МИНИСТЕРСТВО ОБРАЗОВАНИЯ РЕСПУБЛИКИ БЕЛАРУСЬ

УЧРЕЖДЕНИЕ ОБРАЗОВАНИЯ
«БРЕСТСКИЙ ГОСУДАРСТВЕННЫЙ ТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ»

КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

FUNCTIONS
LIMITS
DERIVATIVES

for foreign first-year students

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**
для студентов 1-го курса

Брест 2014

УДК 517.1/.2 (076)=111

Настоящая методическая разработка предназначена для иностранных студентов технических специальностей. Данная разработка содержит необходимый материал по разделам «Основы математического анализа» и «Дифференциальное исчисление функций одной переменной». Изложение теоретического материала по всем темам сопровождается рассмотрением большого количества примеров и задач, некоторые понятия и примеры проиллюстрированы.

Составители: Гладкий И.И., доцент
Дворниченко А.В., старший преподаватель
Каримова Т.И., к.ф.-м.н., доцент
Лебедь С.Ф., к.ф.-м.н., доцент
Шишко Т.В., преподаватель кафедры иностранных языков по техническим специальностям

Рецензент: Мирская Е.И., доцент кафедры алгебры, геометрии и математического моделирования учреждения образования «Брестский государственный университет им. А.С. Пушкина», к.ф.-м.н., доцент.

1 FUNCTIONS

1.1 Functions

Definition Let X and Y be sets. A function from X to Y is a rule or method for assigning to each element in X a unique element in Y.

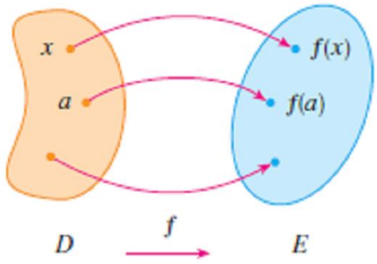


Fig.1.1

A function may be given by a formula. In daily life a function is often indicated by a table.

A function is often denoted by the symbol f. The element that the function assigns to the element x is denoted $f(x)$ (read f of x). In practice, though, almost everyone speaks interchangeably of the function f or the function $f(x)$.

Example 1 Let $f(x) = x^2$ for each real number x. Compute (a) $f(3)$, (b) $f(2)$ and (c) $f(-2)$.

Solution

(a) $f(3) = 3^2 = 9$.

(b) $f(2) = 2^2 = 4$.

(c) $f(-2) = (-2)^2 = 4$.

Definition Let X and Y be sets and let f be a function from X to Y. The set X is called the domain of the function. If $f(x) = y$, y is called the value of f at x. The set of all values of the function is called the range of the function.

When the function is given by a formula, the domain is usually understood to consist of all the numbers for which the formula is defined.

The value $f(x)$ of a function f at x is also called the output; x is called the input or argument. If $y = f(x)$, the symbol x is called the independent variable and the symbol y is called the dependent variable.

If both the inputs and outputs of a function are numbers, we shall call the function numerical. In some more advanced courses such a function is also called a real function of a real variable.

If both the domain and range of a function consist of real numbers, it is possible to draw a picture that displays the behavior of the function.

Definition Graph of a numerical function. Let f be a numerical function. The graph of ***f*** consists of those points (x, y) such that ***y=f (x).***

For instance, the graph of the squaring function $f(x) = x^2$ consist of the points (x, y) such that $y = x^2$. It is the parabola shown later.

Not every curve is the graph of a function. For instance, the curve in Fig. 1.2 is not the graph of a function. The reason is that a function assigns to a given input a single number as the output. A line parallel to the y axis therefore meets the graph of a function in more than one point. This observation provides a visual test for deciding whether a curve in a plane is the graph of a function $y = f(x)$. If some line parallel to the y axis meets the curve more than once, then the curve is not the graph of a function. Otherwise it is the graph of a function. The curve in Fig. 1.3 is the graph of a function.

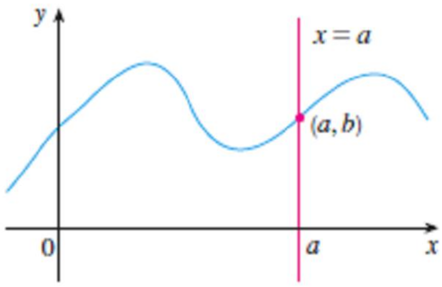


Fig.1.3.

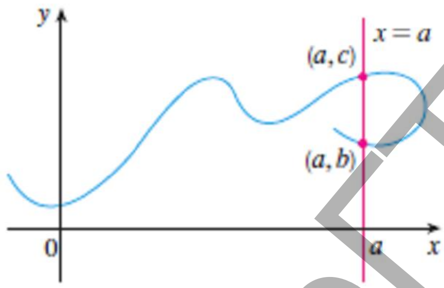


Fig.1.2.

Example 2 Let f be the squaring function $f(x) = x^2$. Compute (a) $f(2+3)$ and (b) $f(2+h)$.

Solution (a) For any number x, $f(x)$ is the square of that number. Thus

$$f(2+3) = (2+3)^2 = 5^2 = 25$$

(b) Similarly,

$$f(2+h) = (2+h)^2 = 4 + 4h + h^2.$$

Warning. A common error is to assume that $f(2+3)$ is somehow related to $f(2) + f(3)$. For most functions there is no relation between the two numbers. In the case of the function x^2, $f(2) + f(3) = 2^2 + 3^2 = 4 + 9 = 13$, but, $f(2+3) = 25$.

Example 3 Let f be the cubing function $f(x) = x^3$. Evaluate the difference $f(2+0.1) - f(2)$.

Solution

$$f(2+0.1) - f(2) = f(2.1) - f(2) = (2.1)^3 - 2^3 = 9.261 - 8 = 1.261.$$

Exercise Set 1

In Exercises 1 to 10 graph the functions.

1. $f(x) = 3x$.
2. $f(x) = -2x$.
3. $f(x) = 3x^2$.
4. $f(x) = 1 + x^2$.
5. $f(x) = 1 - x^2$.
6. $f(x) = 2 - 3x^2$.
7. $f(x) = x^2 - x$.
8. $f(x) = x^2 + 2x + 1$.
9. $f(x) = \dfrac{2}{1+x^2}$.
10. $f(x) = \dfrac{1}{1+2x^2}$.

In Exercises 11 to 20 describe the domain and range of each function.

11. $f(x) = \sqrt{x}$.
12. $f(x) = \sqrt{x+3}$.
13. $f(x) = \sqrt{4 - x^2}$.

14. $f(x)=\sqrt{4+x^2}$.   15. $f(x)=\dfrac{2}{x^2}$.   16. $f(x)=\dfrac{2}{1+x}$.

17. $f(x)=\dfrac{1}{x^3}$.   18. $f(x)=\dfrac{1}{x^4}$.   19. $f(x)=\dfrac{1}{\sqrt{x}}$.

20. $f(x)=\dfrac{2}{1-x^2}$.

In each of Exercises 21 to 24 compute as decimals the outputs of the given function for the given inputs.

21. $f(x)=x+1$: (a)-1; (b) 3; (c) 1.25; (d) 0.

22. $f(x)=\dfrac{2}{1+x}$: (a) -3; (b) 3; (c) 9; (d) 99.

23. $f(x)=x^3$: (a) 1+2; (b) 4-1.

24. $f(x)=\dfrac{2}{x^2}$: (a) 5-3; (b) 4-6.

In Exercises 25 to 30 for the given functions evaluate and simplify the given expressions. (Assume that no denominator is 0.)

25. $f(x)=x^3$: $f(a+1)-f(a)$.

26. $f(x)=\dfrac{1}{x}$: $f(a+h)-f(a)$.

27. $f(x)=\dfrac{1}{x^2}$: $\dfrac{f(d)-f(c)}{d-c}$.

28. $f(x)=\dfrac{1}{2x+1}$: $\dfrac{f(x+h)-f(x)}{h}$.

29. $f(x)=x+\dfrac{1}{x}$: $\dfrac{f(d)-f(c)}{d-c}$.

30. $f(x)=3-\dfrac{1}{x}$: $\dfrac{f(x+h)-f(x)}{h}$.

31. Graph $f(x)=x(x-1)(x+1)$.

(a) For which values of x is $f(x)=0$?

(b) Where does the graph cross the x axis?

(c) Where does the graph cross the y axis?

1.2 Composite Functions

This section describes a way of building up functions by applying one function to the output of another. For instance, the function $y=(1+x^2)^{100}$ is built up by raising $1+x^2$ to the one-hundredth power. That is, $y=u^{100}$, where $u=1+x^2$.

The theme common to these two examples is spelled out in the following definition.

Definition (Composition of functions) Let f and g be functions. Suppose that x is such that $g(x)$ is in the domain of f. Then the function that assigns to x the value $f(g(x))$ is called the composition of f and g. It is denoted $f \circ g$.

Thus if $g(x) = u$ and $f(u) = y$, then $(f \circ g)(x) = y$. ($f \circ g$ is read as f circle g or as f composed with g). In practical terms, the definition says: "To compute $f \circ g$, first apply g and then apply f to the result".

A function can be the composition of more than two functions. For example, $\sqrt{(1+x^2)^5}$ is the composition of three functions. First $1+x^2$ is formed; then the fifth power; then a square root. More formally, the assertion that $y = \sqrt{(1+x^2)^5}$ is the same as saying that $y = \sqrt{u}$, $u = v^5$ and $v = 1 + x^2$.

Example 1 Write $y = 2^{x^2}$ as a composition of functions.

Solution

$$y = 2^u \text{, where } u = x^2.$$

Example 2 Let $f(x) = 2x + 1$ and $g(x) = x^2$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they equal?

Solution

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 1 + 2x^2.$$

$$(g \circ f)(x) = g(f(x)) = g(1+2x) = (1+2x)^2.$$

Since the function $(1+2x)^2$ is not equal to $1+2x^2$, $f \circ g$ is not equal to $g \circ f$. This shows that $f \circ g$ is not necessarily equal to $g \circ f$.

Example 3 Let $f(x) = -x$. Compute $(f \circ f)(x)$.

Solution

$$(f \circ f)(x) = f(f(x)) = f(-x) = -(-x) = x.$$

Thus $(f \circ f)(x) = x$.

Certain functions behave nicely when composed with the function $-x$. That is, their values at $-x$ are closely related to their values at x. The following definitions make this precise.

Definition (Even function) A function f such that $f(-x) = f(x)$ is called an even function (See Figure 1.4).

Consider, for instance, $f(x) = x^4$. We have

$$f(-x) = (-x)^4 = x^4 = f(x).$$

Thus $f(x) = x^4$ is an even function.

In fact, for any even integer n, $f(x) = x^n$ is an even function (hence the name).

Definition (Odd function) A function f such that $f(-x) = -f(x)$ is called an odd function(See Figure 1.5).

The function $f(x) = x^3$ is odd since

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

For any odd integer n, $f(x) = x^n$ is an odd function.

Most functions are neither even nor odd. For instance, $x^3 + x^4$ is neither even nor odd since $(-x)^3 + x^4 = -x^3 + x^4$, which is neither $x^3 + x^4$ nor $-(x^3 + x^4)$. However, many functions used in calculus happen to be even or odd. The graph of such a function is symmetric with respect to the y axis or with respect to the origin, as will now be shown.

Consider an even function f. Assume that the point (a,b) is on the graph of f. That means that $f(a)=b$. Since f is even, $f(-a)=b$. Consequently, the point $(-a,b)$ is also on the graph of f. In other words, the graph of an even function is symmetric with respect to the y axis.

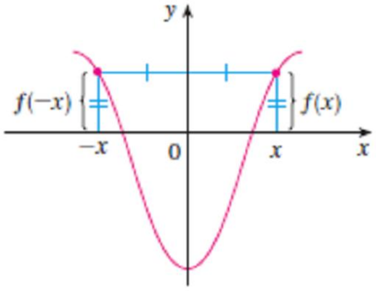


Fig. 1.4

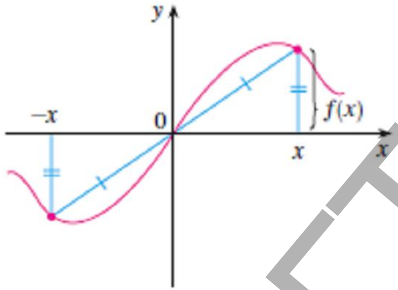


Fig. 1.5

Exercise Set 2

In each of Exercises 1 to 4 find the function $y=f(x)$ defined by the composition of the given functions.

1. $y=u^3$, $u=x^2$.
2. $y=1+u^2, u=1+x$.
3. $y=\sqrt{u}$, $u=1+2v$, $v=x^3$.
4. $y=\frac{1}{u}, u=3+v$, $v=x^2$.

1.3 One-to-one Functions and Their Inverse Functions

With some functions, "the output determines the input". For instance, the cubing function, $f(x)=x^3$, has this property. If we are told that the output of this function is, say, 64, then we know that the input must have been 4. However, the squaring function, $f(x)=x^2$, does not have this property. If we are told that output of this function is, say, 25, then we do not know what the input is. It could be 5 or -5, since $5^2=25$ and $(-5)^2=25$.

Definition A function that assigns distinct outputs to distinct inputs is called a one-to-one function.

For instance, x^3 is a one-to-one function, but x^2 (with domain taken to be the entire axis) is not one-to-one.

The graph of one-to-one numerical function has the property that every horizontal line meets it in more than one point. To see why, consider the line $y=k$ in Fig. 1.6.

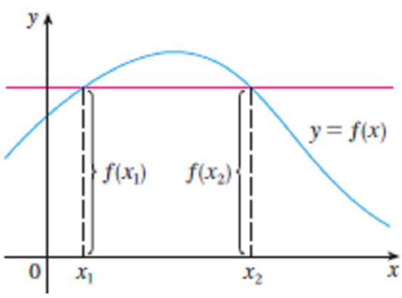


Fig. 1.6

If it meets the graph of a function f in at least two distinct points, say (x_1, k) and (x_2, k), then $f(x_1) = k$ and $f(x_2) = k$. This means that is not a one-to-one function, since the outputs corresponding to the inputs x_1 and x_2 are equal, namely, k.

On the other hand, if each horizontal line meets the graph of a function f in more than one point, then f is one-to-one.

Definition If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is an increasing function. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is a decreasing function.

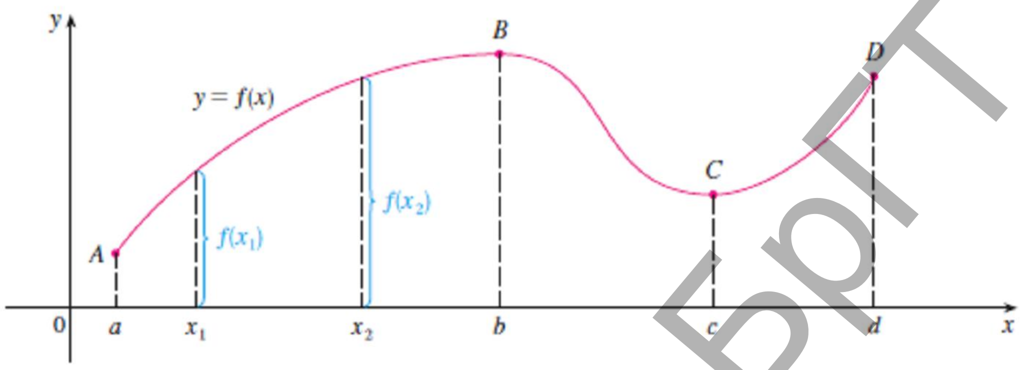


Fig. 1.7

These are illustrated in Fig. 1.7. (These two types of functions are also called monotonic.)

The function $f(x) = x^2$ is not increasing if its domain is taken to be the entire x axis. However, it is an increasing function if it is considered only for $x \geq 0$.

Definition Let $y = f(x)$ be a one-to-one function. The function g that assigns to each output of f the corresponding unique input is called the inverse of f. That is, if $y = f(x)$, then $x = g(y)$.

For example, $y = x^3$ is a one-to-one function. Its inverse is found by solving for x in terms of y; that is, $x = \sqrt[3]{y}$.

