P
P1
m
ql²/16
P2h/4
5/32 P2h
3/16 P2h
MP
d)
Z1 = 1
e)
r11
Z1 = 1
r21
4i01 = 4i
3i12 = 3k/a i
i
M1
2i01 = 2i
f)
r11
3k/a i
4i
g)
Z2 = 1
h)
r12
Z2 = 1
r21
6i01/h = 6i/h
M2
6i/h
3i23/h = 3i/h
i)
r12
6i/h
j)
R1P
m
ql²/8
k)
x
r21
6i/h
l)
x
12i/h²
3i/h²
r22
m)
x
P1
R2P
5/16 P2

Figure 5.10

The frame presented in the figure 5.10*a*, as previously defined (Figure 5.1), has two unknowns of the slope and deflection method (the degree of its kinematic indeterminacy is two). The conjugate system of the deflection method for this frame has the appearance shown on the figure 5.10 *b*. Unit diagrams of $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ bending moments from the action of the $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$ shifts of a unit magnitude ($\boldsymbol{Z_i}$ =1) and M_P diagram due to the actual applied loading (as well as deformation schemes corresponding to unit displacements) are presented on figure 5.10 *n* - 5.10 *g*.

5.7. Determining the Coefficients and Free Terms of the Canonical Equations

Coefficients and free terms of equations in their physical meaning, as already noted, can be two kinds – reactive moments in additional pinched supports (r_{ik} or R_{iP}) and reactive forces in additional hinge supports (r_{ik} or R_{iP}). Their definition, as the definition of any reactions, can be made on the basis of equilibrium equations either as a whole system or a part of it (static way). Experience has shown that it is more convenient to consider the balance of individual parts of the conjugate system, which are under the exposures from which the desired reactive force is determined. At the same time, the following rule of signs is used for reactive forces in additional supports (restrictions) – reactive force is considered positive if its direction coincides with the direction of movement of the appropriate additional restriction. In the process of identifying unknowns, reactive forces should always be directed in positive directions.

According to the above, the definition of the r_{ik} coefficients and R_{iP} terms in the meaning of the reactive moments of the additional pinched supports, is most convenient to perform on the basis of consideration of the equilibrium of nodes, in which were installed appropriate additional pinched supports, preliminarily cutting out these nodes. Then one after another the equilibrium equations for the nodes are derived up, (summing all the moments acting in the ***i***-node, we get one equation). From these equations can be determined the desired reactive moments.

So, for the frame on the figure 5.10 when determining the r_{11} coefficient representing the reactive moment in the ***1-st*** additional fixed support from its rotation by a unit angle, it is necessary to cut the node ***1*** from the $\overline{M}_1$ diagram, (which was built from the turn of the ***1-st*** pinched support to a unit angle) (see Figure 5.10*s*), from the equilibrium of which we will get:

$$\sum M_1 = 0; \quad r_{11} - 4i - \frac{3k}{\alpha} i = 0; \quad r_{11} = 4i + \frac{3k}{\alpha} i.$$

Factor of r_{12} can be found by cutting out ***1*** node on the $\overline{M}_2$ diagram (figure 5.10 *u*):

$$\sum M_1 = 0; \quad r_{12} + \frac{6i}{h} = 0; \quad r_{12} = -\frac{6i}{h}.$$

For definitions of R_{1P} free term, we need to cut the ***1*** node on the M_P diagram (Figure 5.10 *k*):

$$\sum M_1 = 0; \quad R_{1P} + \frac{ql^2}{8} - m = 0; \quad R_{1P} = m - \frac{ql^2}{8}.$$

Thus, the first index of the coefficient or free term here essentially shows the node number which has to be cut to determine this factor, and the second index indicates the diagram from which this node has to be cut.

For example, to determine the r_{22} coefficient, which is being a reactive force in the ***2-nd*** additional hinge support from its own unit displacement, we can cut the ***2*** node on the $\overline{M}_2$ diagram (Figure 5.11); As a result, we will receive:

$$\sum X = 0; \quad r_{22} - Q_{23} - N_{12} = 0; \quad r_{22} = Q_{22} + N_{12}.$$

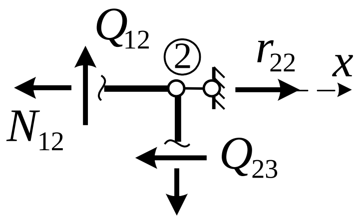


Figure 5.11

The value of Q_{23} is easy to determine by $\overline{M}_2$ diagram, using, for example, a formula:

$$Q = Q_o \pm \left| \frac{M_{right} - M_{left}}{l} \right|, \qquad (5.4)$$

where: Q_0 – takes into account the effect of the q distributed load on the area of frame, i.e. it is a diagram of shear forces on the area, likewise in a simple beam (if q does not exist, then $Q_0 = 0$); M_{right}, M_{left} – ordinates of bending moment on the right and left ends of the area; l – length of the area;

The sign before the module is accepted by the following rule: if the rod of frame on which the ***M*** is plotted must be rotated before combining with the straight line connecting the ordinates of M_{right} and M_{left} on the shortest path clockwise, the sign is taken "+". If it's counterclockwise, it's a "-" sign.

As a result, we will get for area 2-3 (Figure 5.10 *g*):

$$Q_{23} = 0 + \left| \frac{3i/h - 0}{h} \right| = \frac{3i}{h^2}.$$

It should be noted that the value of Q_{23} can be obtained through a support reaction in the 2-3 rod from the action of the accepted of $Z_2 = 1$ linear unit deflection. Considering that the shear force nearby support section is equal to the value of the reaction of the corresponding support, which is given in table diagrams, we can calculate it. The shear force sign is determined by the usual rule of the signs for Q or from *M* diagram. In order to determine the value of the N_{1-2} normal force in the 1-2 rod, it is necessary, first, by use of $\overline{M}_2$ diagram, and the formula (4), to plot a diagram of $\overline{Q}_2$ shear-forces, and then the value of N_{1-2} from the cutting of node can be found. This process, as we can see, is quite labor-intensive.

It will be easier to determine the r_{ik}, R_{iP} coefficients which are the reactions in the linear restrictions (in additional hinged supports), if we cut definite parts of the conjugate system instead of nodes. Thus, the equations of equilibrium should include only shear forces; and it is most convenient to use as equilibrium equations the sum of the projections of forces on the axis, parallel to the unknown reactive force.

So to determine the r_{21} coefficient it is convenient to cut out the top of the $\overline{M}_1$ diagram, shown on figure 10 *l*, out of balance of which we will get:

$$\sum X = 0; \quad r_{21} + \frac{6i}{h} = 0; \quad r_{21} = -\frac{6i}{h}.$$

Similarly, we will find a coefficient r_{22} (Figure 5.10 *m*) and the absolute term R_{2P} (Figure 10 *n*), cutting the corresponding parts of the conjugate system out of the $\overline{M}_2$ and M_P diagrams:

$$\sum X = 0; \quad r_{22} - \frac{12i}{h^2} = \frac{3i}{h^2} = 0; \quad r_{22} = \frac{15i}{h^2};$$

$$\sum X = 0; \quad R_{2P} + P_1 - \frac{5}{16}P_2 = 0; \quad R_{2P} = \frac{5}{16}P_2 - P_1.$$

It should be noted that the r_{ik} and R_{iP} values can also be determined by Mohr's formula. At the same time, the magnitudes of r_{ii} and r_{ik} can be calculated by formulas:

$$r_{ik} = \sum\int \frac{\overline{M}_i\,\overline{M}_k\,ds}{EJ}; \quad r_{ii} = \sum\int \frac{\overline{M}_i^2\,ds}{EJ}; \tag{5.5}$$

The value of R_{iP} absolute term can be calculated according to the formula:

$$R_{iP} = -\sum\int \frac{\overline{M}_i M'_P\,ds}{EJ}, \tag{5.6}$$

where: M'_P – the diagram of bending moment from the action of the applied loads in a statically determinate system obtained from a given system (or conjugate system) by removing redundant restrictions, including necessarily the restriction, the reaction of which is determined.

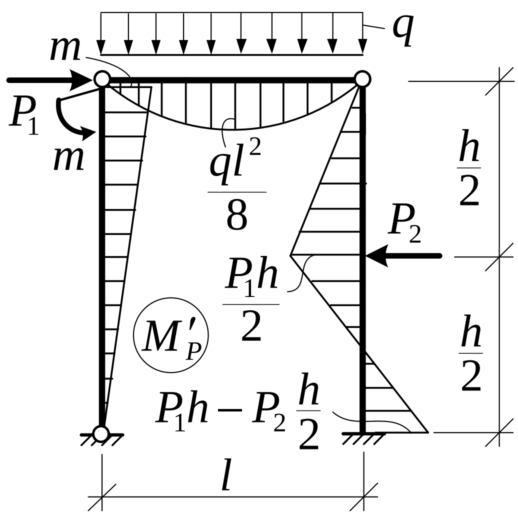


Figure 5.12

Calculate for the example under consideration (Figure 5.10), the r_{12}, r_{22} factors and R_{1P} an absolute term. Unit diagrams in the conjugate system are shown on the figure 10, and one of the possible options for the M'_P diagram on the figure 5.12.

Given that: $\frac{h}{EJ} = \frac{1}{i}$, and $\frac{1}{kEJ} = \frac{\alpha h}{kEJ} = \frac{\alpha}{ki}$, we will find:

$$r_{12}=r_{21}=\sum\int\frac{\overline{M}_1\,\overline{M}_2\,ds}{EJ}=\frac{h}{6EJ}\left(-\frac{6i}{h}\cdot 2i-4i\cdot\frac{6i}{h}\right)=-\frac{6i}{h};$$

$$r_{22}=\sum\int\frac{\overline{M}_2^{\,2}ds}{EJ}=\frac{h}{6EJ}\left[\left(\frac{6i}{h}\right)^2+\left(\frac{6i}{h}\right)^2\right]+\frac{h}{6EJ}\left[\left(\frac{3i}{h}\right)^2+4\cdot\left(\frac{1{,}5i}{h}\right)^2\right]=\frac{15i}{h^2};$$

$$R_{1P}=\sum\int\frac{\overline{M}_1 M'_P\,ds}{EJ}=-\frac{h}{6EJ}\left(-m\cdot 4i-4\cdot\frac{m}{2}\cdot i\right)-\frac{l}{6kEJ}\left(4\cdot\frac{ql^2}{8}\cdot\frac{15k}{\alpha}i\right)=$$

$$=-\frac{1}{6i}(-6mi)-\frac{\alpha}{6ki}\left(\frac{ql^2}{2}\cdot 1{,}5\frac{ki}{\alpha}\right)=m-\frac{ql^2}{8};$$

The values calculated here are the same as values which were found by static method previously.

5.8. Checks of Coefficients and Free Terms of Canonical Equations

Checking the correct calculation of coefficients and absolute terms of canonical equations can be done similarly as the method of force. At the same time, a Unit Total Diagram must be plotted to carry out the checking of $\overline{M}_s$, representing the sum of all unit diagrams in the conjugate system:

$$\overline{M}_s=\overline{M}_1+\overline{M}_2+\overline{M}_3+...+\overline{M}_n. \tag{5.7}$$

The following checks can then be carried out in the method of deflection:

a) Universal verification:

$$\sum\int\frac{\overline{M}_s^{\,2}ds}{EJ}=\sum_{i=1}^{n}\sum_{k=1}^{n}r_{ik}, \tag{5.8}$$

where: $\Sigma\Sigma r_{ik}$ – the sum of all unit deflections (the sum of factors of all equations):

$$\sum\sum r_{ik}=(r_{11}+r_{12}+...+r_{1n})+(r_{21}+r_{22}+...+r_{2n})+$$
$$+...+(r_{n1}+r_{n2}+...+r_{nn}).$$

If a universal check is performed, it confirms the correct calculations of coefficients (principal deflections, and secondary deflections), and absolute terms; if the universal check is not performed, then to find out which group of coefficients is an error, we can make so-called line checks.

b) Line checks, i.e. checks of the coefficients that are included in each of the equations of the method of slope and deflection (*1st*, *2nd*, ... *n*) has:

$$\sum\int\frac{\overline{M}_i\,\overline{M}_s\,ds}{EJ}=\sum_{k=1}^{n}r_{ik}\qquad (i=1...\,n), \tag{5.9}$$

where: $\Sigma\Sigma r_{ik}$ – the amount of coefficients included in i-equation of the method of slope and deflection.

Such checks, as can be seen, should carry out of ***n***-times; some of them will be executed (this will mean that corresponding coefficients were calculated correctly), and some may not; analysis of the results reveals the coefficients in which errors were made.

Note; if a universal check is carried out, there is no need to perform line checks.

c) column check serves to check absolute terms and it is executed according to the formula:

$$\sum\int\frac{\overline{M}_s M'_P\,ds}{EJ}=\sum_{i=1}^{n}R_{iP}, \tag{5.10}$$

where: $R_{iP}=R_{1P}+R_{2P}+...+R_{nP}$ – total number of all absolute terms;

M'_P – the diagram of bending moments from the applied loads in a statically determinate system derived from a given system or a conjugate system by discarding redundant restrictions, including necessarily additional restriction, where are defined reactions R_{iP} (see Figure 5.12).

5.9. Plotting the Final Diagram of Forces and Verifying Them

The found values of coefficients and absolute terms are substituted into the system of canonical equations (5.3), through which we can determine of Z_i $(i=1...n)$ unknowns. After that, the final diagram of bending moment on the base of the principle of independence of force and actions (deflections) can be plotted according to the formula:

$$M=\overline{M}_1 Z_1+\overline{M}_2 Z_2+...+\overline{M}_n Z_n+M_P. \tag{5.11}$$

The final diagram of the shear forces is plotted out on the base of ***M*** diagram by using the formula (5.4). The final normal forces diagram is plotted out on the base of ***Q*** diagram by cutting out the joints. To confirm the correctness of calculations of ***M, Q*** and ***N*** final diagrams, the following checks are carried out:

a) Checking the balance of the nodes on the base of M diagram; this check is important when frames are calculated with the deflection method. Because the balance of joints (in the conjugate system) is not performed on the $\overline{M}_i$ unit diagram or on the M_P diagram (from applied loads), without taking into account additional pinched supports (which the system does not really have); In the ***M*** final diagram the balance of the nodes must be performed;

b) deformation (kinematic) check of the M diagram; this check can be done in the same way as in the method of force, but here to perform it for a given system, choose the primary system of the method of force (preliminarily determining the number of redundant restrictions). Then we need to plot a total unit diagram $\overline{M}_S^{\,force\ meth.}$ from the unit values of all unknowns of method of force (or at least one of the $\overline{M}_i^{\,force\ meth.}$ unit diagrams). In such case, the maximum number of areas of the system should be covered; after that the deformation check is performed by formula:

$$\sum\int\frac{\overline{M}_S^{\,force\ meth.}\cdot Mds}{EJ}=0 \qquad \left(or \quad \sum\int\frac{\overline{M}_i^{\,force\ meth.}\cdot Mds}{EJ}=0\right). \qquad (5.12)$$

The physical meaning of this check here is the same as in the method of force;

c) Checking the equilibrium of the nodes when plotting the N diagram out of the Q diagram; ***N*** diagram is plotted, as already noted, on the base of the ***Q*** diagram in the way of cutting nodes, i.e., normal forces can be found from equations of projections of forces to any two axes. Keeping the balance of all nodes in this way indicates the correctness of the calculation. If at least one of the equilibrium equations is not performed in at least one of the nodes, it means that a mistake has been made in the calculation. Most often, this error is associated with incorrect determination of coefficients or absolute terms, which are reactive forces in additional restrictions (movable hinges);

d) Static verification is performed in the same way as in the method of force. The projections of all the applied loads and support reactions to any two axes and the sum of the moments of forces w.r.t. any point should be zero.

5.10. An Example of Calculation

From the frame depicted in the figure 5.13 *a*, we can see the degree of instability of the system equals three ($n=n_a+n_l=2+1=3$). The conjugate system is presented in figure 5.13 *b*. The relative stiffness of the areas is:

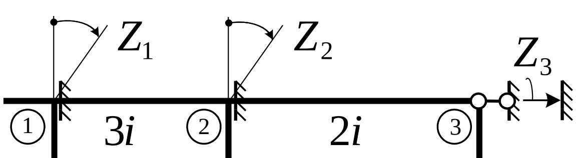
$$i_{01}=\frac{EJ}{2}; \qquad i_{12}=i_{24}=i_{35}=\frac{EJ}{4}; \qquad i_{23}=\frac{EJ}{6}$$