Example 1 Determine the inverse of the "doubling" function f defined by $f(x) = 2x$.

Solution If $y = 2x$, there is only one value of x for each value of y, and it is obtained by solving the equation $y = 2x$ for x: $x = \frac{y}{2}$. Thus f is one-to-one and its inverse function g is the "halving" function: If y is the input in the function g, then the output is $\frac{y}{2}$.

For instance, $f(3) = 6$ and $g(6) = 3$. Thus $(3,6)$ is on the graph of f, and $(6,3)$ is on the graph of g. Since it is customary to reserve the x axis for inputs, we should write the formula for g, the "halving" function, as $g(x) = \frac{x}{2}$.

Example 2 One graph is obtained from the other by reflecting it across the line $y = x$. This can be done because, if (a,b) is on the graph of one function, then (b,a) is on the graph of the other. If you fold the paper along the line $y = x$, the point (b,a) comes together with the point (a,b), as you will note in Fig. 1.8. This relation between the graphs holds for any one-to-one function and its inverse.

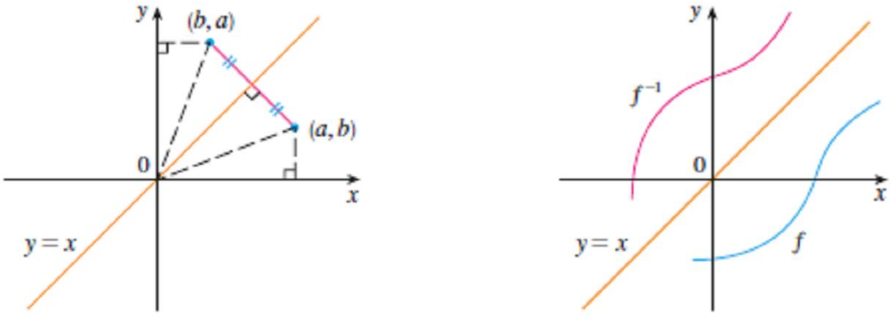


Fig. 1.8

These examples are typical of the correspondence between a one-to-one function and its inverse. Perhaps the word "reverse" might be more descriptive than "inverse". One final matter of notation: we have used the letter g to denote the inverse of f. It is common to use the symbol f^{-1} (read as "f inverse") to denote the inverse function. We preferred to delay its use because its resemblance to the reciprocal notation might cause confusion. It should be clear from the examples that f^{-1}, does not mean to divide 1 by f. The symbol inv f would be unambiguous. However, it is longer than the symbol f^{-1} and the weight of tradition is defied f^{-1}.

Inverse functions come in pairs, each reversing the effect of the other. This table lists some pairs of inversely related functions:

Function f	***Inverse Function*** g
Cubing, $y = x^3$.	Cube root, $x = \sqrt[3]{y}$.
Cube root, $y = \sqrt[3]{x}$.	Cubing, $x = y^3$.
Squaring, $y = x^2$, $x \ge 0$.	Square root, $x = \sqrt{y}$, $y \ge 0$.
Square root, $y = \sqrt{x}$, $x \ge 0$	Squaring, $x = y^2$, $y \ge 0$.

2 LIMITS AND CONTINUOUS FUNCTIONS

2.1 The Limit of a Function

Three examples will introduce the notion of the limit of a numerical function. After them, the concept of a limit will be defined.

Example 1 Let $f(x) = 2x^2 + 1$. What happens to $f(x)$ as x is chosen closer and closer to 3?

Solution Let us make a table of the values of $f(x)$ for some choices of x near 3. When x is close to 3, $2x^2 + 1$ is close to $2 \cdot 3^2 + 1 = 19$. We say that "the limit of $2x^2 + 1$ as x approaches 3 is 19" and write

$$\lim_{x \to 3} \left(2x^2 + 1\right) = 19.$$

Example 1 presented no obstacle. The next example offers a slight challenge.

Example 2 Let $f(x) = \left(x^3 - 1\right) / \left(x^2 - 1\right)$. Note that this function is not defined when $x = 1$, for when x is 1, both numerator and denominator are 0. But we have every right to ask: How does $f(x)$ behave when x is near 1 but is not 1 itself?

Solution First make a brief table of values of $f(x)$, to four decimal places, for x near 1. Choose some x larger than 1 and some x smaller than 1. For instance.

$$f(1.01)=\frac{1.01^3-1}{1.01^2-1}=\frac{1.030301-1}{1.0201-1}=\frac{0.030301}{0.0201}=1.5075.$$

(If you have a calculator handy, evaluate $\left(x^3-1\right)/\left(x^2-1\right)$ at 1.001 and 0.999 as well)

There are two influences acting on the fraction $\left(x^3-1\right)/\left(x^2-1\right)$ when x is near 1. On the one hand, the numerator x^3-1 approaches 0; thus there is an influence pushing the fraction toward 0*. On the other hand, the denominator* x^2-1 *also approaches 0; division by a small number tends to make a fraction large.* How do these two opposing influences balance out?

The algebraic identities

$$x^3-1=\left(x^2+x+1\right)(x-1)$$

$$x^2-1=(x+1)(x-1)$$

enable us to answer the question.

Rewrite the quotient $\left(x^3-1\right)/\left(x^2-1\right)$ as follows: When $x\neq 1$, we have

$$\frac{x^3-1}{x^2-1}=\frac{\left(x^2+x+1\right)(x-1)}{(x+1)(x-1)}=\frac{x^2+x+1}{x+1},$$

so the behavior of $\left(x^3-1\right)/\left(x^2-1\right)$ for x near 1, but not equal to 1, is the same as the behavior of $\left(x^2+x+1\right)/(x+1)$ for x near1, but not equal to 1. Thus

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{x^2+x+1}{x+1}.$$

Now, as x approaches 1, x^2+x+1 approaches 3 and $x+1$ approaches 2. Thus

$$\lim_{x\to 1}\frac{x^2+x+1}{x+1}=\frac{3}{2}$$

from which it follows that

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\frac{3}{2}$$

Note that $\frac{3}{2}=1.5$, which is closely approximated by $f(1.01)$ and $f(0.99)$.

The arrow $\to$ will stand for “approaches”. According to Example 2, as

$$x\to 1,\ \frac{x^3-1}{x^2-1}\to\frac{3}{2}.$$

This notation will be used in the next example and often later.

Example 3 Consider the function f defined by $f(x)=\dfrac{x}{|x|}$.

The domain of this function consists of every number except 0. For instance,

$$f(3)=\frac{3}{|3|}=\frac{3}{3}=1$$

and

$$f(-3)=\frac{-3}{|-3|}=\frac{-3}{3}=-1$$

When x is positive, $f(x)=1$. When x is negative, $f(x)=-1$. This is shown in Fig. 2.1. The graph does not intersect the y axis, since f is not defined for $x=0$. The hollow circles at $(0,1)$ and $(0,-1)$ indicate that those points are not on the graph. What happens to $f(x)$ as $x \to 0$?

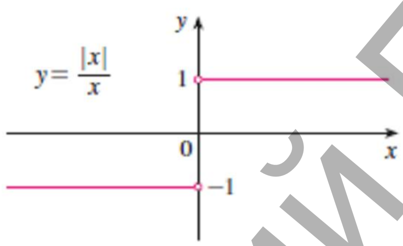


Fig. 2.1

Solution As $x \to 0$ through positive numbers, $f(x)\to 1$ since $f(x)=1$ for any positive number. When x is near 0, it is not the case that $f(x)$ is near one specific number.

Thus $\lim\limits_{x\to 0} f(x)$ does not exist, that is, $\lim\limits_{x\to 0}\dfrac{x}{|x|}$ does not exist. However, if $a\neq 0$, $\lim\limits_{x\to a} f(x)$ does exist, being 1 when a is positive and -1 when a is negative. Thus $\lim\limits_{x\to a} f(x)$ exists for all a other than 0.

Whether a function f has a limit at a has nothing to do with $f(a)$ itself. In fact, ***a*** might not even be in the domain of f. See, for instance, Examples 2 and 3. In Example 1, $a=3$ happened to be in the domain of f, but that fact did not influence the reasoning. It is only the behavior of $f(x)$ for x *near* a that concerns us.

These three examples provide a background for describing the limit concept which will be used throughout the text.

Consider a function f and a number a which may or may not be in the domain of f. In order to discuss the behavior of $f(x)$ for x near a, we must know that domain of f contains numbers arbitrarily close to a. Note how this assumption is built into each of the following definitions.

Definition(*Limit of* $f(x)$ *at* ***a****)* Let f be a function and a some fixed number. Assume that domain of f contains open intervals (c,a) and (a,b), as shown in Fig. 2.2. If there is a number L such that as x approaches a, either from the right or from the left, $f(x)$ approaches L, then L is called the limit of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to a.$$

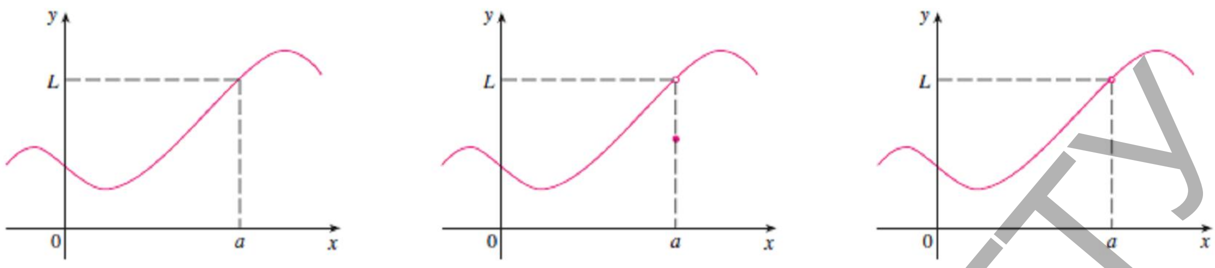


Fig. 2.2

Definition(*Right-hand limit of* $f(x)$ *at* a) Let f be a function and a some fixed number. Assume that the domain of f contains as open interval (a,b). If, as x approaches a from the right, $f(x)$ approaches a specific number L, then L is called the right-hand of $f(x)$ as x approaches a.

This is written

$$\lim_{x\to a^+} f(x) = L$$

or

$$\text{as } x \to a^+, \quad f(x) \to L.$$

The assertion that

$$\lim_{x\to a^+} f(x) = L$$

is read "the limit of f of x as x approaches a".(See Fig.2.3)

The left-hand limit is defined similarly. The only differences are that the domain of f must contain an open interval of the form (c,a) and $f(x)$ is examined as x approaches a from the left.(See Fig.2.4)

The notations for the left-hand limit are

$$\lim_{x\to a^-} f(x) = L$$

$$\text{or} \quad \text{as } x \to a^-, \quad f(x) \to L.$$

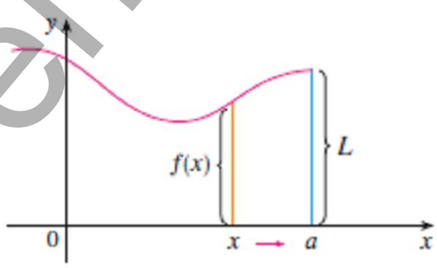


Fig. 2.3

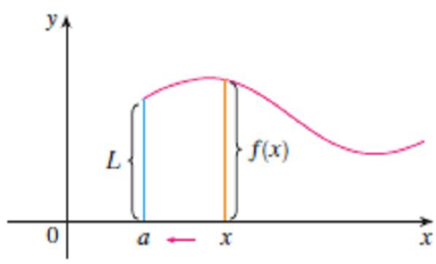


Fig 2.4

As Example 3 showed $\lim\limits_{x\to 0^+} \dfrac{x}{|x|} = 1$ and $\lim\limits_{x\to 0^-} \dfrac{x}{|x|} = -1.$

We could also write, for instance.

As $x \to 0^+$, $\frac{x}{|x|} \to 1$. Note that if both the right-hand and the left-hand limits of f exist at a and are equal, then $\lim_{x\to a} f(x)$ exists. But if the right-hand limits are not equal, then $\lim_{x\to a} f(x)$ does not exist. For instance, $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

By contrast, the tamest functions are the "constant" functions. If, say, a function L is constant, then $f(x) = L$ for all x. We have

$$\lim_{x\to a} f(x) = L.$$

It may seem strange to say that "the limit of L is L," but in practice this offers no difficulty. For instance,

$$\lim_{x\to 5} \frac{1+x^2}{1+x^2} = \lim_{x\to 5} 1 = 1$$

and $\lim_{x\to 3} 1^x = \lim_{x\to 3} 1 = 1.$

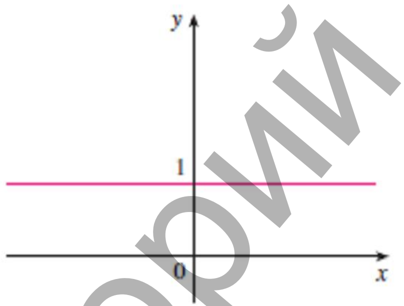


Fig. 2.5

Exercise Set 3

In Exercises 1 to 14 find the limits, all of which exist. Use intuition and, if needed, algebra.

1. $\lim_{x\to 5}(x+7)$	2. $\lim_{x\to 1}(4x-2)$	3. $\lim_{x\to 2}\frac{x^2-4}{x-2}$	4. $\lim_{x\to 3}\frac{x^2-9}{x-3}$
5. $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$	6. $\lim_{x\to 1}\frac{x^6-1}{x^3-1}$	7. $\lim_{x\to 3}\frac{1}{x+2}$	8. $\lim_{x\to 5}\frac{3x+5}{4x}$
9. $\lim_{x\to 3} 25$	10. $\lim_{x\to 3}\pi^2$	11. $\lim_{x\to 0^+}\sqrt{x}$	12. $\lim_{x\to 1^+}\sqrt{4x-4}$
13. $\lim_{x\to 1^+}\frac{x-1}{\lvert x-1\rvert}$	14. $\lim_{x\to 1^-}\frac{x-1}{\lvert x-1\rvert}$		

In Exercises 15 to 22 decide whether the limits exist and, if they do, evaluate them.

15. $\lim_{h\to 1}\frac{(1+h)^2-1}{h}$	16. $\lim_{h\to 0}\frac{(1+h)^2-1}{h}$
17. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$	18. $\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

19. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$ (Hint: Rationalize the numerator.)	20. $\lim_{x\to 0}\frac{\sqrt{x+4}+2}{x}$
21. $\lim_{x\to 4^+}\left(\sqrt{x-4}+2\right)$	22. $\lim_{x\to 0} 64^x$

2.2 Computations of Limits

Certain frequently used properties of limits should be put on the record.

Theorem Let f and g be two functions and assume that

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

both exist. Then

1. $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$.
2. $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$.
3. $\lim_{x\to a}kf(x)=k\lim_{x\to a}f(x)$ for any constant k.
4. $\lim_{x\to a}f(x)g(x)=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$.
5. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$ if $\lim_{x\to a}g(x)\neq 0$.
6. $\lim_{x\to a}f(x)^{g(x)}=\left(\lim_{x\to a}f(x)\right)^{\lim_{x\to a}g(x)}$ if $\lim_{x\to a}f(x)>0$.

Example 1 Suppose that $\lim_{x\to 3}f(x)=4$ and $\lim_{x\to 3}g(x)=5$; discuss $\lim_{x\to 3}f(x)/g(x)$.

Solution By property 5, $\lim_{x\to 3}f(x)/g(x)$ exists and $\lim_{x\to 3}\frac{f(x)}{g(x)}=\frac{4}{5}$.

No further information about f and g is needed to determine the limit of $f(x)/g(x)$ as $x\to 3$.

Example 2 Suppose that $\lim_{x\to 3}f(x)=0$ and $\lim_{x\to 3}g(x)=0$; discuss $\lim_{x\to 3}f(x)/g(x)$.

Solution In contrast to Example 1, in this case property 5 gives no information, since $\lim_{x\to 3}g(x)=0$. It is necessary to have more information about f and g.

For instance, if

$$f(x)=x^2-9 \quad \text{and} \quad g(x)=x-3,$$

then

$$\lim_{x\to 3}f(x)=0 \quad \text{and} \quad \lim_{x\to 3}g(x)=0$$

and the limit of the quotient is

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

Loosely put, "when x is near 3, x^2-9 is about 6 times as large as $x-3$."

A different choice of f and g could produce a different limit for the quotient $f(x)/g(x)$. To be specific, let

$$f(x)=(x-3)^2 \quad \text{and} \quad g(x)=x-3.$$

Then $\lim\limits_{x\to 3} f(x)=0$ and $\lim\limits_{x\to 3} g(x)=0$.

And the limit of the quotient is

$$\lim_{x\to 3}\frac{(x-3)^2}{x-3}=\lim_{x\to 3}(x-3)=0.$$

In this case we could say "$(x-3)^2$ approaches 0 much faster than does $x-3$, when $x\to 3$."

In short, the information that $\lim\limits_{x\to 3} f(x)=0$ and $\lim\limits_{x\to 3} g(x)=0$ is not enough to tell us how $f(x)/g(x)$ behaves as $x\to 3$.

Sometimes it is useful to know how $f(x)$ behaves when x is a very large positive number (or a negative number of large absolute value). Example 3 serves as an illustration and introduces a variation on the theme of limits.

Example 3 Determine how $f(x)=1/x$ behaves for (a) large positive inputs and (b) negative inputs of large absolute value.

Solution First make a table of values as shown in the margin. As x gets arbitrarily large, $\frac{1}{x}$ approaches 0.

(a) This is similar to (a). For instance,

$$f(-1000)=-0.001.$$

As negative numbers x are chosen of arbitrarily large absolute value, $\frac{1}{x}$ approaches 0. (See Fig.2.6)

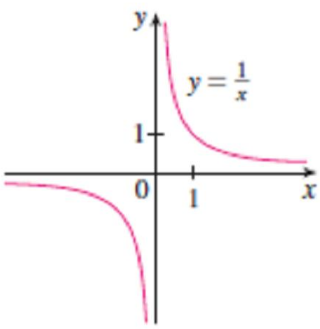


Fig.2.6

Rather than writing "as x gets arbitrarily large through positive values, $f(x)$ approaches the number L," it is customary to use the shorthand

$$\lim_{x\to\infty} f(x) = L.$$

This is read: "as x approaches infinity, $f(x)$ approaches L," or "the limit of $f(x)$ as x approaches infinity is L." For instance,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

More generally, for any fixed positive exponent a,

$$\lim_{x\to\infty} \frac{1}{x^a} = 0.$$

Similarly, the assertion that "as negative numbers x are chosen of arbitrarily large absolute value, $f(x)$ approaches the number L" is abbreviated to

$$\lim_{x\to-\infty} f(x) = L.$$

For instance, $\lim\limits_{x\to-\infty} \frac{1}{x} = 0.$

The six properties of limits stated at the beginning of the section hold when "$x \to a$" is replaced by "$x \to \infty$" or by "$x \to -\infty$."

It could happen that as $x \to \infty$, a function $f(x)$ becomes and remains arbitrarily large and positive. For instance, as $x \to \infty, x^3$ gets arbitrarily large. The shorthand for this is

$$\lim_{x\to\infty} f(x) = \infty.$$

For instance,

$$\lim_{x\to\infty} x^3 = \infty.$$

It is important, when reading the shorthand

$$\lim_{x\to\infty} f(x) = \infty.$$

to keep in mind that "∞" is not a number. The limit does not exist.

Properties 1 to 6 cannot, in general, be applied in such cases. Other notations, such as $\lim\limits_{x\to\infty} f(x) = -\infty$ or $\lim\limits_{x\to\infty} f(x) = \infty$ are defined similarly. For instance,

$$\lim_{x\to-\infty} x^3 = -\infty.$$

It can be shown that if, as $x \to \infty, f(x) \to \infty$ and $g(x) \to L > 0$, then $\lim\limits_{x\to\infty} f(x)g(x) = \infty$. This fact is used in the next example.

Example 4 Discuss the behavior of $2x^3 - 11x^2 + 12x$ when x is large.

Solution First consider x positive and large. The three terms, $2x^3, -11x^2$, and $12x$, all become of large absolute value. To see how the function $2x^3 - 11x^2 + 12x$ behaves for large positive x, factor out x^3:

$$2x^3 - 11x^2 + 12x = x^3\left(2 - \frac{11}{x} + \frac{12}{x^2}\right).$$

Now, since $\frac{11}{x}$ and $\frac{12}{x^2} \to 0$ as $x \to \infty$,

$$\lim_{x\to\infty}\left(2 - \frac{11}{x} + \frac{12}{x^2}\right) = 2.$$

Moreover, as $x \to \infty, x^3 \to \infty$. Thus

$$\lim_{x\to\infty} x^3\left(2 - \frac{11}{x} + \frac{12}{x^2}\right) = \infty;$$

Hence $\lim_{x\to\infty}\left(2x \;\; - 11x^2 + 12x\right) = \infty.$

Now consider x negative and of large absolute value. The argument is similar. Use Eq. (1), and notice that $\lim_{x\to-\infty} x^3 = -\infty$ and

$$\lim_{x\to-\infty}\left(2 - \frac{11}{x} + \frac{12}{x^2}\right) = 2.$$

It follows that $\lim_{x\to-\infty}\left(2x^3 - 11x^2 + 12x\right) = -\infty.$

Example 5 Determine how $f(x) = \left(x^3 + 6x^2 + 10x + 2\right) / \left(2x^3 + x^2 + 5\right)$ behaves for arbitrarily large positive number x.

Solution As x gets large, the numerator $x^3 + 6x^2 + 10x + 2$ grows large, influencing the quotient to become large. On the other hand, the denominator also grows large, influencing the quotient to become small. An algebraic device will help reveal what happens to the quotient. We have

$$f(x) = \frac{x^3 + 6x^2 + 10x + 2}{2x^3 + x^2 + 5} = \frac{x^3\left(1 + \frac{6}{x} + \frac{10}{x^2} + \frac{2}{x^3}\right)}{x^3\left(2 + \frac{1}{x} + \frac{5}{x^3}\right)} = \frac{1 + \frac{6}{x} + \frac{10}{x^2} + \frac{2}{x^3}}{2 + \frac{1}{x} + \frac{5}{x^3}} \quad \text{for } x \neq 0.$$

Now we can see what happens to $f(x)$ when x is large.

As x increases, $6/x \to 0, 10/x^2 \to 0, 2/x^3 \to 0, 1/x \to 0$ and $5/x^3 \to 0$. Thus

$$f(x) \to \frac{1 + 0 + 0 + 0}{2 + 0 + 0} = \frac{1}{2}.$$

So, as x gets arbitrarily large through positive values, the quotient $\left(x^3 + 6x^2 + 10x + 2\right) / \left(2x^3 + x^2 + 5\right)$ approaches $\frac{1}{2}$. In short,

$$\lim_{x\to\infty}\frac{x^3 + 6x^2 + 10x + 2}{2x^3 + x^2 + 5} = \frac{1}{2}.$$

The technique used in Example 5 applies to any function that can be written as the quotient of two polynomials. Such a function is called a rational function.

Let $f(x)$ be a polynomial and let ax^n be its term of highest degree. Let $g(x)$ be another polynomial and let bx^m be its term of highest degree.

Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{ax^n}{bx^m} \quad \text{and} \quad \lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{x\to-\infty}\frac{ax^n}{bx^m}.$$

(The proofs of these facts are similar to the argument used in Example 5.) In short, when working with the limit of a quotient of two polynomials as $x\to\infty$ or as $x\to-\infty$, disregard all terms except the one of highest degree in each of the polynomials. The next example illustrates this technique.

Example 6 Examine the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{3x^4+5x^2}{-x^4+10x+5}$ (b) $\lim\limits_{x\to\infty}\dfrac{x^3-16x}{5x^4+x^3-5x}$

(c) $\lim\limits_{x\to-\infty}\dfrac{x^4+x}{6x^3-x^2}$

Solution By the preceding observations,

$$\text{(a)}\ \lim_{x\to\infty}\frac{3x^4+5x^2}{-x^4+10x+5}=\lim_{x\to\infty}\frac{3x^4}{-x^4}=\lim_{x\to\infty}(-3)=-3.$$

$$\text{(b)}\ \lim_{x\to\infty}\frac{x^3+16x}{5x^4+x^3-5x}=\lim_{x\to\infty}\frac{x^3}{5x^4}=\lim_{x\to\infty}\frac{1}{5x}=0.$$

$$\text{(c)}\ \lim_{x\to-\infty}\frac{x^4+x}{6x^3-x^2}=\lim_{x\to-\infty}\frac{x^4}{6x^3}=\lim_{x\to-\infty}\frac{x}{6}=-\infty.$$

Example 7 Examine $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$.

Solution As $x\to\infty$, both $\sqrt{x^2+x}$ and x approach ∞. It is not immediately clear how their difference $\sqrt{x^2+x}-x$ behaves. It is necessary to use a little algebra and rationalize the expression:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\frac{\left(\sqrt{x^2+x}-x\right)}{\left(\sqrt{x^2+x}+x\right)}=$$

$$=\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2(1+1/x)}+x}=\lim_{x\to\infty}\frac{x}{x\left(\sqrt{1+1/x}+1\right)}=$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{2}.$$

Example 8 How does $f(x)=1/x$ behave when x is near 0?

Solution The reciprocal of a small number x has a large absolute value.

For instance, when $x=0.01, 1/x=100$; when $x=-0.01, 1/=-100$. Thus, as x approaches 0 from the right, $\frac{1}{x}$, which is positive, becomes arbitrarily large. The notation for this is $\lim\limits_{x \to 0+} \frac{1}{x} = \infty$.

As x approaches 0 from the left, $\frac{1}{x}$, which is negative, has arbitrarily large absolute values. The notation for this is $\lim\limits_{x \to 0} \frac{1}{x} = -\infty$.

The many different types of limits all have the same flavor. Rather than spell each out in detail, we list some typical cases.

Notation	*In Words*	*Concept*	*Example*
$\lim\limits_{x \to a} f(x) = 1.$	As x approaches a, $f(x)$ approaches L.	$f(x)$ is defined in some open intervals (c, a) and (a, b) and, as x approaches ***a*** from the right or from the left, $f(x)$ approaches L.	$\lim\limits_{x \to 3} (2x+1) = 7$
$\lim\limits_{x \to \infty} f(x) = L.$	As x approaches (positive) infinity, $f(x)$ approaches L	$f(x)$ is defined for all x beyond some number and, as x gets large through positive values, $f(x)$ approaches L.	$\lim\limits_{x \to \infty} \frac{1}{x} = 0$
$\lim\limits_{x \to -\infty} f(x) = L$	As x approaches negative infinity, $f(x)$ approaches L	$f(x)$ is defined for all x to the left of some number and, as the negative number x takes on large absolute values, $f(x)$ approaches L.	$\lim\limits_{x \to -\infty} \frac{x+1}{x} = 1$
$\lim\limits_{x \to \infty} f(x) = \infty$	As x approaches infinity, $f(x)$ approaches positive infinity.	$f(x)$ is defined for all x beyond some number and, as x gets large through positive values, $f(x)$ becomes and remains arbitrarily large and positive.	$\lim\limits_{x \to \infty} x^3 = \infty$

$\lim_{x\to a^+} f(x) = \infty$	As x approaches a from the right, $f(x)$ approaches (positive) infinity.	$f(x)$ is defined in some open interval (a,b), and, as x approaches a from the right, $f(x)$ becomes and remains arbitrarily large and positive.	$\lim_{x\to 0^+} \frac{1}{x} = \infty$
$\lim_{x\to a^+} f(x) = -\infty$	As x approaches a from the right, $f(x)$ approaches negative infinity.	$f(x)$ is defined in some open interval (a,b), and, as x approaches a from the right, $f(x)$ becomes negative and $\lvert f(x)\rvert$ becomes and remains arbitrarily large.	$\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$
$\lim_{x\to a} f(x) = \infty$	As x approaches a, $f(x)$ approaches (positive) infinity.	$f(x)$ is defined for some open intervals (c,a) and (a,b), and, as x approaches a from either side, $f(x)$ becomes and remains arbitrarily large and positive.	$\lim_{x\to 0} \frac{1}{x^2} = \infty$

Exercise Set 4

In Exercises 1 to 26 examine the given limits and compute those which exist.

1. $\lim_{x\to\infty} \left(x^5 - 100x^4\right)$	2. $\lim_{x\to\infty} \left(-4x^5 + 35x^2\right)$
3. $\lim_{x\to-\infty} \left(6x^5 + 21x^3\right)$	4. $\lim_{x\to-\infty} \left(19x^6 + 5x - 300\right)$
5. $\lim_{x\to-\infty} \left(-x^3\right)$	6. $\lim_{x\to-\infty} \left(-x^4\right)$
7. $\lim_{x\to\infty} \frac{6x^3 - x}{2x^{10} + 5x + 8}$	8. $\lim_{x\to\infty} \frac{100x^9 + 22}{x^{10} + 21}$
9. $\lim_{x\to\infty} \frac{x^5 + 1066x^2 - 1492x}{2x - 1984}$	10. $\lim_{x\to\infty} \frac{6x^3 - x^2 + 5}{3x^3 - 100x + 1}$
11. $\lim_{x\to\infty} \frac{x^3 + 1}{x^4 + 2}$	12. $\lim_{x\to-\infty} \frac{5x^3 + 2x}{x^{10} + x + 7}$
13. $\lim_{x\to 0^+} \frac{1}{x^4}$	14. $\lim_{x\to 0^-} \frac{1}{x^4}$

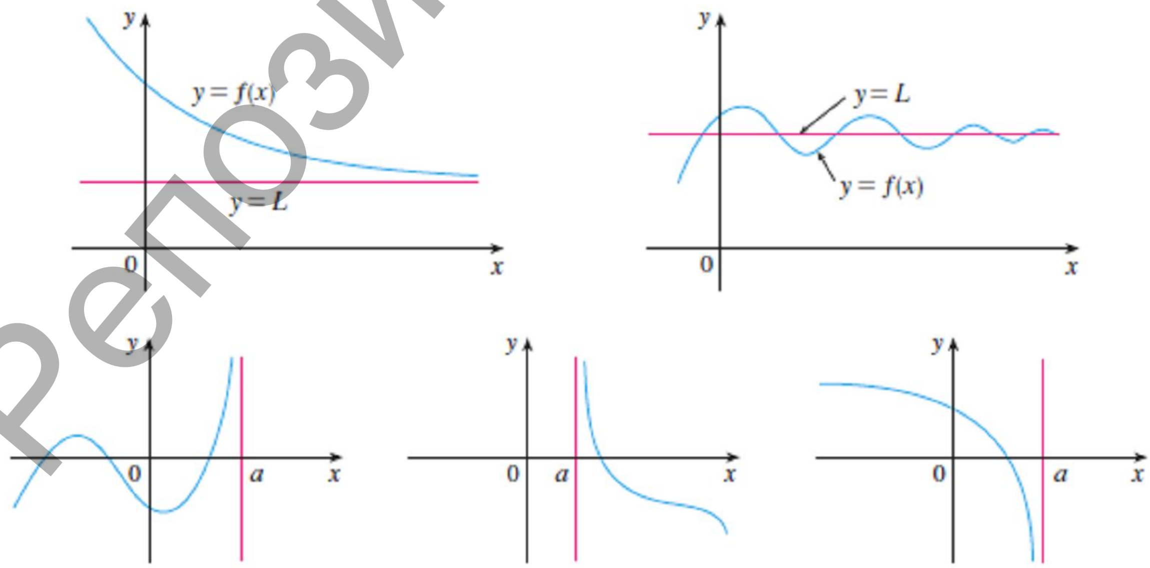
15. $\lim\limits_{x\to 0^+} \dfrac{1}{x^3}$	16. $\lim\limits_{x\to 0^-} \dfrac{1}{x^3}$
17. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+100}-x\right)$	18. $\lim\limits_{x\to 1} \left(\sqrt{x^2+5}-\sqrt{x^2+3}\right)$
19. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+100x}-x\right)$	20. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+100x}-\sqrt{x^2+50x}\right)$
21. $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+2x+1}}{3x}$	22. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^2+x+3}}{6x}$
23. $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+x}}{\sqrt{9x^2-3x}}$	24. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+3x+1}}{\sqrt{16x^2+x+2}}$
25. $\lim\limits_{x\to 1^+} \dfrac{1}{x-1}$	26. $\lim\limits_{x\to -1^-} \dfrac{1}{(x+1)}$

2.3 Asymptotes And Their Use In Graphing

If $\lim\limits_{x\to\infty} f(x) = L$, where L is a real number, the graph of $y = f(x)$ gets arbitrarily close to the horizontal line $y = L$ as x increases. The line $y = L$ is called a horizontal asymptote of the graph of f. An asymptote is defined similarly if $f(x) \to L$ as $x \to -\infty$.