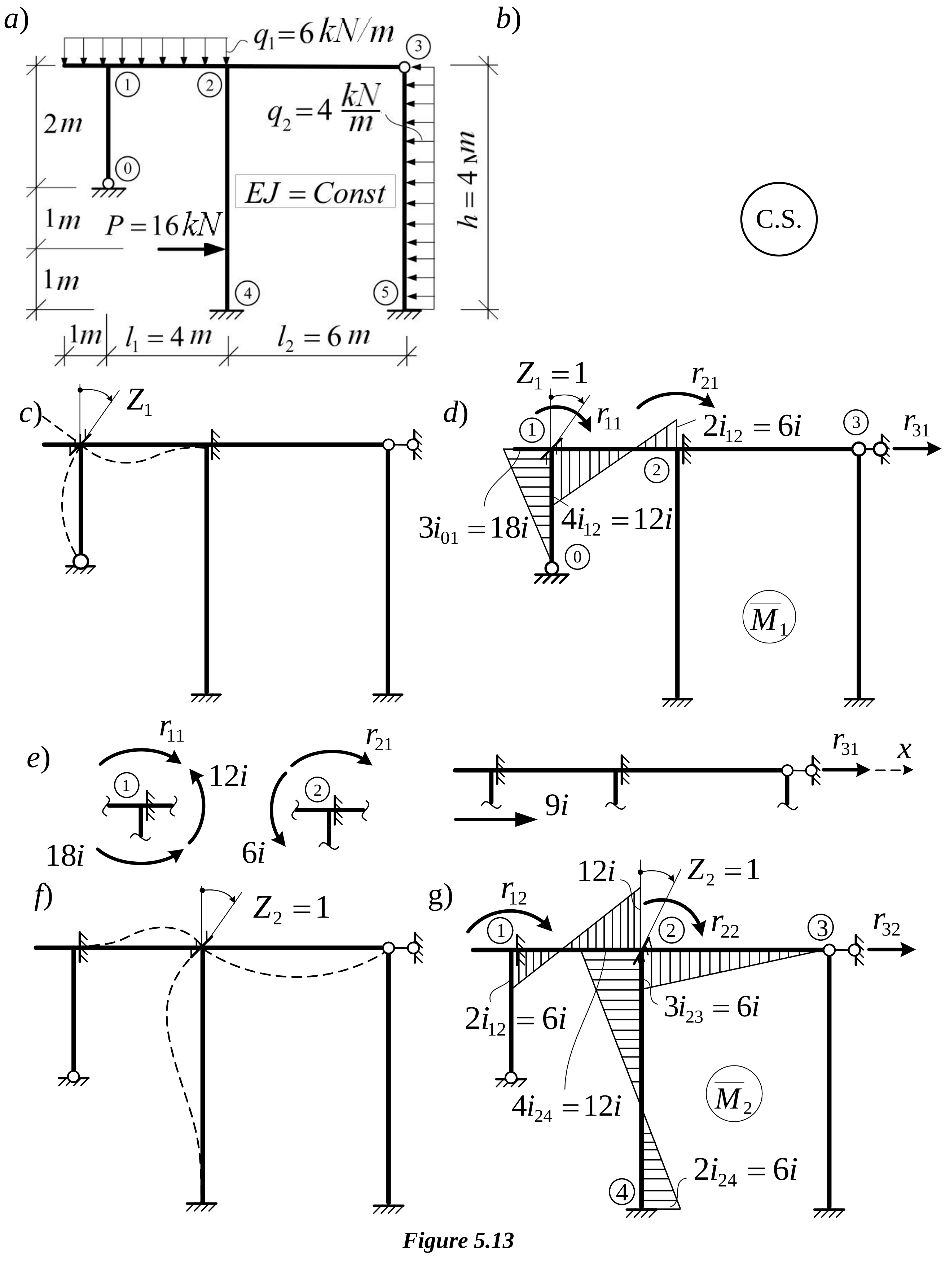
Through a common value for all areas – $i=EJ/12$ ($EJ=12i$), we can receive:

$$i_{01}=6i; \quad i_{12}=i_{24}=i_{35}=3i; \quad i_{23}=2i;$$

To be clear, this relative stiffness of areas is shown in the conjugate system of the deflection method (Figure 5.13 *b*). The $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$ unit bending moments in conjugate system as shown respectively on figure 5.13 *d*, 5.13 *g*, 5.13 *j*, are plotted on the basis of deformation schemes of conjugate system from deflections of Z_1, Z_2, Z_3 nodes on unit values (Figure 5.13 *c*, 5.13 *f*, 5.13 *i*) using table diagrams (table.1). Unit coefficients or absolute terms of canonical equations are determined in a static way: reactive moments r_{1k}, r_{2k} can be derived by cutting out 1 and 2 nodes of the $\overline{M}_k$ diagram. The r_{3k} reactive force can be derived from cutting out the top part of the frame on the $\overline{M}_k$ diagram (Figure 5.13 *i*, 5.13 *h*, 5.13 *k*).

– figure 5.13*e*:
$$\begin{cases} \sum M_1=0; & r_{11}-12i-18i=0; & r_{11}=30i; \\ \sum M_2=0; & r_{21}-6i=0; & r_{21}=6i; \\ \sum X=0; & r_{31}+9i=0; & r_{31}=-9i; \end{cases}$$





a)
b)
$q_1 = 6\,kN/m$
$q_2 = 4\,\frac{kN}{m}$
$2m$
$1m$
$1m$
$P = 16\,kN$
$EJ = Const$
$h = 4\,m$
$1m$
$l_1 = 4\,m$
$l_2 = 6\,m$
C.S.
c)
Z_1
d)
$Z_1 = 1$
r_{21}
r_{11}
$2i_{12} = 6i$
r_{31}
$3i_{01} = 18i$
$4i_{12} = 12i$
$\overline{M}_1$
e)
r_{11}
$12i$
$18i$
r_{21}
$6i$
r_{31}
x
$9i$
f)
$Z_2 = 1$
g)
$12i$
$Z_2 = 1$
r_{12}
r_{22}
r_{32}
$2i_{12} = 6i$
$3i_{23} = 6i$
$4i_{24} = 12i$
$\overline{M}_2$
$2i_{24} = 6i$

Figure 5.13

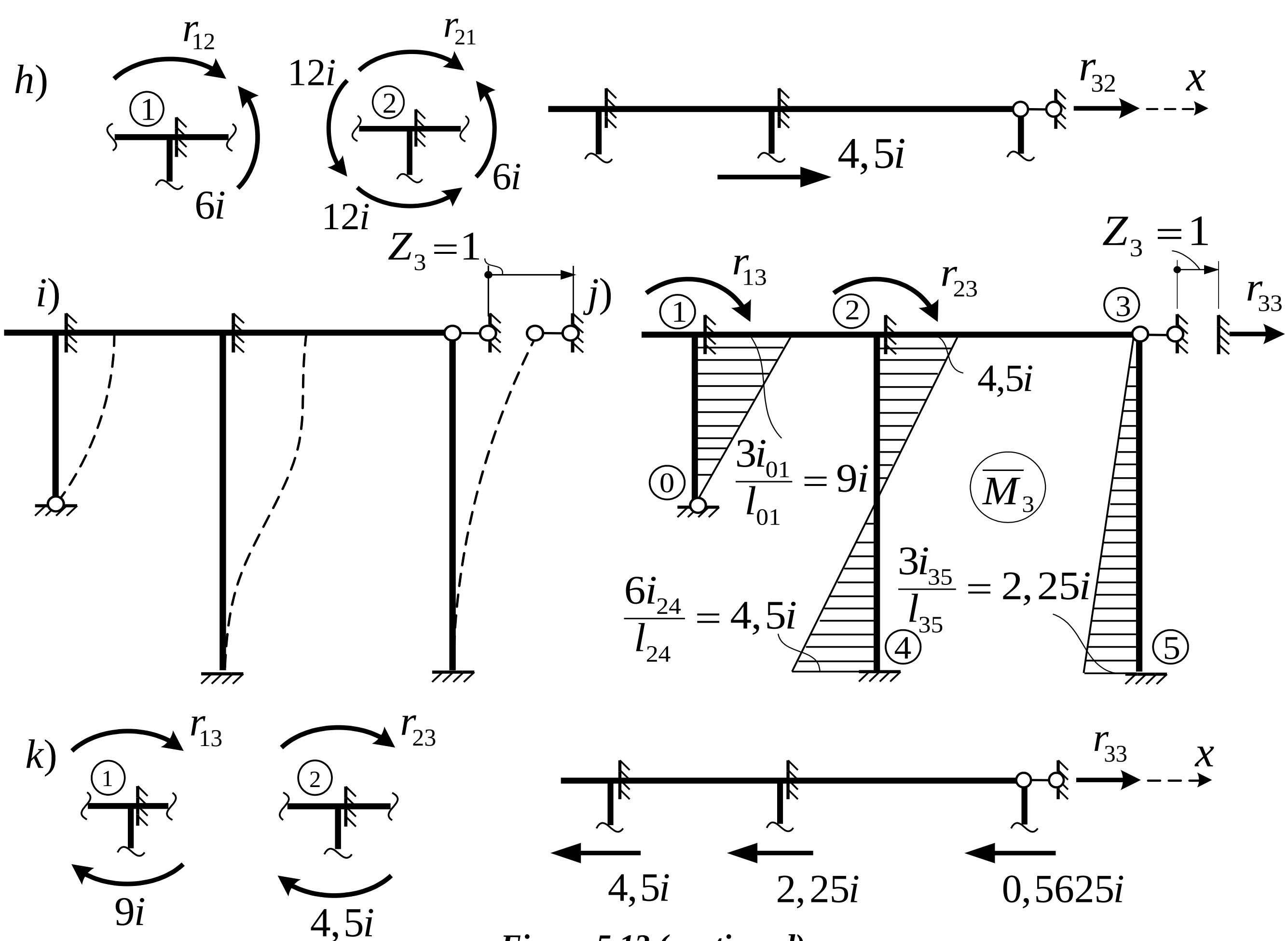


Figure 5.13 (continued)

$$\text{– figure 5.13}h\text{:}\begin{cases} \sum M_1 = 0; & r_{12} - 6i = 0; & r_{12} = 6i; \\ \sum M_2 = 0; & r_{22} - 6i - 12i - 12i = 0; & r_{22} = 30i; \\ \sum X = 0; & r_{32} + 4,5i = 0; & r_{32} = -4,5i; \end{cases}$$