If $\lim\limits_{x\to a^+} f(x) = \infty$ or if $\lim\limits_{x\to a^-} f(x) = \infty$, the graph of $y = f(x)$ resembles the vertical line $x = a$ for x near a. The line $x = a$ is called a vertical asymptote of the graph of f. A similar definition holds if $\lim_{x\to a^+} f(x) = -\infty$ or if $\lim_{x\to a^-} f(x) = -\infty$.

Fig. 2.7 shows some of these asymptotes.



(a) $\lim\limits_{x\to a^-} f(x) = \infty$ (b) $\lim\limits_{x\to a^+} f(x) = \infty$ (c) $\lim\limits_{x\to a^-} f(x) = -\infty$

Fig. 2.7

Horizontal and Vertical Asymptotes in Graphing

Some examples of graphing rational functions will show the usefulness of asymptotes.

Example 1. Using asymptotes, graph $f(x) = 1/x^2$.

Solution. When $x = 0$, the function is undefined. However, when x is near 0, $1/x^2$ is a large positive number, since x^2 is a small positive number, Thus

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

This means that the graph of $f(x) = 1/x^2$ approaches the upper part of the vertical asymptote $x = 0$ both from the right and from the left.

Since

$$\lim_{x \to \infty} \frac{1}{x^2} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x^2} = 0.$$

The x axis is a horizontal asymptote. All this information is incorporated in Fig. 2.8.

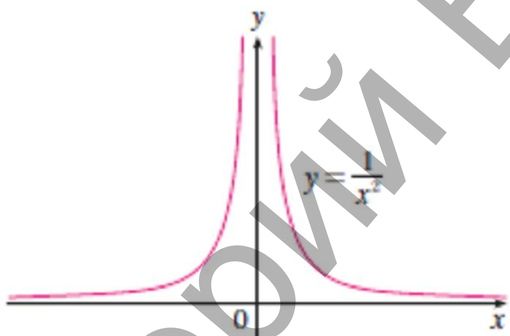


Fig. 2.8

Example 2. Graph $f(x) = \left(x^2 + 1\right)/x$.

Solution. After dividing x into $x^2 + 1$, we can write

$$f(x) = x + \frac{1}{x}.$$

When $|x|$ is large, $f(x)$ differs from x by the small quantity $\frac{1}{x}$. So when $|x|$ is large, the graph of f is close to the line $y = x$. When x is negative, $f(x) = x + 1/x$ is smaller than x, since $\frac{1}{x}$ is negative. So for x negative the graph of f lies below the line $y = x$. Similar reasoning shows that for x positive the graph of f lies above the line $y = x$.

Next search for any vertical asymptotes. Near $x = 0$ the function becomes arbitrarily large. In fact.

$$\lim_{x \to 0^+} \left(x + \frac{1}{x}\right) = \infty \text{ and } \lim_{x \to 0} \left(x + \frac{1}{x}\right) = -\infty.$$

The y axis is a vertical asymptote. The graph in Fig. 2.9 incorporates the information about the tilted and vertical asymptotes.

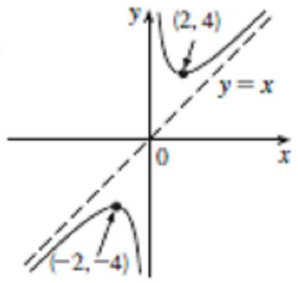


Fig. 2.9

Exercise Set 5

In Exercises 1 to 14 use asymptotes to sketch the graph of the functions.

1. $f(x)=\frac{1}{x-2}$	2. $f(x)=\frac{1}{x+3}$
3. $f(x)=\frac{1}{(x+1)^2}$	4. $f(x)=\frac{1}{(x+1)^3}$
5. $y=\frac{1}{x^2-x}$	6. $y=\frac{1}{x^3-x}$
7. $y=\frac{1}{x^4-x^2}$	8. $y=\frac{1}{x^3+x^2}$
9. $y=\frac{x(x-1)}{x^2+1}$	10. $y=\frac{(x-1)(x-2)}{x^2(x-3)}$
11. $y=\frac{x^3+2x^2+x+4}{x^2}$	12. $y=\frac{x^2-4}{x+4}$
13. $y=\frac{x^3}{x^2+1}$	14. $y=\frac{x^3}{x^2-1}$

2.4 Equivalent Infinitesimal Functions. The Table of Equivalent Infinitesimal Functions

The limits evaluated in Sect. 2.1 and 2.2 were found by algebraic means, such as factoring, rationalizing, or canceling. But some of the most important limits in calculus cannot be found so easily. To reinforce the concept of a limit and also to prepare for the calculus of trigonometric functions, we shall determine

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.$$

Since both the numerator, $\sin\theta$, and the denominator, θ, approaches 0, this is a challenging limit.

Theorem 1. Let sin θ denote the sine of an angle of θ radians. Then

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

The Squeeze Principle If $g(x)\le f(x)\le h(x)$ and

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

then

$$\lim_{x\to a} f(x) = L.$$

Example 1. Find $\lim\limits_{x\to\infty} (\sin 5x)/5x.$

Solution. Observe that as $x \to 0, \quad 5x \to 0.$ Let $\theta = 5x.$ Thus

$$\lim_{x\to 0} \frac{\sin 5x}{5x} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Example2. Find $\lim\limits_{x\to 0} (\sin 5x)/2x.$

Solution. A little algebra permits one to exploit the result found in Example 1:

$$\lim_{x\to 0} \frac{\sin 5x}{2x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{5x}{2x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{2} = 1 \cdot \frac{5}{2} = \frac{5}{2}.$$

From a practical point of view this section showed that if angles are measured in radians, then the sine of a small angle is “roughly” the angle itself: that is

$$\sin x \approx x.$$

This is another way of saying that x is small, the quotient $(\sin x)/x$ is close to 1. In engineering and physics $\sin x$ is often replaced by x when x is small. Moreover, tgx may also be replaced by x for small x. This being a reasonable estimate is justified by the fact that

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to} \frac{\frac{\sin x}{\cos x}}{x} = \left(\lim_{x\to 0} \frac{\sin x}{x} \right) \left(\lim_{x\to 0} \frac{x}{\cos x} = 1 \cdot 1 = 1. \right)$$

So $tgx \approx x$ for small x. Similarly, $tgx \approx \sin x$ for small x.

Exercise Set 6

1. $\lim\limits_{x\to 0} \frac{\sin x}{2x}$	2. $\lim\limits_{x\to 0} \frac{\sin 2x}{x}$	3. $\lim\limits_{x\to 0} \frac{\sin 3x}{5x}$
4. $\lim\limits_{x\to 0} \frac{2x}{\sin 3x}$	5. $\lim\limits_{\theta\to 0} \frac{\sin^2\theta}{\theta}$	6. $\lim\limits_{h\to\infty 0} \frac{\sinh^2}{h^2}$
7. $\lim\limits_{\theta\to 0} \frac{\tan^2\theta}{\theta}$	8. $\lim\limits_{\theta\to 0} \theta\cot\theta$	9. $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}$
10. $\lim\limits_{\theta\to 0^-} \frac{1-\cos\theta}{\theta^3}$	11. $\lim\limits_{\theta\to 0^+} \frac{1-\cos\theta}{\theta^3}$	12. $\lim\limits_{x\to 0} \frac{\sin^2 x}{x^2}$
13. $\lim\limits_{\theta\to 0^+} \frac{1}{\sin\theta}$	14. $\lim\limits_{\theta\to 0^-} \frac{1}{\sin\theta}$	

Example 3. Find $\lim\limits_{x\to 0} \sqrt[3]{\frac{\sin x}{x}}.$

Solution. As $x \to 0, \frac{\sin x}{x} \to 1.$ Moreover, the cube root function is continuous. Therefore,

$$\lim_{x\to 0}\sqrt[3]{\frac{\sin x}{x}}=\sqrt[3]{\lim_{x\to 0}\frac{\sin x}{x}}=\sqrt[3]{1}=1.$$

Example 3 generalizes as follows.

Let f be a continuous function. If g is some other function for which $\lim_{x\to a} g(x)$ exists and is in the domain of f and $g(x)$ is in the domain of f for x near a, then

$$\lim_{x\to a} f\left(g(x)\right)=f\left(\lim_{x\to a} g(x)\right).$$

In Example 4, $f(x)=\sqrt[3]{x}$ and $g(x)=(\sin x)/x$.

Definition. A function $f(x)$ is called infinitesimal function in the point $x=x_0$, if $\lim_{x\to x_0} f(x)=0$

These functions are denoted by small Greek letters $(\alpha(x),\beta(x),\gamma(x))$

Definition. Infinitesimal functions $\alpha(x)$ and $\beta(x)$ are called equivalent infinitesimal functions in the point $x=x_0$, if $\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=1$

Equivalent infinitesimal functions in the point $x=x_0$ are denoted as $\alpha(x)\approx\beta(x)$

Theorem. Let $\alpha(x)\approx\alpha_1(x),\beta(x)\approx\beta_1(x)$. If $\lim_{x\to x_0}\frac{\alpha_1(x)}{\beta_1(x)}=L$, then $\lim_{x\to x_0}\frac{\alpha\ (x)}{\beta\ (x)}=L$

The table of Equivalent infinitesimal functions is often used in solving problems.

$\sin\ \alpha(x)\approx\alpha(x)$
$tg\ \alpha(x)\approx\alpha(x)$
$\arcsin\alpha(x)\approx\alpha(x)$
$arctg\ \alpha(x)\approx\alpha(x)$
$e^{\alpha(x)}-1\approx\alpha(x)$
$\ln(1+\alpha(x))\approx\alpha(x)$
$\sqrt[n]{1+\alpha(x)}-1\approx\frac{1}{n}\alpha(x)$

Example 4. Find a) $\lim_{x\to 0}\frac{\sin^2 3x}{\arcsin^2 5x}$ b) $\lim_{x\to 0}\frac{\ln(1-4x)}{e^{7x}-1}$ c) $\lim_{x\to 0}\frac{\sqrt{1+7x^3}-1}{arctg^2 2x}$

Solution. As $x\to 0$, then $\sin^2 3x\to 0,\arcsin^2 5x\to 0,\ln(1-4x)\to 0,e^{7x}-1\to 0$

$\sqrt{1+7x^3}-1\to 0, arctg^2 2x\to 0$

a)

$$\lim_{x\to 0}\frac{\sin^2 3x}{\arcsin^2 5x}=\left(\frac{0}{0}\right)=\left[\sin^2 3x\approx(3x)^2=9x^2,\arcsin^2 5x\approx(5x)^2=25x^2\right]=$$

$$=\lim_{x\to 0}\frac{9x^2}{25x^2}=\frac{9}{25}$$

b)

$$\lim_{x\to 0}\frac{\ln(1-4x)}{e^{7x}-1}=\left(\frac{0}{0}\right)=\left[\ln(1-4x)=\ln(1+(-4x))\approx -4x,e^{7x}-1\approx 7x\right]=$$

$$=\lim_{x\to 0}\frac{-4x}{7x}=-\frac{4}{7}$$

c)

$$\lim_{x\to 0}\frac{\sqrt{1+7x^3}-1}{arctg^2 2x}=\left(\frac{0}{0}\right)=\left[\sqrt{1+7x^3}-1\approx 7x^3, arctg^2 2x\approx(2x)^2=4x^2\right]=\lim_{x\to 0}\frac{7x^3}{4x^2}=$$

$$=\frac{7}{4}\lim_{x\to 0}x=0$$

Exercise Set 7

1. $\lim\limits_{x\to 0}\frac{\sin^3 2x}{\arcsin^2 3x^2}$	2. $\lim\limits_{x\to 0}\frac{\sin^2 3x}{tg^2 9x}$	3. $\lim\limits_{x\to 0}\frac{\arcsin x}{arctg 5x}$	4. $\lim\limits_{x\to 0}\frac{\ln(1+2x)}{e^{4x}-1}$
5. $\lim\limits_{x\to 0}\frac{\ln(1+4x^3)}{e^x-1}$	6. $\lim\limits_{x\to 0}\frac{tg^2 3x}{e^{7x^2}-1}$	7. $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{arctg^2 x}$	8. $\lim\limits_{x\to 0}\frac{\sqrt{1-8x^4}-1}{tg^2 2x^2}$
9. $\lim\limits_{x\to 0}\frac{5x^2}{arctg^2 2x}$	10. $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+4x^2}-1}{tg^2 x}$	11. $\lim\limits_{x\to 0}\frac{\sqrt{1+7x^3}-1}{\ln(1-6x^3)}$	12. $\lim\limits_{x\to 0}\frac{7x}{arctg 5x^2}$

2.5 Continuity

Definition *(Continuity from the right at a number a)*. Assume that $f(x)$ is defined at a and in some open interval (a,b). Then the function f is continuous at a from the right if

$$\lim_{x\to a^+} f(x)=f(a).$$

This means that

1. $\lim\limits_{x\to a^+} f(x)$ exists and

2. that limit is $f(a)$.

Definition *(Continuity from the left at a number a).* Assume that $f(x)$ is defined at a and in some open interval (c,a). Then the function f is continuous at a from the left if

$$\lim_{x\to a^-} f(x)=f(a).$$

This means that

1. $\lim\limits_{x\to a^-} f(x)$ exists and

2. that limit is $f(a)$.

The next definition applies if the function is defined in some open interval that includes the number a. It essentially combines the first two definitions.

Definition *(Continuity at a number a).* Assume that $f(x)$ is defined in some open interval (b,c) that contains the number a. Then the function f is continuous at a if

$$\lim_{x\to a} f(x)=f(a).$$

This means that

1. $\lim\limits_{x\to a} f(x)$ exists and

2. that limit is $f(a)$.

This third definition amounts to asking that the function can be continuous both from the right and from the left at a.

Example 1 Let $f(x)=x^2$ for all x. Show that f is continuous at $a=3$.

Solution. As $x\to 3$, $f(x)=x^2$ approaches 9; that is,

$$\lim_{x\to 3} x^2 = 9.$$

Next, compute $f(3)$, which is 3^2 or 9. Since $\lim_{x\to 3} f(x)$ exists and equals $f(3)$, f is continuous at 3. (In fact, f is continuous at each real number.)

Example 2 Let $f(x)=\sqrt{x}$ for $x\ge 0$. Show that f is continuous from the right at $a=0$.

Solution. As the graph of $f(x)=\sqrt{x}$ in Fig.2.10 reminds us, the domain of f does not contain an open interval around 0. It is meaningful to speak of "continuity from the right" at 0 but not of "continuity from the left."

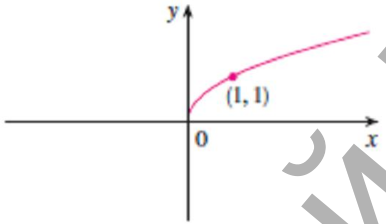


Fig.2.10

Since $\sqrt{x}$ approaches 0 as x approaches 0, $\lim_{x\to 0^+} f(x)=0$. Is this limit the same as $f(0)$? Since $f(0)=\sqrt{0}=0$ the answer is "yes." In short, f is continuous from the right at 0.

Definition *(Continuous function).* Let f be a function whose domain is the x axis or is made up of open intervals. Then f is a continuous function if it is continuous at each number a in its domain.