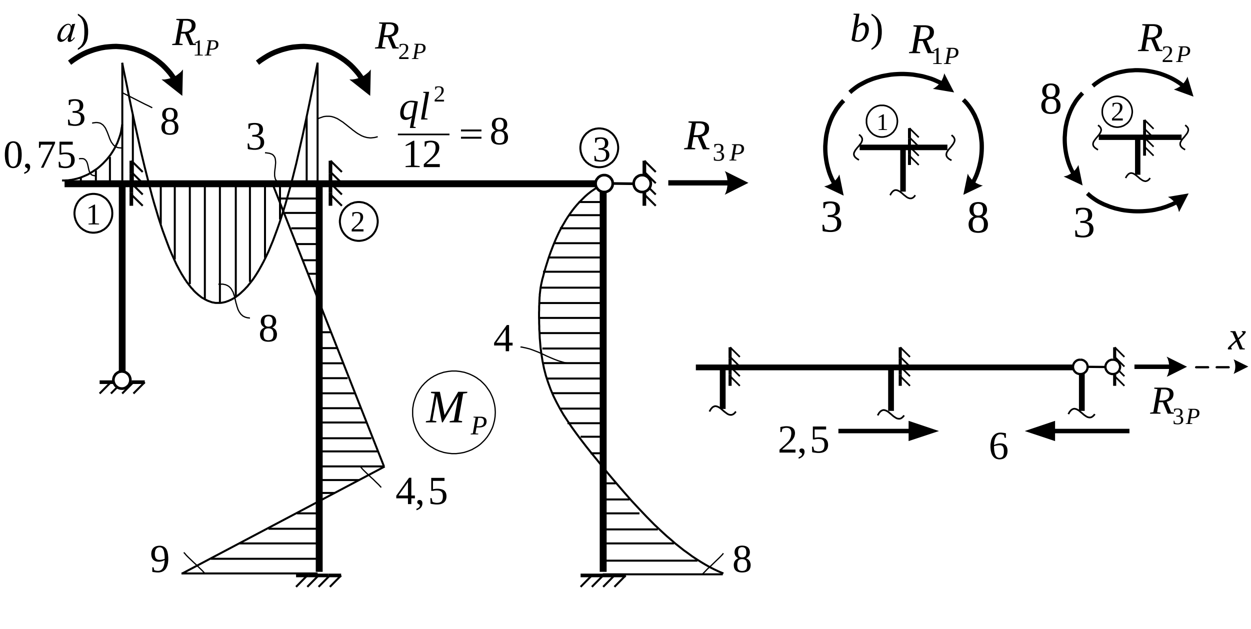


Figure 5.14

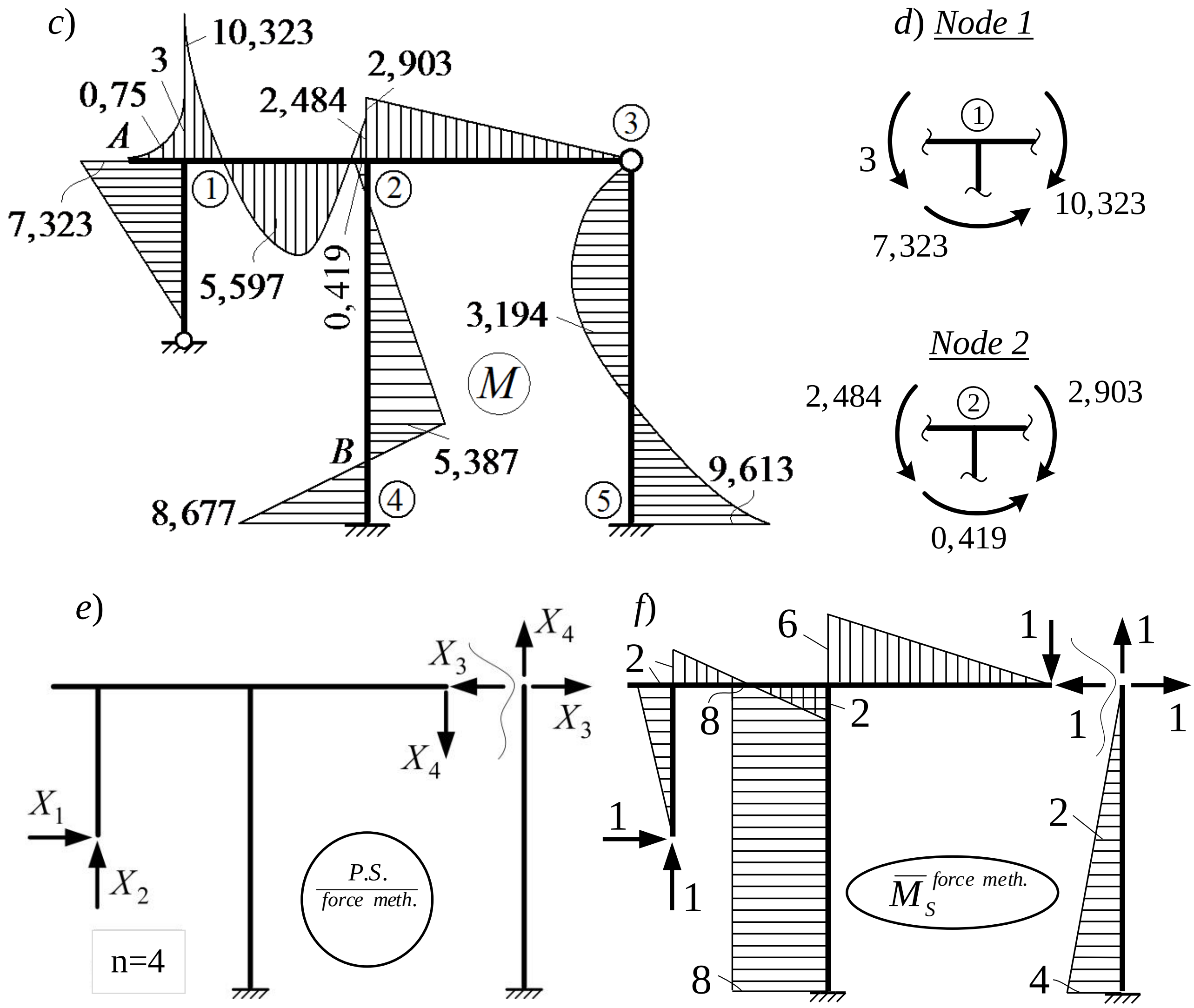


Figure 5.14 (continued)

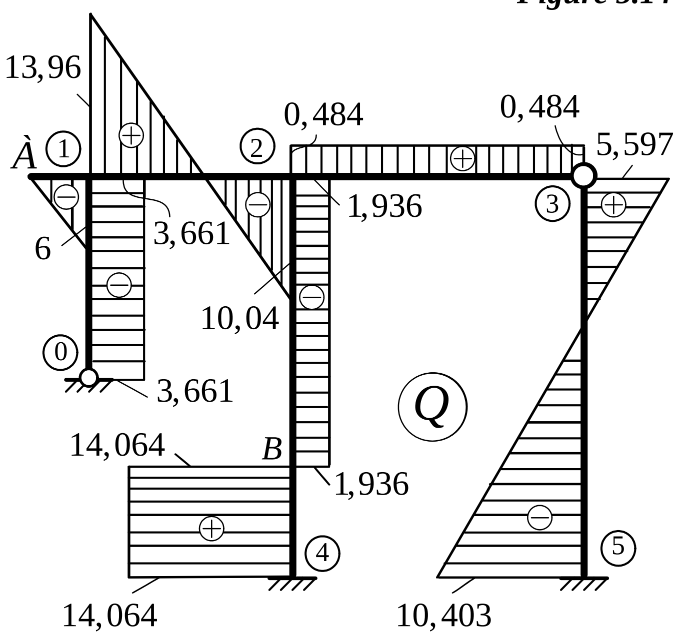


Figure 5.15

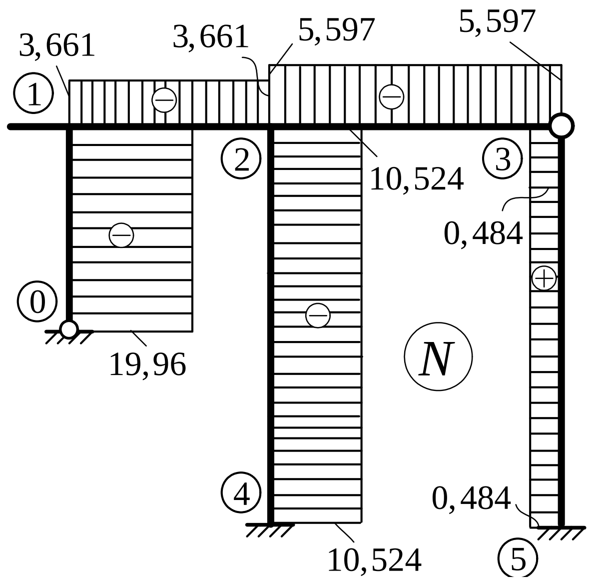


Figure 5.16

– Figure 5.13*k*: $\begin{cases} \sum M_1 = 0; & r_{13} + 9i = 0; & r_{13} = -9i; \\ \sum M_2 = 0; & r_{23} + 4,5i = 0; & r_{23} = -4,5i; \\ \sum X = 0; & r_{33} - 4,5i - 2,25i - 0,5625i = 0; & r_{33} = 7,3125i. \end{cases}$

The diagram M_P in the conjugate system is presented in figure 5.14 *a*. Absolute terms are defined by analogy with coefficients (principal deflections and secondary deflections) (see Figure 5.14 *b*):

$$\sum M_1 = 0; \qquad R_{1P} + 8 - 3 = 0; \qquad R_{1P} = -5;$$

$$\sum M_2 = 0; \qquad R_{2P} - 8 - 3 = 0; \qquad R_{2P} = 11;$$

$$\sum X = 0; \qquad R_{3P} + 2,5 - 6 = 0; \qquad R_{3P} = 3,5.$$

After substituting the found values of coefficients and absolute terms in the system of equations (3) we get the form of system:

$$\begin{cases} 30i \cdot Z_1 + 6i \cdot Z_2 - 9i \cdot Z_3 - 5 = 0; \\ 6i \cdot Z_1 + 30i \cdot Z_2 - 4,5i \cdot Z_3 + 11 = 0; \\ -9i \cdot Z_1 - 4,5i \cdot Z_2 + 7,3125i \cdot Z_3 + 3,5 = 0; \end{cases}$$

Having solved this system of equations, we will find unknown deflections of frame joints:

$$Z_1 = \frac{0,0484}{i}; \qquad Z_2 = -\frac{0,4839}{i}; \qquad Z_3 = -\frac{0,7169}{i},$$

After that, the final diagram of bending moment is plotted according to the formula:

$$M = \overline{M}_1 Z_1 + \overline{M}_2 Z_2 + \overline{M}_3 Z_3 + M_P.$$

And it will have the look presented in figure 5.14 *c*; Figure 5.14 *d* shows the balance of nodes 1 and 2 on the final diagram *M*.

To perform a deformation, check of diagram ***M*,** we have to select conjugate system in the form shown in figure 5.14 *e* with four redundant restrictions (n = 4). Total unit diagram $\overline{M}_S^{\text{force meth.}}$ is plotted at once from all the unknowns $X_1 \ldots X_4$ of unit values, shown in figure 5.14 *f*; then the deformation check can be recorded as :

$$\sum \int \frac{\overline{M}_S^{\text{force meth.}} \cdot M ds}{EJ} = 0;$$

$$\frac{1}{EJ}\left[\frac{7,323 \cdot 2}{2} \cdot \frac{2}{3} 2 + \frac{4}{6}(10,323 \cdot 2 + 4 \cdot 5,597 \cdot 0 - 2 \cdot 2,484) + \frac{2,903 \cdot 6}{2} \cdot \frac{2}{3} 6 + \frac{4}{6}(4 \cdot 2 \cdot 3,194 - \right.$$

$$-9,613\cdot 4)-(8\cdot 3)\cdot\frac{5,387-0,419}{2}+(8\cdot 1)\cdot\frac{8,677-5,387}{2}\Bigg]=\frac{1}{EJ}(9,764+13,764-$$

$$-3,312+34,838+17,032-25,634-59,612+13,161)=\frac{1}{EJ}(88,559-88,558)\approx 0;$$

$$\text{Discrepancy }\left(\left|\frac{88,559-88,558}{88,558}\right|\cdot 100\%=0,1\%\right)\text{ is negligible.}$$

The diagram of the shear force $\boldsymbol{Q}$ (Figure 5.15) can be plotted by use of areas, where diagram $\boldsymbol{M}$ changes continuously according to formula (4):

$$Q_{0-1}=-\left|\frac{7,323-0}{2}\right|=-3,661\ kN;\qquad Q_{A-1}^{left}=+\frac{6\cdot 1}{2}-\left|\frac{3-0}{1}\right|=0;$$

$$Q_{A-1}^{right}=-\frac{6\cdot 1}{2}-\left|\frac{3-0}{1}\right|=-6\ kN;$$

$$Q_{1-2}^{left}=\frac{6\cdot 4}{2}+\left|\frac{10,323-2,484}{4}\right|=12+1,96=13,96\ kN;$$

$$Q_{1-2}^{right}=-\frac{6\cdot 4}{2}+\left|\frac{10,323-2,484}{4}\right|=-12+1,96=-10,04\ kN;$$

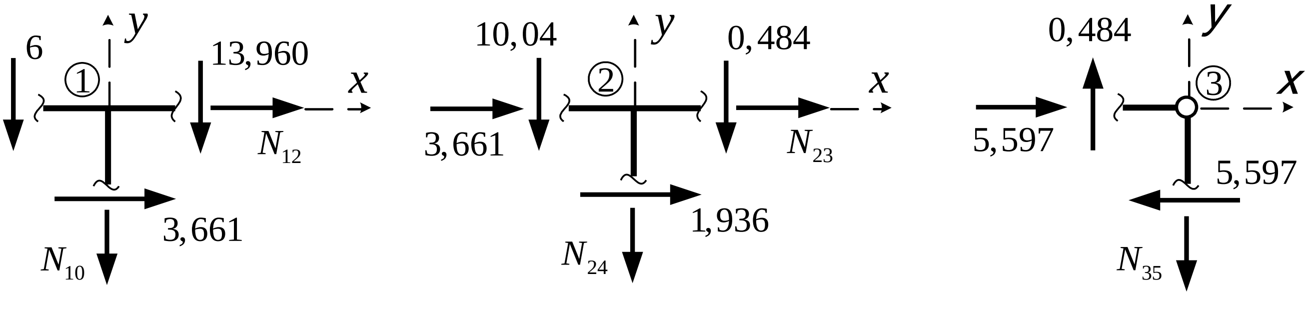
$$Q_{2-3}=+\left|\frac{2,903-0}{6}\right|=0,484\ kN;\qquad Q_{4-B}=+\left|\frac{8,677-(-5,387)}{1}\right|=14,064\ kN;$$

$$Q_{B-2}=-\left|\frac{5,387-(-0,419)}{3}\right|=-1,936\ kN;\qquad Q_{3-5}^{left}=+\frac{4\cdot 4}{2}-\left|\frac{9,613-0}{4}\right|=5,597\ kN;$$

$$Q_{3-5}^{right}=-\frac{4\cdot 4}{2}-\left|\frac{9,613-0}{4}\right|=-8-2,403=-10,403\ kN.$$

Normal force diagram (Figure 5.16) can be plotted by cutting out the nodes (from the diagram Q):

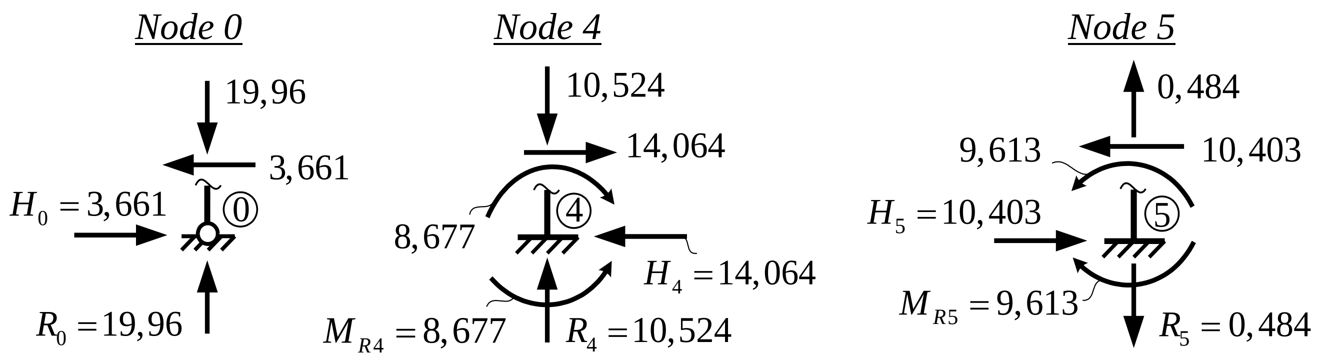
Node 1 *Node 2* *Node 3*



$\Sigma X=0;\ N_{12}=-3.661\ kN;\quad \Sigma X=0;\ N_{23}=-5,597\ kN;\quad \Sigma X=0;\ 5,597-5,597=0;$