Thus x^2 is a continuous function. So is $\frac{1}{x}$, whose domain consists of the intervals $(-\infty,0)$ and $(0,+\infty)$. Although this function explodes at 0, this does not prevent it from being a continuous function. *The key to being continuous is that the function is continuous at each number in its domain*. The number 0 is not in the domain of $\frac{1}{x}$.

Only a slight modification of the definition is necessary to cover functions whose domains involve closed intervals. We will say that a function whose domain is the closed interval $[a,b]$ is continuous if it is continuous at each point in the open interval (a,b), continuous from the right at a, and continuous from the left at b. Thus $\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$.

In a similar spirit, we say that a function with domain $[a,+\infty)$ is continuous if it is continuous at each point in $(a,+\infty)$ and continuous from the right at a. Thus $\sqrt{x}$ is a continuous function. A similar definition covers functions whose domains are of the form $(-\infty,b]$.

Definition (*Sum, difference, product, and quotient of functions)*. Let f and g be two functions. The functions $f+g$, $f-g$, $f\cdot g$ and $\dfrac{f}{g}$ are defined as follows.

$(f+g)(x)=f(x)+g(x)$ for x in the domains of both f and g.
$(f-g)(x)=f(x)-g(x)$ for x in the domains of both f and g.
$(f\cdot g)(x)=f(x)\cdot g(x)$ for x in the domain of both f and g.
$\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ for x in the domains of both f and g, $g(x)\neq 0$.

If f and g are defined at least in an open interval that includes the number a and if f and g are continuous at a, then so are $f+g$, $f-g$ and $f\cdot g$. Moreover, if $g(a)\neq 0$, $\dfrac{f}{g}$ is also continuous at a.

A function obtained by the composition of continuous functions is also continuous. That is, if the function g is continuous at a and the function f is continuous at $g(a)$, then the composition, $f\circ g$ is continuous at a. For instance, the function $\sqrt[3]{1+x^2}$ is continuous since both the polynomial $1+x^2$ and the cube root function are continuous.

THEOREM The following types of functions are continuous at every number in their domains: Polynomials, rational functions ,root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that it is **discontinuous at** a (or has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Example 3 Where are each of the following functions discontinuous?

(a) $f(x)=\dfrac{x^2-x-2}{x-2}$

(b) $f(x)=\begin{cases}\dfrac{1}{x^2}, x\neq 0\\ 1, x=0\end{cases}$

(c) $f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, x\neq 2\\ 1, x=2\end{cases}$

(d) $f(x)=[|x|]$

Solution

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0)=1$ is defined but $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{1}{x^2}$ does not exist. (See Example 1 in Section 2.3.) So f is discontinuous at 0.

(c) Here $f(2)=1$ is defined and

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)}=\lim_{x\to 2}(x+1)=3$$

exists. But

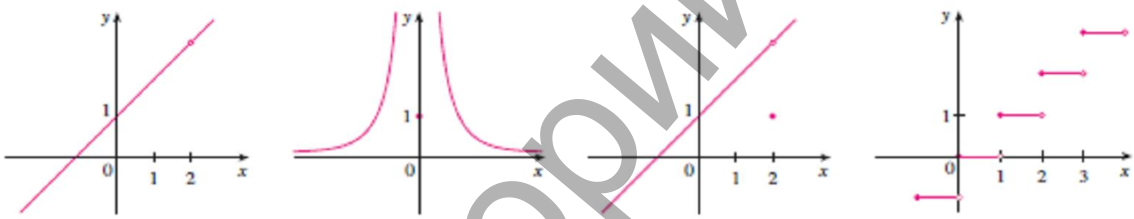
$$\lim_{x\to 2} f(x)\neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x)=[\![x]\!]$ has discontinuities at all of the integers because $\lim_{x\to n}[\![x]\!]$ does not exist if n is an integer.

Figure 2.11 shows the graphs of the functions in Example 3. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2.

[The function $g(x)=x+1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x)=\dfrac{x^2-x-2}{x-2}$ (b) $f(x)=\begin{cases}\dfrac{1}{x^2} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ (c) $f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2} & \text{if } x\neq 2\\ 1 & \text{if } x=2\end{cases}$ (d) $f(x)=[\![x]\!]$

Fig. 2.11

2.6 The Maximum-Value Theorem and The Intermediate-Value Theorem

Continuous functions have two properties of particular importance in calculus: the "maximum-value" property and the "intermediate-value" property. Both are quite plausible, and give a glance at the graph of a "typical" continuous functions. No proofs will be offered; they depend on the precise definitions of limits given in Secs. 2.7 and 2.8 and are part of an advanced calculus course.

The first theorem asserts that a function is continuous throughout the closed interval $[a,b]$ at which f takes on a maximum value. That is, for some number c in $[a,b]$

$$f(c)\geq f(x)$$

for all x in $[a,b]$.

Similarly, f takes on a minimum value somewhere in the interval.

To persuade yourself that this theorem is plausible, imagine sketching the graph of a continuous function. As your pencil moves along the graph from some point on the graph to some other point on the graph, it passes through the highest point and also through the lowest point. (See Fig. 2.12.)

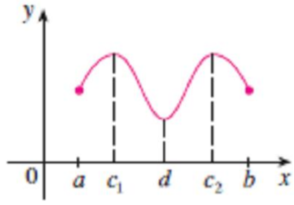


Fig. 2.12

The maximum-value theorem guarantees that a maximum value exists, but it does not tell how to find it. The problem of finding the maximum value (and minimum value) is discussed in Chap. 4.

Example 1 Let $f(x) = \cos x$ and $[a,b] = [0,3\pi]$. Find all numbers in $[0,3\pi]$ at which f takes on a maximum value. Also find all numbers in $[0,3\pi]$ at which f takes on a minimum value.

Solution. Fig. 2.13 is a graph of $\cos x$ for x in $[0,3\pi]$. Inspection of the graph shows that the maximum value of $\cos x$ for $0 \le x \le 3\pi$ is 1, and it is attained when $x = 0$ and when $x = 2\pi$. The minimum value is -1, which is attained when $x = \pi$ and when $x = 3\pi$.

The maximum and minimum values of a function are frequently called its extreme values or extremum. Thus the extreme values $\cos x$ of for x in $[0,3\pi]$ are 1 and -1.

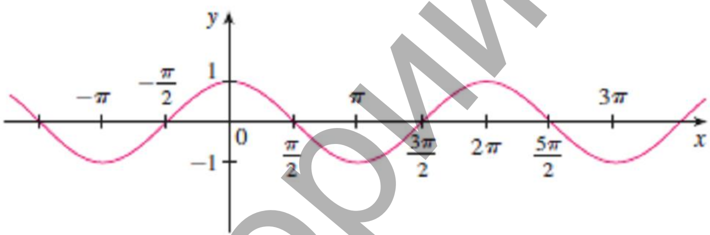


Fig. 2.13

To apply the maximum-value theorem, we must know that the function is continuous and the interval is closed (that is, contains its endpoints). The next two examples show that if either of these assumptions is deleted, the conclusion no longer needs hold. In Examples 2 and 3 the interval is not closed.

Example 2 Let $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$ and let (a,b) be the open interval $(2,4)$. Show that f does not have a maximum value for x in (a,b).

Solution. For x near 4, $f(x)$ gets arbitrarily large since the denominator $(x-2)^2(x-4)^4$ is close to 0. The graph of f for x in $(2,4)$, is shown in Fig. 2.14. This function is continuous throughout the open interval $(2,4)$, but there is no number c in $(2,4)$ at which f has a maximum value. However, f has a minimum value, $f(3) = 576$.

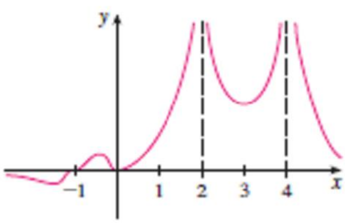


Fig. 2.14

Example 3 Let (a,b) be the open interval $(0,1)$. Show that $f(x)=\frac{1}{x}$ does not have a maximum value in (a,b).

Solution. Fig. 2.15 shows the pertinent part of the graph of $f(x)=\frac{1}{x}$. Since $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$, the function has no minimum value for x in $(0,1)$. It does take on values arbitrarily close to 1 for inputs that are close to 1, but there is no number in the open interval $(0,1)$ at which $f(x)$ is equal to1.

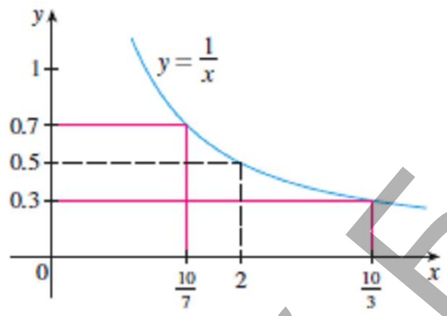


Fig. 2.15

The next theorem says that a function which is continuous throughout an interval takes on all values between any two of its values.

Intermediate-Value Theorem

Let f be continuous throughout the closed interval $[a,b]$. Let N be any number $f(a)$ and $f(b)$. [That is, $f(a)\le N\le f(b)$ if $f(a)\le f(b)$ or $f(a)\ge N\ge f(b)$ if $f(a)\ge f(b)$] Then there is at least one number c in $[a,b]$ such that $f(c)=N$.

In ordinary English, the intermediate-value theorem reads: a continuous function defined on $[a,b]$ takes on all values between $f(a)$ and $f(b)$. Pictorially, it asserts that a horizontal line of height N must meet the graph of f at least once if N is between $f(a)$ and $f(b)$, as shown in Fig. 2.16. In other words, when you move a pencil it passes through all intermediate heights.

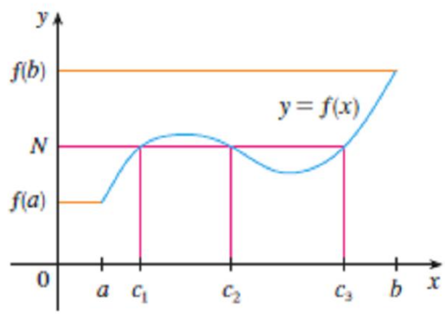


Fig. 2.16

Even though the theorem guarantees the existence of c, it does not tell how to find it. To find c, we must solve an equation, namely, $f(c)=N$.

Example 4 Use the intermediate-value theorem to show that the $2x^3+x^2-x+1=5$ has a solution in the interval $[1,2]$.

Solution Let $P(x) = 2x^3 + x^2 - x + 1$. Then

$$P(1) = 2 \cdot 1^3 + 1^2 - 1 + 1 = 3$$

and

$$P(2) = 2 \cdot 2^3 + 2^2 - 2 + 1 = 19.$$

Since $P(x)$ is continuous and 5 is between $P(1) = 3$ and $P(2) = 19$ we may apply the Intermediate-value theorem to $P(x)$ in the case $a = 1, b = 2$ and $N = 5$. Thus there is at least one number c between 1 and 2 such that $P(c) = 5$. This completes the answer. (To get a more accurate estimate for a number c such that $P(c) = 5$ find a shorter interval for which the intermediate-value theorem can be applied. For instance, $P(1.2) \approx 4.7$ and $P(1.3) \approx 5.8$. By the intermediate-value theorem, there is a number c in $[1.2, 1.3]$ such that $P(c) = 5$).

Example 5 Show that the equation $x^5 - 2x^2 + x + 11 = 0$ has at least one real root.

Solution. For x large and positive the polynomial $P(x) = x^5 - 2x^2 + x + 11$ is positive [since $\lim\limits_{x \to \infty} P(x) = \infty$]. Thus there is a number b such that $P(b) > 0$. Similarly, for x negative and large value, $P(x)$ is negative [since $\lim\limits_{x \to +\infty} P(x) = -\infty$]. Select a number a such that $P(a) < 0$.

The number 0 is between $P(a)$ and $P(b)$. Since $P(x)$ is continuous on the interval $[a, b]$ there is a number c in $[a, b]$ such that $P(c) = 0$. This number c is a real solution to the equation $x^5 - 2x^2 + x + 11 = 0$.

2.7 Precise Definitions of " $\lim\limits_{x \to \infty} f(x) = \infty$ " and " $\lim\limits_{x \to \infty} f(x) = L$ "

In the definitions of the limits considered in Secs. 2.1 and 2.2 appear such phrases as "x approaches a", "$f(x)$ approaches a specific number," "as x gets large," and "$f(x)$ becomes and remains arbitrarily large". Such phrases, although appealing to the intuition, seem to suggest moving objects and call to mind the motion of a pencil point as it traces out the graph of a function.

In this section we examine how Weierstrass would define the concepts:

$$\lim_{x \to \infty} f(x) = \infty \text{ and } \lim_{x \to \infty} f(x) = L$$

Throughout, "f" refers to a numerical function. Use the next section we consider "$\lim\limits_{x \to a} f(x) = L$"

Recall the definition of "$\lim\limits_{x \to \infty} f(x) = \infty$" given in the table in Sec. 2.2.

Informal definition of $\lim\limits_{x \to \infty} f(x) = \infty$, $f(x)$ is defined for all x beyond some number and, as x gets large through positive values, $f(x)$ becomes and remains arbitrarily large and positive.

To take us part way to the precise definition, let us reword the informal definition, paraphrasing it in the following definition, which is still informal.

Reworded informal definition of $\lim_{x\to\infty} f(x) = \infty$ [assume that $f(x)$ is defined for all x greater than some number c].

If x is sufficiently large and positive, then $f(x)$ is necessarily large and positive.

For each number E there is a number D such that for all $x > D$ it is true that

$$f(x) > E.$$

Think of the number E as a challenge and D as the reply. The larger E is, the larger D must usually be. Only if a number D (which depends on E) can be found for every number E can we make the claim that " $\lim_{x\to\infty} f(x) = \infty$ ".

Example 1 Using the precise definition, show that $\lim_{x\to\infty} 2x = \infty$.

Solution. Let E be any number. We must show that there is a number D such that whenever $x > D$, it follows that $2x > E$. For example, if $E = 50$, then $D = 25$ would do. It is indeed on E .

Now, the inequality $2x > E$ is equivalent to

$$x > \frac{E}{2}.$$

In other words, if $x > \frac{E}{2}$ then $2x > E$. So $D = \frac{E}{2}$ suffices. That is, for $x > D \Rightarrow 2x > E$. We conclude immediately that

$$\lim_{x\to\infty} 2x = \infty.$$

Informal definition of $\lim_{x\to\infty} f(x) = L$ [assume that $f(x)$ is defined for all x beyond some number c]. As x gets large through positive values, $f(x)$ approaches L.

Again we reword this definition before offering the precise definition.

Reworded informal definition of $\lim_{x\to\infty} f(x) = L$ [assume that there is a number c such that $f(x)$ is defined for all $x > c$]

If x is sufficiently large and positive, then $f(x)$ is necessarily near L.

Again, the precise definition parallels the reworded informal definition. In order to make the phrase "$f(x)$ is necessarily near L" precise, we shall use the absolute value of $f(x) - L$ to measure the distance from $f(x)$ to L. The following definition says that "if x is large enough, then $|f(x) - L|$ is as small as we please."

Precise definition of $\lim_{x\to\infty} f(x) = L$ [assume that $f(x)$ is defined for all x beyond some number c].

For each positive number ε there is a number D such that for all $x > D$ it is true that

$$|f(x) - L| < \varepsilon.$$

The positive number ε is the challenge, and D is a response. The smaller ε is, the larger D usually must be chosen. The geometric meaning of the precise definition of $\lim_{x\to\infty} f(x) = L$ is shown in Fig. 2.17.