$\Sigma Y=0;\ N_{10}=-19,96\ kN;\quad \Sigma Y=0;\ N_{24}=-10,524\ kN;\quad \Sigma Y=0;\ N_{35}=0,484\ kN;$

Cutting out the supporting joints, we can determine the supporting reactions:



Static check:

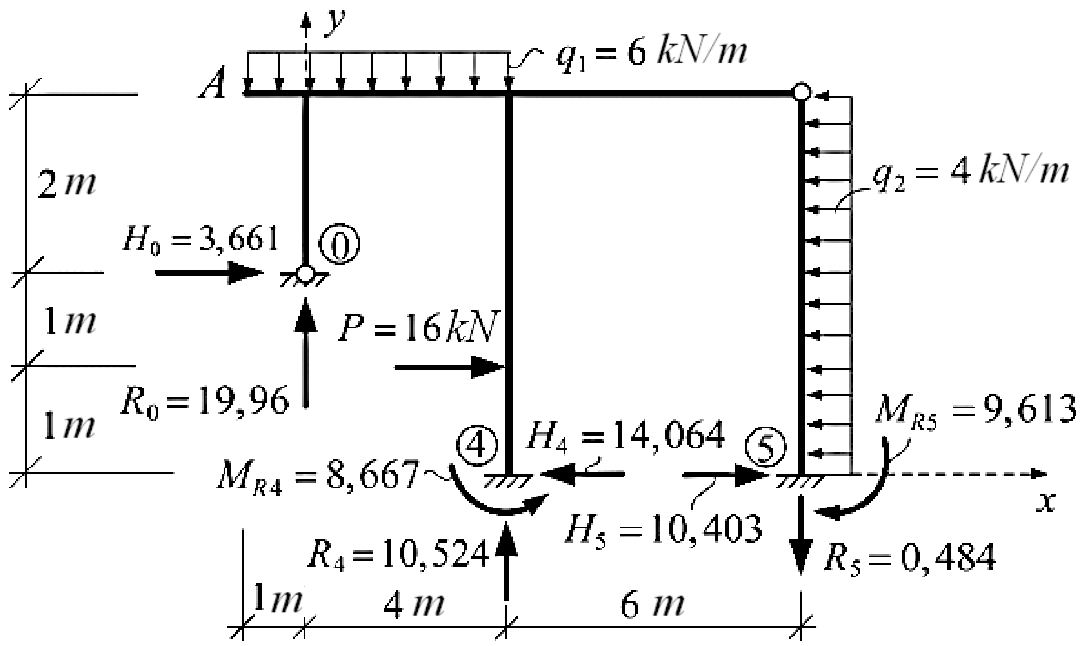


Figure 5.17

$$\sum X = 0; \quad 3,661 - 14,064 + 10,403 - 4 \cdot 4 + 16 = 0; \quad 30,064 - 30,064 = 0;$$

$$\sum Y = 0; \quad 19,96 + 10,524 - 0.484 - 6 \cdot 5 = 0; \quad 30,484 - 30,484 = 0;$$

$$\sum M_A = 0; \quad 6 \cdot 5 \cdot 2,5 - 16 \cdot 3 + 4 \cdot 4 \cdot 2 - 3,661 \cdot 2 + 14,064 \cdot 4 - 10,403 \cdot 4 -$$

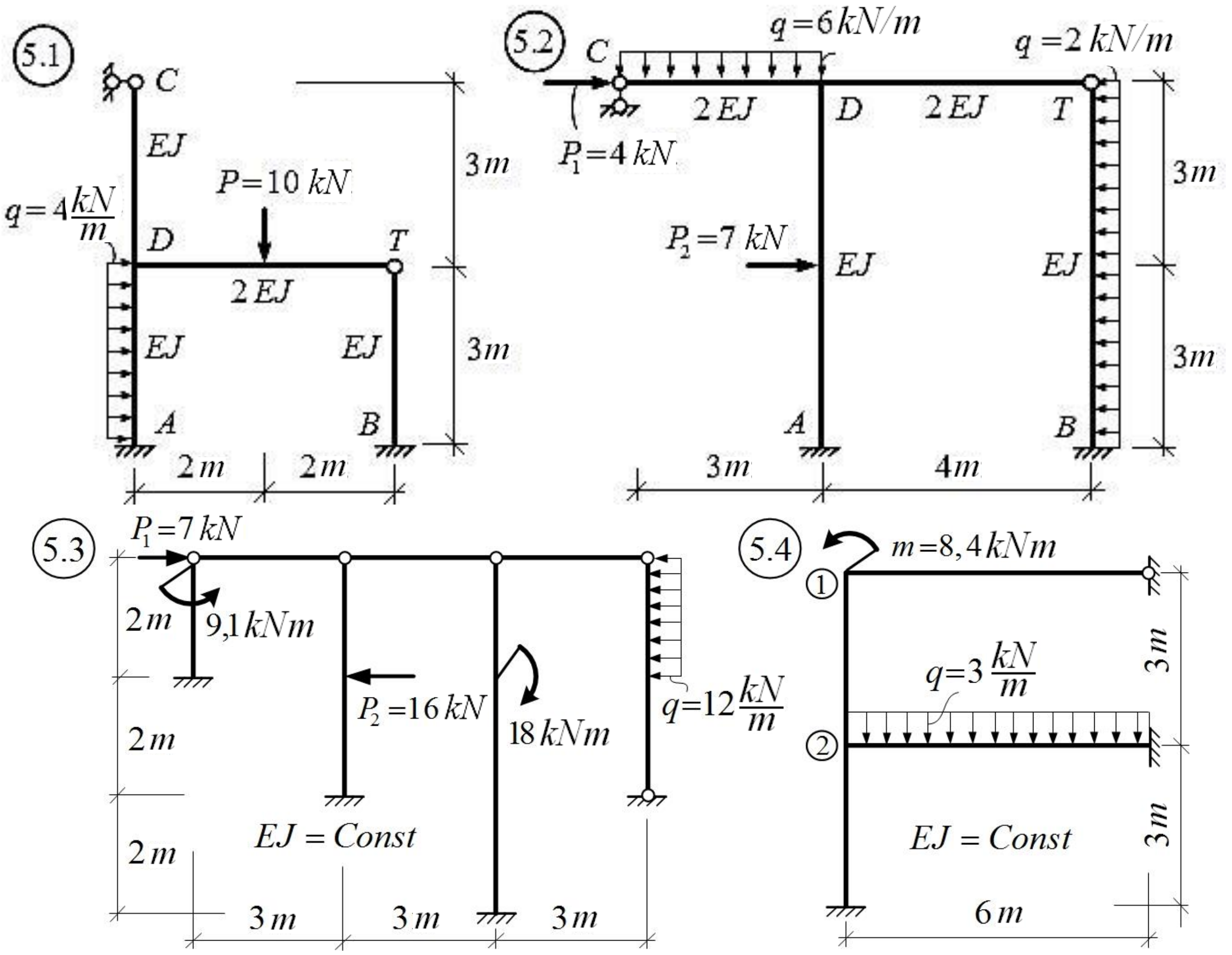
$$-19,96 \cdot 1 - 10,524 \cdot 5 + 0,484 \cdot 11 - 8,667 + 9,613 = 0;$$

$$178,193 - 178,191 \approx 0.$$

All checks carried out, the frame calculation is done.

5.11. Tasks to Self-Solution

Plot the diagrams of bending moment, shear and normal forces according to the given frames. Perform their calculations by means of the slope and deflection method (key to tasks is presented at figure on pages 63-66).