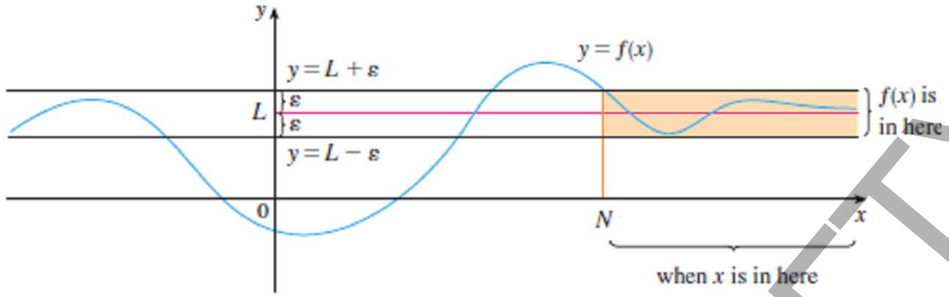


Fig. 2.17

Example 2 Use the precise definition of “ $\lim_{x\to\infty} f(x) = L$ ” to show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$.

Solution. Here $f(x) = 1 + 1/x$, which is defined for all $x > 0$. The number L is 1. We must show that for each positive number ε however small, there is a number D such that, for all $x > D$,

$$\left|\left(1+\frac{1}{x}\right)-1\right| < \varepsilon.$$

Inequality reduces to

$$\left|\frac{1}{x}\right| < \varepsilon.$$

Since we shall consider only $x > 0$, this inequality is equivalent to

$$\frac{1}{x} < \varepsilon.$$

Multiplying inequality by the positive number x gets the equivalent inequality

$$1 < \varepsilon x.$$

Division of inequality by the positive number ε gets

$$\frac{1}{\varepsilon} < x \text{ or } x > \frac{1}{\varepsilon}.$$

These steps are reversible. This shows that $D = 1/\varepsilon$ is a suitable reply to the challenge ε. If $x > 1/\varepsilon$, then

$$\left|\left(1+\frac{1}{x}\right)-1\right| < \varepsilon.$$

According to the precise definition of “ $\lim_{x\to\infty} f(x) = L$" ,we may conclude that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1.$$

2.8 Precise Definition of " $\lim_{x\to a} f(x) = L$ "

Recall the informal definition given in Sec. 2.2.

Informal definition of $\lim_{x\to a} f(x) = L$

Let f be a function and a is some fixed number. Assume that the domain of f contains open intervals (c,a) and (a,b) for some number $c < a$ and some number $b > a$.

If, as x approaches a, either from the left or from the right, $f(x)$ approaches a specific number L, then L is called the limit of $f(x)$ as x approaches a. This is written

$$\lim_{x\to a} f(x) = L.$$

Keep in mind that ***a*** need not be in the domain of f. Even if ***a*** happens to be in the domain of f, the value $f(a)$ plays no role in determining whether $\lim_{x\to a} f(x) = L$.

Reworded informal definition of $\lim_{x\to a} f(x) = L$ [assume that $f(x)$ is defined for all x in some intervals (c,a) and (a,b)].

If x is sufficiently close to ***a*** but not equal to a, then $f(x)$ is necessarily near L.

The precise definition parallels the reworded informal definition. The letter δ that appears in it is lower case Greek "delta," equivalent to the English letter d.

Precise definition of $\lim_{x\to a} f(x) = L$ [assume that $f(x)$ is defined in some intervals (c,a) and (a,b)]:

For each positive number ε there is a positive number δ such that for all x that satisfy the inequality

$$0 < |x - a| < \delta$$

it is true that

$$|f(x) - L| < \varepsilon.$$

Example 1 Use the precise definition of " $\lim_{x\to a} f(x) = L$ " to show that $\lim_{x\to 0} x^2 = 0$.

Solution. In this case $a = 0$ and $L = 0$. Let ε be a positive number. We wish to find a positive number δ such that for $0 < |x - 0| < \delta$ it follows that $|x^2 - 0| < \varepsilon$

Since $|x^2| = |x^2|$, we are asking, "for which x is $|x| < \varepsilon$"? This inequality is satisfied when

$$|x| < \sqrt{\varepsilon}.$$

In other words, when $|x| < \sqrt{\varepsilon},$ it follows that $|x^2 - 0| < \varepsilon$. Thus $\delta = \sqrt{\varepsilon}$ suffices. (For instance, when $\varepsilon = 1, \delta = \sqrt{1} = 1$ is a suitable response. When $\varepsilon = 0.01, \delta = 0.1$ suffices.)

Example 2 Use the precise definition of "$\lim_{x \to a} f(x) = L$" to show that $\lim_{x \to 2}(3x+5) = 11$.

Solution. Here $a = 2$ and $L = 11$. Let ε be a positive number. We wish to find a number $\delta > 0$ such that for $0 < |x-2| < \delta$ it follows that $|(3x+5)-11| < \varepsilon$.

So let us find out for which x it is true that $|(3x+5)-11| < \varepsilon$. This inequality is equivalent to

$$|3x-6| < \varepsilon$$

or

$$3|x-2| < \varepsilon$$

or

$$|x-2| < \frac{\varepsilon}{3}.$$

Thus $\delta = \varepsilon / 3$ is an adequate response. If $0 < |x-2| < \varepsilon / 3$, then $|(3x+5)-11| < \varepsilon$.

2.9 The Number e

Definition *(The number e).*

$$e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = \left(1^\infty\right) = 2.718281828$$

Observe that for large n the expression $\left(1+\frac{1}{n}\right)^n$ is of the form

$$(1+small\_number)^{big\_number}.$$

So we may consider

$$(1+x)^{1/x}$$

When x is near 0, even if x is not of the form $1/n$, that is, not the reciprocal of an integer. It can be shown, that $\lim_{x \to 0}(1+x)^{1/x}$ exists and equals e:

$$\lim_{x \to 0}(1+x)^{1/x} = e. \qquad (*)$$

Often $(*)$ is taken as the definition of e. It is this expression for e that will be used in the next section, where we will find derivatives of the logarithm functions.

From the fact that $\lim_{x \to 0}(1+x)1/x = e$, we can obtain order of closely related limits. For instance, $\lim_{h \to 0}(1+2h)^{1/2h} = e$. (Note that $2h \to 0$ and the exponent is the reciprocal of $2h$).

Example 1 Find $\lim_{h \to 0}(1+2h)^{1/h}$.

Solution. The expression $(1+2h)^{1/h}$ is not of the form $(1+h)^{1/h}$.

Since $1/h$ is not the reciprocal of $2h$. A little algebra gets around this obstacle:

$$\lim_{h\to 0}(1+2h)^{1/h} = \lim_{h\to 0}(1+2h)^{2/2h} = \lim_{h\to 0}\left((1+2h)^{1/2h}\right)^2 = \left(\lim_{h\to 0}(1+2h)^{1/2h}\right)^2 = e^2$$

Example 2 Find $\lim\limits_{x\to\infty}\left(\dfrac{3x-1}{3x-5}\right)^{2x+3}$.

Solution.

$$\lim_{x\to\infty}\left(\frac{3x-1}{3x-5}\right)^{2x+3} = \left(1^\infty\right) = \lim_{x\to\infty}\left(1+\frac{3x-1}{3x-5}-1\right)^{2x+3} = \lim_{x\to\infty}\left(1+\frac{3x-1-(3x-5)}{3x-5}\right)^{2x+3} =$$

$$= \lim_{x\to\infty}\left(1+\frac{4}{3x-5}\right)^{2x+3} = \lim_{x\to\infty}\left(\left(1+\frac{4}{3x-5}\right)^{\frac{3x-5}{4}}\right)^{\frac{4}{3x-5}\cdot(2x+3)} = \lim_{x\to\infty} e^{\frac{8x+12}{3x-5}} = e^{\lim\limits_{x\to\infty}\frac{8x+12}{3x-5}} = e^{\frac{8}{3}}$$

Exercise Set 8

Examine the following limits:

(a) $\lim\limits_{x\to 1}\left(x^2+5x\right)$

(b) $\lim\limits_{x\to\infty}\dfrac{3x^4-100x+3}{5x^4+7x-1}$

(c) $\lim\limits_{x\to 0}\dfrac{3x^4-100x+3}{5x^4+7x-1}$

(d) $\lim\limits_{x\to -\infty}\dfrac{500x^3-x^2-5}{x^4+x}$

(e) $\lim\limits_{x\to 0}\dfrac{\sin 3t}{6t}$

(f) $\lim\limits_{x\to -\infty}\dfrac{-6x^5+4x}{x\ \ +x+5}$

(g) $\lim\limits_{x\to\infty} 2^{-x}$

(h) $\lim\limits_{x\to 0}\dfrac{x^3+8}{x+2}$

(i) $\lim\limits_{x\to -2}\dfrac{x^3+8}{x+2}$

(j) $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$

(k) $\lim\limits_{x\to\infty}\sin x$

(l) $\lim\limits_{x\to\infty}\dfrac{1+3\cos x}{x^2}$

(m) $\lim\limits_{x\to 16}\dfrac{\sqrt{x}-4}{x-16}$

(n) $\lim\limits_{x\to\infty}\left(\sqrt{4x^2+5x}-\sqrt{4x^2+x}\right)$

In Exercises 1 to 52 examine the limits. Evaluate those which exist. Determine those which do not exist and, among these, the ones that are infinite.

1. $\lim\limits_{x\to 1}\dfrac{x^3+1}{x^2+1}$	2. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}$	3. $\lim\limits_{x\to 2}\dfrac{x^4-16}{x^3-8}$
4. $\lim\limits_{x\to 0}\dfrac{x^4-16}{x^3-8}$	5. $\lim\limits_{x\to\infty}\dfrac{x^7-x^2+1}{2x^7+x^3+300}$	6. $\lim\limits_{x\to -\infty}\dfrac{x^9+6x+3}{x^{10}-x-1}$
7. $\lim\limits_{x\to -\infty}\dfrac{x^3+1}{x^2+1}$	8. $\lim\limits_{x\to -\infty}\dfrac{x^4+x^2+1}{3x^2+4}$	9. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$
10. $\lim\limits_{x\to 81}\dfrac{x-81}{\sqrt{x}-9}$	11. $\lim\limits_{x\to 1^+}\dfrac{1}{x-1}$	12. $\lim\limits_{x\to 1^-}\dfrac{1}{x-1}$
13. $\lim\limits_{x\to\infty}\left(\sqrt{2x^2}-\sqrt{2x^2-6x}\right)$	14. $\lim\limits_{x\to 0^+} 2^{1/x}$	15. $\lim\limits_{x\to 0^-} 2^{1/x}$

16. $\lim\limits_{x\to\infty} 2^{1/x}$	17. $\lim\limits_{x\to\infty} \dfrac{(x+1)(x+2)}{(x+3)(x+4)}$	18. $\lim\limits_{x\to-\infty} \dfrac{(x+1)^{100}}{(2x+50)^{100}}$
19. $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{1+\sin x}$	20. $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{1-\sin x}$	21. $\lim\limits_{x\to0} \dfrac{\sin x}{3x}$
22. $\lim\limits_{x\to\infty} \dfrac{\sin x}{3x}$	23. $\lim\limits_{x\to\pi/2^+} \cos x$	24. $\lim\limits_{x\to\frac{\pi}{2}^+} \sec x$
25. $\lim\limits_{x\to0^-} \sin x$	26. $\lim\limits_{x\to0^-} \csc x$	27. $\lim\limits_{x\to\infty} \sin\dfrac{1}{x}$
28. $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x}$	29. $\lim\limits_{x\to\infty} x^2\cos x$	30. $\lim\limits_{x\to\pi/4} x^2\cos x$
31. $\lim\limits_{x\to0}\dfrac{\sin^3 2x}{\arcsin^2 3x^2}$	32. $\lim\limits_{x\to0}\dfrac{\sin^2 3x}{tg^2 9x}$	33. $\lim\limits_{x\to0}\dfrac{\arcsin x}{arctg 5x}$
34. $\lim\limits_{x\to0}\dfrac{\ln(1+2x)}{e^{4x}-1}$	35. $\lim\limits_{x\to0}\dfrac{\ln(1+4x^3)}{e^{x}-1}$	36. $\lim\limits_{x\to0}\dfrac{tg^2 3x}{e^{7x^2}-1}$
37. $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1}{arctg^2 x}$	38. $\lim\limits_{x\to0}\dfrac{\sqrt{1-8x^4}-1}{tg^2 2x^2}$	39. $\lim\limits_{x\to0}\dfrac{5x^2}{arctg^2 2x}$
40. $\lim\limits_{x\to0}\dfrac{\sqrt[3]{1+4x^2}-1}{tg^2 x}$	41. $\lim\limits_{x\to0}\dfrac{\sqrt{1+7x^3}-1}{\ln(1-6x^3)}$	42. $\lim\limits_{x\to0}\dfrac{\cos 2x-\cos 5x}{5x^2}$
43. $\lim\limits_{x\to0}\dfrac{\sin^2 3x-\sin^2 5x}{tg^2 9x}$	44. $\lim\limits_{x\to0}\dfrac{1-\cos 6x}{arctg^2 5x}$	45. $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+1}\right)^{0.5x-1}$
46. $\lim\limits_{x\to4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$	47. $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+3}\right)^{4-0.5x}$	48. $\lim\limits_{x\to-8}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$
49. $\lim\limits_{x\to\infty}\left(\dfrac{3x+2}{3x+1}\right)^{5-9x}$	50. $\lim\limits_{x\to1}\dfrac{\sqrt{x-1}}{\sqrt[3]{x^2-1}}$	51. $\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x-5}\right)^{4x+3}$

52-69. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

52. $f(x)=\begin{cases} x+4, x<-1 \\ x^2+2, -1\le x<1 \\ 2x, x\ge1 \end{cases}$	53. $f(x)=\begin{cases} x+1, x<0 \\ (x+1)^2, 0\le x<2 \\ -x+4, x\ge2 \end{cases}$
54. $f(x)=\begin{cases} x+2, x\le-1 \\ x^2+1, -1<x<1 \\ -x+3, x\ge1 \end{cases}$	55. $f(x)=\begin{cases} \sqrt{1-x}, x\le0 \\ 0, 0<x\le2 \\ x-2, x>2 \end{cases}$

56. $f(x)=\begin{cases}\sin x, x<0\\ x, 0\le x<2\\ 0, x\ge 2\end{cases}$	57. $f(x)=\begin{cases}1, x\le 0\\ 2^x, 0<x\le 2\\ x+3, x>2\end{cases}$
58. $f(x)=\begin{cases}3x+4, x\le -1\\ x^2-2, -1<x<2\\ x, x\ge 2\end{cases}$	59. $f(x)=\begin{cases}x^3, x<-1\\ x-1, -1\le x<3\\ -x+5, x\ge 3\end{cases}$
60. $f(x)=\begin{cases}-1, x<0\\ \cos x, 0\le x<\pi\\ 1-x, x\ge \pi\end{cases}$	61. $f(x)=\begin{cases}2, x<-1\\ 1-x, -1\le x<1\\ \ln x, x\ge 1\end{cases}$
62. $f(x)=\begin{cases}-x; \quad x\le 0\\ \sin x; \ 0<x\le \pi\\ x-2; \quad x>\pi\end{cases}$	63. $f(x)=\begin{cases}3-x; \quad x<-2\\ x^2-5; \ -2\le x<3\\ 7-2x; \quad x\ge 3\end{cases}$
64. $f(x)=\begin{cases}-x^2; \quad x\le 0\\ tgx; \ 0<x\le \pi/4\\ 2; \quad x>\pi/4\end{cases}$	65. $f(x)=\begin{cases}-2x; \quad x\le 0\\ \sqrt{x}; \ 0<x\le 4\\ 1; \quad x>4\end{cases}$
66. $f(x)=4^{\frac{4}{x-2}}$	67. $f(x)=7^{\frac{3}{x-3}}$
68. $f(x)=25^{\frac{1}{2x-2}}$	69. $f(x)=2^{-\frac{2}{1-x}}$

3 DERIVATIVES

3.1 The Derivative

Definition (*The derivative of a function at the number x*) Let f be a function that is defined at least in some open interval that contains the number x. If

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists, it is called the derivative of f at x and is denoted $f'(x)$. The function is said to be differentiable at x.

If the function f is defined only to the right of x, in an interval of the form $[x,b)$ then in the definition of the derivative "$h\to 0$" would be replaced by "$h\to 0^+$".The function is then said to be "differentiable on the right." A similar stipulation is made if f is defined only in an interval of the from $(a,x]$ and the function is said to be "differentiable on the left."