List of Literature Recommended for the Study of the Discipline “Building Mechanics”

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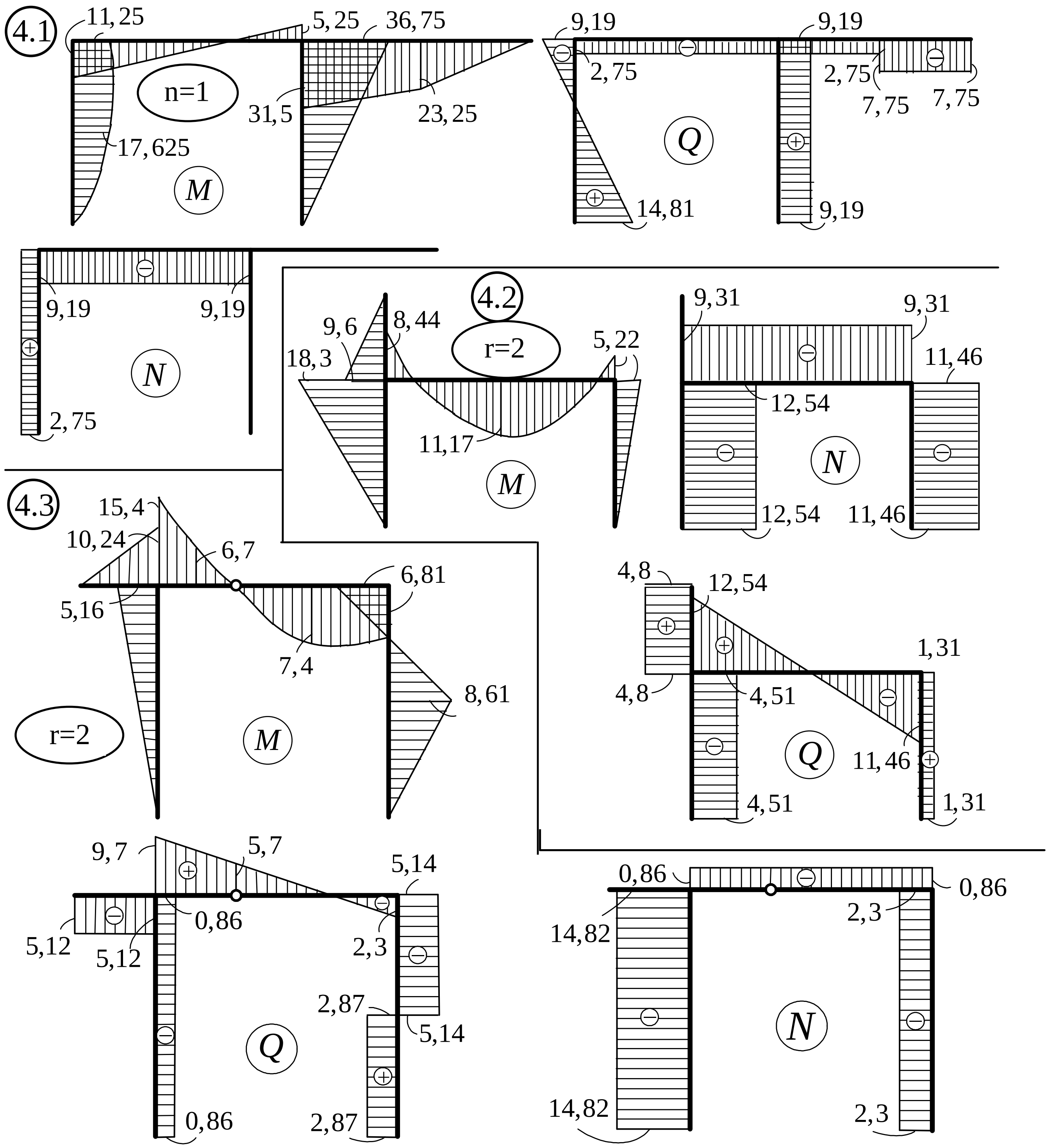
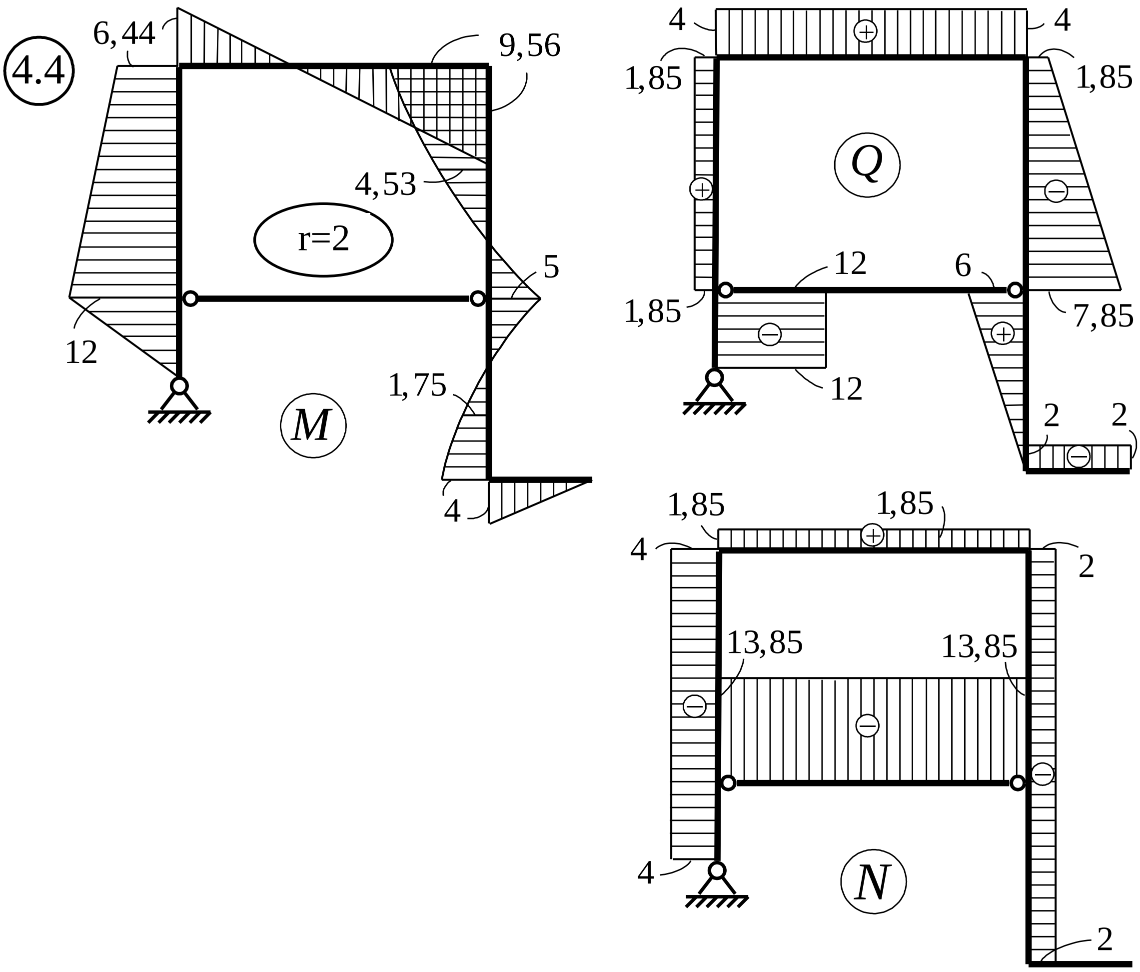
4. Анохин, Н.Н. Строительная механика в примерах и задачах : учебное пос.
– Ч. 1 : Статически определимые системы. – М. : Изд-во АСВ, 1999. – 335 с.
– Ч. 2 : Статически неопределимые системы. – М. : Изд-во АСВ, 2000. – 464 с.
(в интернете можно найти электронную версию этого учебного пособия).

5. Строительная механика : учебник : в 2 кн. / Под ред. В.Д. Потапова. – Кн. 1 : Статика упругих систем. – М. : Высшая школа, 2007. – 511 с.

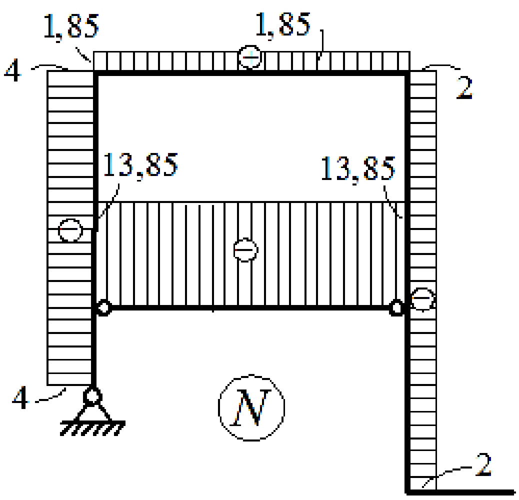
Key to Tasks for Self-Solution

Section 4. Calculating Statically Redundant Frames by Method of Force



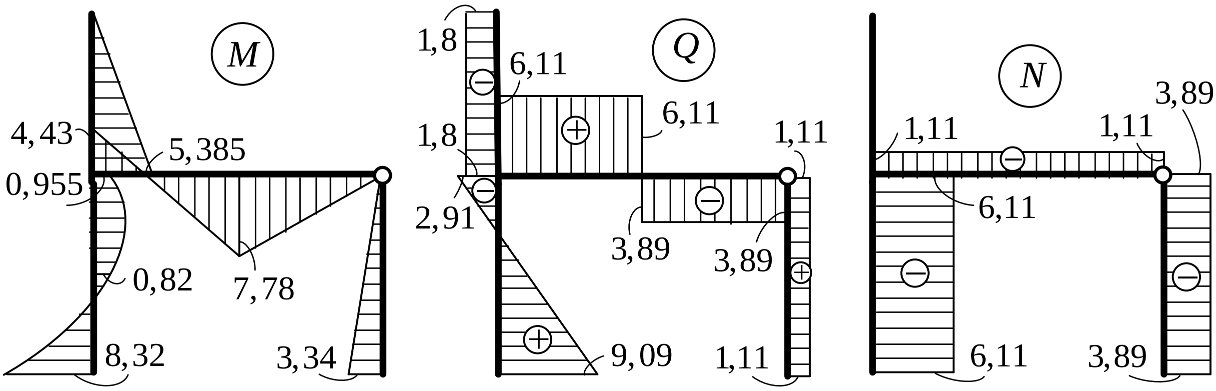


Section 5. Calculation of the Statically Redundant Frames by Slope and Deflection Method