The numerator, $f(x+h)-f(x)$ is the change, or difference, in the outputs; the denominator, h, is the change in the inputs. Keep in mind that $x+h$ can be either to the right or left of x. Similarly, $f(x+h)$ can be either larger or smaller than $f(x)$.

A few examples will illustrate the concept of the derivative.

Example 1 Find the derivative of the squaring function at the number 2.

Solution. In this case, $f(x) = x^2$ for any input x. By definition, the derivative of this function at 2 is

$$\lim_{h\to 0}\frac{(2+h)^2-2^2}{h} = \lim_{h\to 0}\frac{2^2+4h+h^2-2^2}{h} = \lim_{h\to 0}(4+h) = 4.$$

We say that "the derivative of the function $f(x) = x^2$ at 2 is 4."

The next example determines the derivative of the squaring function at any input, not just at 2.

Example 2 Find the derivative of the function $f(x) = x^2$ at any number x.

Solution By definition, the derivative at x is

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

The derivative of the squaring function at x is $2x$.That the derivative of the function $f(x) = x^2$ is the function $2x$ is denoted

$$(x^2)' = 2x.$$

This notation is convenient when dealing with a specific function. [*Warning*: don't replace x by a specific number in this notation. For instance, do not write that $(3^2)'$ equals $2\cdot 3$. This is not correct].

The result in Example 2 can be interpreted in terms of each of the four problems in Sec. 3.1. For example, we know from Example 2 that the slope of the tangent line to the parabola $y = x^2$ at the point (x, x^2) is $2x$. In particular, the slope of the tangent line at $(1, 1^2)$ is $2\cdot 1 = 2$ a result found in Sec 3.1. Also, according to the formula for the derivative, $(x^2)' = 2x$ the slope of the tangent line to $y = x^2$ at $(-1, (-1)^2)$ is $2\cdot(-1) = -2$ and at $(0,0)$ is $2\cdot 0 = 0$ A glance at Fig. 3.1 shows that these are reasonable results. The derivative of $f(x) = x^2$ is a function. It assigns to the number x the slope of the tangent line to the parabola $y = x^2$ at the point (x, x^2).

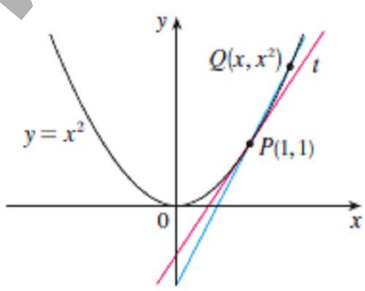


Fig. 3.1

The next two examples illustrate the idea of the derivative with functions other than x^2.

Example 3 Find $f'(x)$ if $f(x) = x^3$.

Solution. In this case, $f(x+h) = (x+h)^3$ and $f(x) = x^3$. The derivative of the function at x is therefore

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.$$

The derivative of x^3 at x is $3x^2$.

Theorem 1 For each positive integer n,

$$\left(x^n\right)' = nx^{n-1}.$$

Direct application of this theorem yields, for instance:

the derivative of x^4 is $4x^{4-1} = 4x^3$;

the derivative of x^3 is $3x^{3-1} = 3x^2$;

the derivative of x^2 is $2x^{2-1} = 2x$;

the derivative of x^1 is $1 \cdot x^{1-1} = 1$.

(in agreement with the fact that the line given by the formula $y = x$ has slope 1).

The next theorem generalizes the fact that $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$ which can be written

$$\left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{\frac{1}{2}-1}.$$

Theorem 2 For each positive integer n,

$$\left(x^{\frac{1}{n}}\right)' = \frac{1}{n}x^{\frac{1}{n}-1}$$

(for those x at which both $x^{\frac{1}{n}}$ and $x^{\frac{1}{n}-1}$ are defined).

Now that we have the concept of the derivative, we are in a position to define tangent line, speed, magnification, and density, terms used only intuitively until now. These definitions are suggested by the similarity of the computations made in the four problems in sec. 3.1.

The slope of a nonvertical line equals the quotient $\frac{(y_2 - y_1)}{(x_2 - x_1)}$ where $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two distinct points on the line. Now it is possible to define the slope of a curve at a point on the curve.(In all five definitions it is assumed that the derivative exists.)

Definition (*Slope of a curve*). The slope of the graph of the function f at $(x, f(x))$ is the derivative of f at x.

Definition (*Tangent line to a curve*). The tangent line to the graph of the function f at the point $P(x, y)$ is the line through P that has a slope equal to the derivative of f at x.

Definition (*Velocity and speed of a particle moving on a line*). The velocity at time t of an object whose position on a line at time t is given by $f(t)$ is the derivative of f at time t. The speed of the particle is the absolute value of the velocity.

Note the distinction between velocity and speed. Velocity can be negative; speed is either positive or 0.

Definition (*Magnification of a linear projector*). The magnification at x of a lens that projects the point x of one line onto the point $f(x)$ of another line is the derivative of f at x.

Definition (*Density of material*). The density at x of material distributed along a line in such a way that the left-hand x centimeters have a mass of $f(x)$grams is equal to the derivative of

f at x.

Exercise Set 9

In Exercises 1 to 16 use the definition of the derivative to find the derivatives of the given functions.

1. x^4	2. x^5	3. $3x$	4. $7x$
5. x^2+3	6. $4x^2+5$	7. $-5x^2+3x$	8. x^3-2x^2+3
9. $7\sqrt{x}$	10. $x^2+3\sqrt{x}$	11. $\frac{1}{x}$	12. $\frac{1}{x+2}$
13. $\frac{1}{x^2}$	14. $\sqrt{x}+\frac{4}{x}$	15. $3-\frac{1}{x}$	16. $\frac{1}{x^3}$

In Exercises 17 to 20, use Theorems 1 and 2 to find the derivatives of the given functions at the given numbers.

17. x^4 at $x=-1$	18. x^4 at $x=\frac{1}{2}$	19. x^5 at $x=a$	20. x^5 at $x=\sqrt{2}$

3.2 Differentiation Rules

After presenting another $f'(x)$ this section shows the relation between "having a derivative" and "being continuous." It concludes by introducing the notion of an "antiderivative."

It is also common to give the difference or change h the name Δx "delta x"). The difference quotient then takes the form $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ and the derivative is defined as

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Furthermore, the difference in the outputs is often named Δf or Δy:

$$f(x+\Delta x)-f(x)=\Delta f$$

and so $f(x+\Delta x)=f(x)+\Delta f$.

The latter equation says that "the value of the function at $x+\Delta x$ is equal to the value of the function at x plus the change in the function". With Δx denoting the change in the inputs and Δf denoting the change in the outputs, we have

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}.$$

Fig. 3.2 illustrate the Δ notation for the difference quotient.

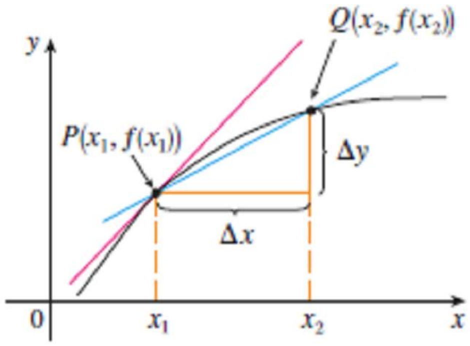


Fig. 3.2

Example 1 Find $\left(x^2\right)'$ using the Δ notation.

Solution. By the definition of the derivative, the derivative of the squaring function at x is

$$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

So the derivative of x^2 is $2x$, in agreement with the result in Example 2 of the preceding section.

This section develops methods for finding derivatives of functions, or what is called differentiating functions. With these methods it will be a routine matter to find, for instance, the derivative of $f(x)=\dfrac{\left(x^2-6\right)\sqrt{\left(1+x^2\right)^3}}{120x^5}$ without going back to the definition of the derivative and (at great effort) finding the limit of a difference quotient.

Before developing the methods, it will be useful to find the derivative of any constant function.

Theorem 1 The derivative of a constant function is 0; in symbols,

$$\left(c\right)'=0 \quad \text{or} \quad \frac{dc}{dx}=0 \text{ or } dc=0.$$

Theorem 2 If f and g are differentiable functions, then so is $f+g$. Its derivative is given by the formula

$$\left(f+g\right)'=f'+g'.$$

Similarly, $\left(f-g\right)'=f'-g'$.

Proof. Give the function $f+g$ the name u. That is,

$$u(x)=f(x)+g(x)$$

Then $u(x+\Delta x)=f(x+\Delta x)+g(x+\Delta x)$ so,

$$\Delta u=u(x+\Delta x)-u(x)=f(x+\Delta x)+g(x+\Delta x)-\left(f(x)+g(x)\right)=\left(f(x+\Delta x)-f(x)\right)+$$
$$+\left(g(x+\Delta x)-g(x)\right)=\Delta f+\Delta g$$

Thus $u'(x)=\displaystyle\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f+\Delta g}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x}=f'+g'$.

Hence $f+g$ is differentiable, and

$$\left(f+g\right)'=f'+g'.$$

A similar argument applies to $f-g$.

Example 2. Using Theorem 2, differentiate x^2+x^3.

Solution. $(x^2+x^3)'=\left(x^2\right)'+\left(x^3\right)'=2x+3x^2$.

Example 3. Differentiate $x^4-\sqrt{x}-6$.

Solution.

$$(x^4-\sqrt{x}-6)' = \left(x^4\right)' - \left(\sqrt{x}\right)' - (6)' = 4x^3 - \frac{1}{2\sqrt{x}} - 0 = 4x^3 - \frac{1}{2\sqrt{x}}.$$

Theorem 3. If f and g are differentiable functions, then so is $f \cdot g$. Its derivative is given by the formula

$$(f \cdot g)' = f' \cdot g + g' \cdot f.$$

Proof. Call the function $f \cdot g$ simply u. That is,

$$u(x) = f(x) \cdot g(x)$$

Then $u(x+\Delta x) = f(x+\Delta x) \cdot g(x+\Delta x)$.

Rather than subtract directly, first write

$$f(x+\Delta x) = f(x) + \Delta f \quad \text{and} \quad g(x+\Delta x) = g(x) + \Delta g.$$

Then $u(x+\Delta x) = \left(f(x)+\Delta f\right) \cdot \left(g(x)+\Delta g\right) = f \cdot g + f \cdot \Delta g + g \cdot \Delta f + \Delta f \cdot \Delta g$

Hence

$$\Delta u = u(x+\Delta x) - u = f \cdot g + f \cdot \Delta g + g \cdot \Delta f + \Delta f \cdot \Delta g - f \cdot g = f \cdot \Delta g + g \cdot \Delta f + \Delta f \cdot \Delta g$$

and

$$\frac{\Delta u}{\Delta x} = f \cdot \frac{\Delta g}{\Delta x} + g \cdot \frac{\Delta f}{\Delta x} + \Delta f \cdot \frac{\Delta g}{\Delta x}.$$

As $\Delta x \to 0$, $\frac{\Delta g}{\Delta x} \to g'$, $\frac{\Delta f}{\Delta x} \to f'$ and, because f is differentiable (hence continuous), $\Delta f \to 0$. It follows that

$$u'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f' \cdot g + g' \cdot f + 0 \cdot g' = f' \cdot g + g' \cdot f$$

Therefore, u is differentiable and

$$(u)' = f' \cdot g + g' \cdot f.$$

Example 4. Find $\left(\left(x^2+x^3\right)\cdot\left(x^4-\sqrt{x}-6\right)\right)'$.

Solution. (Note that in Example 2 and 3 the derivatives of both factors, x^2+x^3 and $x^4-\sqrt{x}-6$ were found.)

By theorem 3,

$$\left(\left(x^2+x^3\right)\cdot\left(x^4-\sqrt{x}-6\right)\right)' = \left(x^2+x^3\right)' \cdot \left(x^4-\sqrt{x}-6\right) + \left(x^4-\sqrt{x}-6\right)' \cdot \left(x^2+x^3\right) =$$

$$= \left(2x+3x^2\right)\cdot\left(x^4-\sqrt{x}-6\right) + \left(4x^3 - \frac{1}{2\sqrt{x}}\right)\cdot\left(x^2+x^3\right)$$

A special case of the formula $(f \cdot g)' = f' \cdot g + g' \cdot f$ occurs so frequently, that it is singled out in Theorem 4.

Theorem 4. If c is a constant function and f is a differentiable function then $c \cdot f$ is differentiable and its derivative is given by formula

$$(c \cdot f)' = c \cdot f'.$$

Any polynomial can be differentiated by the methods already developed, as Example 5 illustrates.

Example 5 Differentiate $5x^7-3x^2+8-\dfrac{1}{x^3}-\pi^3$

Solution.

$$\left(5x^7-3x^2+8-\frac{1}{x^3}-\pi^3\right)'=\left(5x^7\right)'-\left(3x^2\right)'+(8)'-\left(\frac{1}{x^3}\right)'-\left(\pi^3\right)'=$$

$$=5\left(x^7\right)'-3\left(x^2\right)'-\left(x^{-3}\right)'=7x^6-3\cdot 2x-(-3)\cdot x^{-4}=7x^6-6x+3x^{-4}$$

It will next be shown that if the functions f and g are differentiable at a number x, and if $g(x)\neq 0$ then $\dfrac{f}{g}$ is differentiable at x.

Theorem 5. If f and g are differentiable functions, then so is $\dfrac{f}{g}$ and

$$\left(\frac{f}{g}\right)'=\frac{f'\cdot g-g'\cdot f}{g^2}\quad \text{(where } g(x)\neq 0\text{)}.$$

Example 6. Compute $\left(\dfrac{x^3-4}{x^2+3}\right)'$.

Solution.

$$\left(\frac{x^3-4}{x^2+3}\right)'=\frac{\left(x^3-4\right)'\cdot\left(x^2+3\right)-\left(x^3-4\right)\cdot\left(x^2+3\right)'}{\left(x^2+3\right)^2}=\frac{3x^2\cdot\left(x^2+3\right)-2x\cdot\left(x^3-4\right)}{\left(x^2+3\right)^2}=$$

$$=\frac{3x^4+9x^2-2x^4+8x}{\left(x^2+3\right)^2}=\frac{x^4+9x^2+8x}{\left(x^2+3\right)^2}$$

Corollary 1

$$\left(\frac{1}{g}\right)'=-\frac{g'}{g^2}\ \text{(where } g(x)\neq 0\text{)}.$$

The differentiation techniques obtained so far do not enable us to differentiate such functions as $(1+2x)^{100}$, $\sqrt{1+x^2}$, $\sin x^3$.

We could differentiate $(1+2x)^{100}$, but only with great effort, by first expanding $(1+2x)^{100}$ to form a polynomial of degree 100 and then differentiating that polynomial. This section develops a shortcut for differentiating composite functions, such as $(1+2x)^{100}$, $\sqrt{1+x^2}$ and $\sin x^3$ which are built up from simpler functions by composition.

If f and g are differentiable functions, is the composite function $f\circ g=f(g(x))$ also differentiable? If so, what is its derivative? More concretely: If $y=f(u)$ and $u=g(x)$ then y

is a function of x. How can we find $y' = \frac{dy}{dx}$?

Take the simple case, $y = 3u$ and $u = 2x$. Hence $y = 6x$. In this case,

$$\frac{dy}{du} = 3, \quad \frac{du}{dx} = 2 \quad \text{and} \quad \frac{dy}{dx} = 6.$$

So $\frac{dy}{dx}$ is the product of the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$. This observation suggests the all-important chain rule, which will be proved at the end of this section after several examples showing how it is used.

The Chain Rule (Informal Statement)

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ or } y'_x = y'_u \cdot u'_x.$$

The equation $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ is read as "derivative of y with respect to x equals derivative of y with respect to u times derivative of u with respect to x".

Example 7. Differentiate $\sqrt{1+x^2}$

Solution. $y = \sqrt{1+x^2}$ is a composite function $y = \sqrt{u}$ where $u = 1 + x^2$.