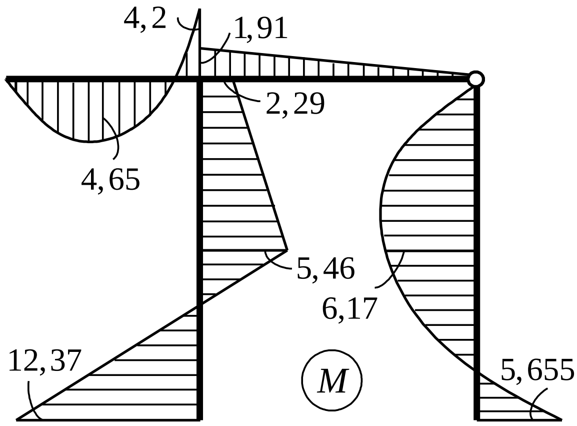


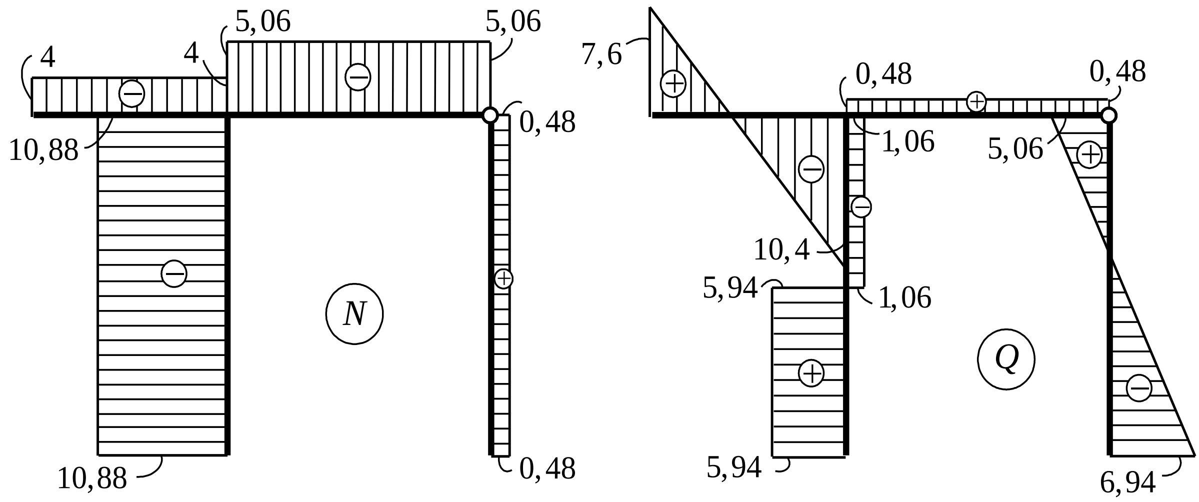
5.1. The degree of instability of the frame is equal to two ($r = 2$) – the turn of the D node (Z_1) and the linear shift (horizontally) of the D and T nodes (Z_2); at $EJ = 1$ deflections are equal to: Z_1= 2,045 (clockwise), Z_2= 10,020 (to the right). The internal forces are shown in the pictures.

5.2. The degree of instability of the frame is equal to two ($r = 2$) – turning of D node (Z_1) and a linear shift (horizontally) D and T nodes (Z_2). When $EJ = 1$ the unknowns of the slope and deflection method are equal: Z_1= 2,045 (counterclockwise), Z_2= 10,020 (to the right).



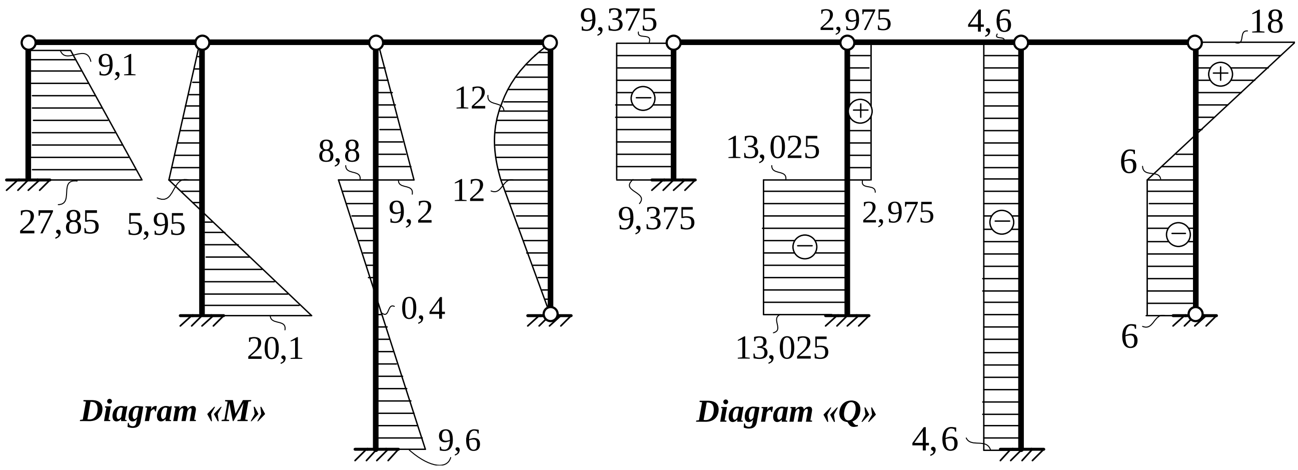
The internal forces are shown in the pictures.



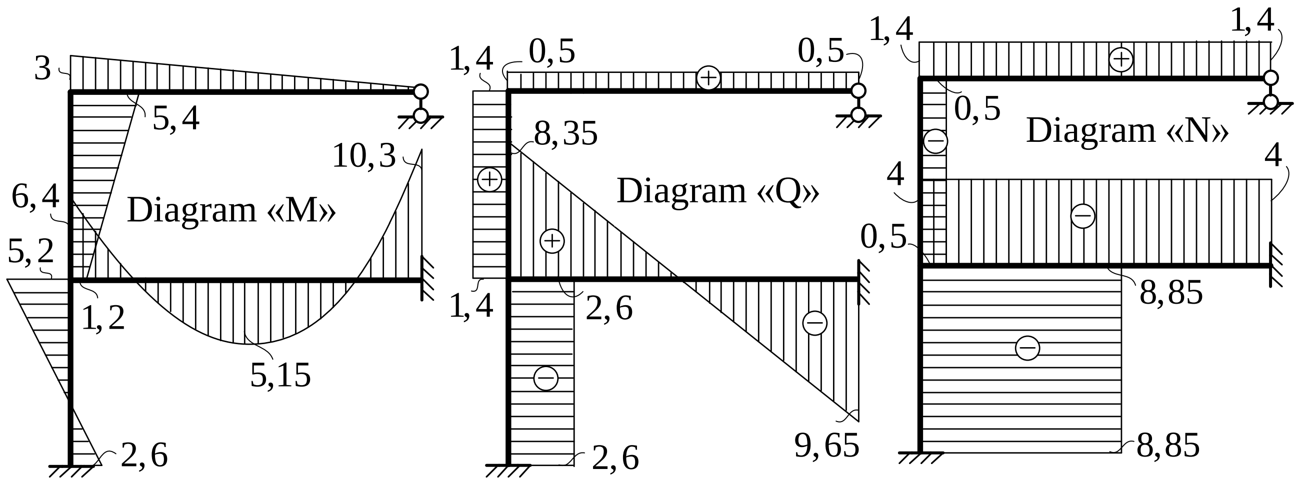


5.3 The degree of instability (freedom) of the frame is one (r =1) – the upper hinge node can move linearly horizontally.

When EJ = 108 the displacement of their nodes: Z_1 is 0,4 (to the left). Internal force diagrams are shown in the pictures (N diagram is zero).



5.4. The degree of instability (freedom) of the frame is equal to two ($r = 2$) – unknowns are the turns of 1 and 2 rigid nodes. When $EJ = 12$ these angles of turns of nodes are: $Z_1 = -\,0{,}5$ (counterclockwise), $Z_2 = 0{,}325$ (clockwise). Diagrams of internal forces are shown in figures.



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Zheltkovich Andrey
Zhao Leiyang (赵磊阳)
Gao Jian (老师)

Structural Mechanics

Part 2: STATICALLY INDETERMINATE FRAMES

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as a manual
on discipline "Structural Mechanics"
for students of building specialties

*The text is printed in the author's edition,
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