By the chain rule, $y'_x = y'_u \cdot u'_x$

$$\left(\sqrt{1+x^2}\right)' = \left(\sqrt{u}\right)'_u \cdot \left(1+x^2\right)'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}.$$

Example 8. Differentiate $\sin x^3$.

Solution. $y = \sin x^3$ can be expressed as $y = \sin u$ where $u = x^3$.

By the chain rule,

$$\left(\sin x^3\right)' = (\sin u)'_u \cdot \left(x^3\right)'_x = \cos u \cdot 3x^2 = \cos x^3 \cdot 3x^2$$

Example 9. Differentiate $(1+2x)^{100}$.

Solution. $y = (1+2x)^{100}$ is the composition of $y = u^{100}$ and $u = 1 + 2x$.

By the chain rule,

$$\left((1+2x)^{100}\right)' = \left(u^{100}\right)'_u \cdot (1+2x)'_x = 100u^{99} \cdot 2 = 200 \cdot (1+2x)^{100}.$$

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

1. $(c)' = 0$

5. $(c \cdot f)' = c \cdot f'$

2. $(f+g)' = f' + g'$

3. $(f-g)' = f' - g'$

4. $(f\cdot g)' = f'\cdot g + g'\cdot f$

6. $\left(\dfrac{f}{g}\right)' = \dfrac{f'\cdot g - g'\cdot f}{g^2}$

7. $\left(\dfrac{1}{g}\right)' = -\dfrac{g'}{g^2}$

8. $\left(f(g(x))\right)' = f'_g \cdot g'_x$

Using the definition of derivative, limit and rules of differentiation, we can get the table of derivatives of elementary functions.

1. $\left(x^n\right)' = nx^{n-1}$	8. $(\cos x)' = -\sin x$
2. $\left(x^{\frac{1}{n}}\right)' = \dfrac{1}{n}x^{\frac{1}{n}-1}$	9. $(tgx)' = \dfrac{1}{\cos^2 x}$
3. $\left(e^x\right)' = e^x$	10. $(ctgx)' = -\dfrac{1}{\sin^2 x}$
4. $\left(a^x\right)' = a^x \cdot \ln a, a>0$	11. $(arc\sin x)' = \dfrac{1}{\sqrt{1-x^2}}$
5. $(\ln x)' = \dfrac{1}{x}$	12. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$
6. $(\log_a x)' = \dfrac{1}{x\cdot \ln a}$	13. $(arctgx)' = \dfrac{1}{1+x^2}$
7. $(\sin x)' = \cos x$	14. $(arcctgx)' = -\dfrac{1}{1+x^2}$

Example 10. Differentiate $\dfrac{e^{\sqrt{x^2-3x-9}}}{\sin x^3}$.

Solution.

$$\left(\frac{e^{\sqrt{x^2-3x-9}}}{\sin x^3}\right)' = \frac{\left(e^{\sqrt{x^2-3x-9}}\right)'\cdot\left(\sin x^3\right) - \left(e^{\sqrt{x^2-3x-9}}\right)\cdot\left(\sin x^3\right)'}{\left(\sin x^3\right)^2} =$$

$$= \frac{e^{\sqrt{x^2-3x-9}}\cdot\dfrac{1}{2\sqrt{x^2-3x-9}}\cdot(2x-3)\cdot\sin x^3 - e^{\sqrt{x^2-3x-9}}\cdot\cos x^3\cdot 3x^2}{\left(\sin x^3\right)^2}$$

Exercise Set 10

In Exercises 1 to 48 differentiate with the aid of formulas, not by using the definition of the derivative.

1. $y=3x^3+4x^2-x-2$

2. $y=x-\ln 2+e^x$

3. $y=\sin\sqrt{3}+3\cos 6x$;

4. $y=\frac{tgx-3ctgx}{\sqrt{2}}$;

5. $y=\frac{1}{4\sqrt{5}}\ln 7x$;

6. $y=arctg 4x+3\pi$;

7. $y=\frac{(2x^2-1)}{3x^3}$;

8. $y=e^{2x}\cdot\sin x$;

9. $y=\frac{\cos^2 3x}{3\sin 6x}$;

10. $y=x^5\cdot\cos x$;

11. $y=\left(3x^2-6x+8\right)^{21}$;

12. $y=\sin\sqrt{x}$;

13. $y=\frac{x^4-8x^2}{2(x^2-4)}$

14. $y=\frac{2x-1}{4}\cdot\sqrt{2+x-x^2}$

15. $y=\frac{\sin^2 4x}{4\cos 8x}$;

16. $y=\frac{1}{2}arctg\frac{e^x}{2}$;

17. $y=e^{-2x}\cdot\arcsin(e^{2x})$;

18. $y=\sin^6 7x$

19. $y=\frac{2x^2-x-1}{3\sqrt{2+4x}}$;

20. $y=\ln\left(1+2^x\right)$

21. $y=\frac{\cos^2 4x}{\sin 8x}$;

22. $y=arctg\frac{\sqrt{1+x^2}}{x}$;

23. $y=\ln\left(3x^2-7x-8\right)$;

24. $y=\arcsin e^x$

25. $y=\frac{\left(x^8+1\right)}{12x^{12}}$;

26. $y=2\sqrt{e^x+1}$

27. $y=\frac{\sin^2 2x}{2\cos 4x}$;

28. $y=\frac{x^2-4}{\sqrt{x^4+16}}$;

29. $y=\frac{1}{2}tg^3 x-\cos 4x$;

30. $y=\ln(6x^2-x)$

31. $y=\frac{x^2}{2\sqrt{1-3x^4}}$;

32. $y=\frac{2}{3}\sqrt{\left(tg\, e^x\right)}$

33. $y=\frac{\cos^2 2x}{4\sin 4x}$;

34. $y=\sqrt{\frac{2}{3}}arctg(3x-1)$;

35. $y=-\frac{1}{2}\ln tg\frac{x}{2}$;

36. $y=x\cdot\arcsin 7x$

37. $y=\frac{\sqrt{\left(1+x^2\right)^3}}{120x^5}$;

38. $y=\frac{1}{2}\ln(e^{2x}+1)$

39. $y=\ln(3x^5-7x)^2$;

40. $y=\frac{\sin^2 7x}{7\cos 14x}$;

41. $y=\ln\frac{x-1}{x+1}-arctg\, x$

42. $y=(ctg 3x)\cdot 2e^x$

43. $y=\frac{\sqrt{x^2-8}}{6x^3}$;

44. $y=\frac{3x-1}{3x^2-2x+1}$

45. $y=\frac{\cos^2 8x}{16\sin 16x}$

46. $y=\frac{1}{\sqrt{2}}\ln\frac{1+x}{1-x^2}$

47. $y=\frac{(x-4)\sqrt{8x-x^2-7}}{2}$

48. $y=x\cdot e^{tg\, x}$

In each of Exercises 49 to 51 find the slope of the given curve at the point with the given x coordinate.

49. $y=3x^3+4x^2-x-2$ at $x=2$

50. $y=\frac{1}{2x+1}$ at $x=3$

51. $y=\sqrt{x}\cdot(x+2)$ at $x=4$.

3.3 Implicit Differentiation. Logarithmic Differentiation. Calculus With Parametric Curves

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable — for example,

$$y = \sqrt{x^2 - 3} \text{ or } y = \sin 5x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 4 \text{ or } x^3 + y^2 = 4xy \quad .$$

We don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example 1

(a) If $x^2 + y^2 = 4$, find y'.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 4$ at the point $(3,4)$.

Solution.

(a) Differentiate both sides of the equation $x^2 + y^2 = 4$:

$$\left(x^2 + y^2\right)'_x = (4)'_x$$

$$\left(x^2\right)'_x + \left(y^2\right)'_x = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$2x + 2y \cdot y' = 0$$

Now we solve this equation for y':

$$y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

(b) At the point $(3,4)$ we have $x = 3$ and $y = 4$, so $y' = -\dfrac{3}{4}$

An equation of the tangent to the circle at $(3,4)$ is therefore

$$y - 4 = -\frac{3}{4}(x - 3) \text{ or } 3x + 4y = 4.$$

Example 2 If $x^3 + y^2 = 4xy$, find y'.

Solution. Differentiating both sides $x^3 + y^2 = 4xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^2 and the Product Rule on the term $4xy$, we get

$$3x^2 + 2yy' = 4y + 4xy'.$$

We now solve for y':

$$2yy' - 4xy' = 4y - 3x^2$$

$$y' \cdot (2y - 4x) = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(2y - 4x)}.$$

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

Example 3. Differentiate

$$y = \frac{(2x^2 - 1)\sqrt{1 + x^2}}{3x^3}.$$

Solution. We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \ln(2x^2 - 1) + \frac{1}{2}\ln(1 + x^2) - \ln(3x^3)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \cdot y' = \frac{1}{2x^2 - 1} \cdot 4x + \frac{1}{2} \cdot \frac{1}{1 + x^2} \cdot 2x - \frac{1}{3x^3} \cdot 9x^2$$

Solving for y', we get

$$y' = y \cdot \left(\frac{4x}{2x^2 - 1} + \frac{x}{1 + x^2} - \frac{3}{x} \right).$$

Because we have an explicit expression for y, we can substitute and write

$$y' = \frac{(2x^2 - 1)\sqrt{1 + x^2}}{3x^3} \cdot \left(\frac{4x}{2x^2 - 1} + \frac{x}{1 + x^2} - \frac{3}{x} \right).$$

Example 4. Differentiate $y = (\sin x)^{5e^x}$.

Solution. We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = 5e^x \cdot \ln(\sin x)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \cdot y' = 5e^x \cdot \ln(\sin x) + 5e^x \cdot \frac{1}{\sin x} \cdot \cos x$$

Solving for y', we get

$$y' = y \cdot 5e^x \cdot (\ln(\sin x) + ctgx).$$

Because we have an explicit expression for y, we can substitute and write

$$y' = (\sin x)^{5e^x} \cdot 5e^x \cdot (\ln(\sin x) + ctgx).$$

Some curves defined by parametric equations $x = x(t)$ and $y = y(t)$ can also be expressed, by eliminating the parameter, in the form $y = F(x)$.

If we substitute $x = x(t)$ and $y = y(t)$ in the equation $y = F(x)$, we get

$$y(t) = F(x(t))$$

and so, if $x = x(t)$, $y = y(t)$, and $y = F(x)$ are differentiable, the Chain Rule gives

$$y'_t(t) = F'_x(x) \cdot x'_t(t)$$

If $x'(t) \neq 0$, we can solve for $F'(x)$:

$$F'(x) = \frac{y'_t(t)}{x'_t(t)}$$

Using Leibniz notation, we can rewrite last equation in an easily remembered form:

$$y'_x = \frac{y'_t(t)}{x'_t(t)} \quad (1).$$

Example 5 Differentiate $\begin{cases} x = \sqrt{1-t^2}, \\ y = \dfrac{1}{t}. \end{cases}$.

Solution. Using formula(1), we get

$$x'_t = \left(\sqrt{1-t^2}\right)' = \frac{1}{2\sqrt{1-t^2}} \cdot (-2t)$$

$$y'_t = \left(\frac{1}{t}\right)'_t = -\frac{1}{t^2}$$

$$y'_x = \frac{y'_t(t)}{x'_t(t)} = \frac{-\dfrac{1}{t^2}}{\dfrac{1}{2\sqrt{1-t^2}} \cdot (-2t)} = \frac{\sqrt{1-t^2}}{t^3}.$$

Exercise Set 11

In Exercises 1 to 24 differentiate with the aid of formulas.

1. $y = \dfrac{2(3x^3-2)\cdot(2x-7)}{15\sqrt{1+x}}$	2. $y = (arctg 5x)^{arctg\, x}$	3. $y = \dfrac{(2x^3-9)\sqrt{1+3x^2}}{x^3}$
4. $y = (\sin\sqrt{x})^{\ln\sqrt{x}}$	5. $y = \dfrac{(x^3-1)\sqrt{5+x^2}}{4x^3}$	6. $y = (\arcsin x)^{e^x}$
7. $y = \dfrac{\left(x^8+1\right)\sqrt{x^8+1}}{12x^{12}}$	8. $y = (\ln x)^{3x}$	9. $y = \dfrac{(2x^2-x-1)}{3\sqrt{2+4x}} \cdot \sin 5x$
10. $x^3 + 4y^2 = 5xy$	11. $e^{x+y} = 5x - 6y$	12. $arc\sin(3x-y) = x^3 - y$
13. $tg(xy) = \sin 2x - 3\cos y$	14. $x^2 + 2y^2 = \dfrac{x}{y}$	15. $e^{x+y} = x^4 + 2y$
16. $\begin{cases} x = \sqrt{1-t^2}, \\ y = \sqrt{tg(1+t)}. \end{cases}$	17. $\begin{cases} x = \sqrt{2t-t^2}, \\ y = \dfrac{1}{\sqrt[3]{(t-1)^2}}. \end{cases}$	18. $\begin{cases} x = t + \sin t, \\ y = 2 - \cos t. \end{cases}$

19. $\begin{cases} x=\sqrt{2t-t^2}, \\ y=\arcsin(t-1). \end{cases}$	20. $\begin{cases} x=arctge^t, \\ y=\sqrt{e^t+1}. \end{cases}$	21. $\begin{cases} x=\sqrt{t}, \\ y=\dfrac{1}{\sqrt{1-t}}. \end{cases}$
22. $\begin{cases} x=\dfrac{1}{t}, \\ y=\dfrac{1}{t^2+1}. \end{cases}$	23. $\begin{cases} x=tgt, \\ y=\dfrac{1}{\sin 2t}. \end{cases}$	24. $\begin{cases} x=\sqrt{t-1}, \\ y=\dfrac{t}{\sqrt{t-1}}. \end{cases}$

3.4 Linear Approximations And Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$.

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y - f(a) = f'(a)(x-a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a.

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials.*

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

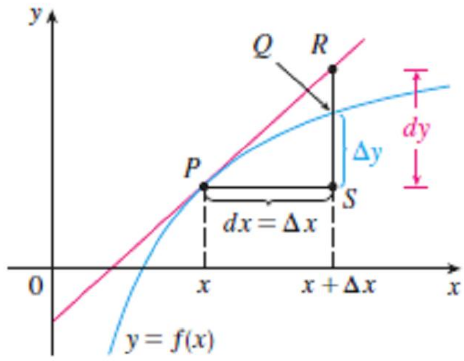


Fig.3.3

The geometric meaning of differentials is shown in Figure 3.3. Let $P(x, f(x))$ and $Q(x+\Delta x, f(x+\Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x+\Delta x) - f(x)$$

The slope RP of the tangent line is the derivative $f'(x)$. Thus the directed distance from S to R is $dy = f'(x)dx$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller. Notice also that dy was easier to compute than Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation can be written as

$$f(x+\Delta x) \approx f(x) + dy \quad \text{or} \quad f(a+\Delta x) \approx f(a) + f'(a)\Delta x.$$

Example 1. Use a differential to estimate $\sqrt{67}$.

Solution. The object is to estimate the value of the square root function $f(x) = \sqrt{x}$ at the input $x = 67$. In this case, $f(64)$ is know. We have

$$f(64) = 8 \text{ and } f'(x) = \frac{1}{2\sqrt{x}}, f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16}.$$

Since $67 = 64 + 3$, $f(x+\Delta x) \approx f(x) + dy$, $\Delta x = 3$. Therefore,

$$\sqrt{67} \approx f(64) + dy = f(64) + f'(64)\cdot 3 = 8 + \frac{1}{16}\cdot 3 = 8.1875.$$

A calculator shows that to four decimal places, $\sqrt{67} \approx 8.1854$. So the estimate obtained by the differential is not far off.

Exercise Set 12

In Exercises 1 to 12 use a differential to estimate

1. $\sqrt{37}$	2. $\sqrt[3]{10}$	3. $\sqrt[10]{1026}$	4. $\sqrt[3]{25}$
5. $\sin 31^\circ$	6. $\cos 61^\circ$	7. $tg 47^\circ$	8. $ctg 31^\circ 30'$
9. $\ln(tg 46^\circ)$	10. $\sin 131^\circ$	11. $arcctg 0.95$	12. $arctg 1.02$

3.5 Higher Derivatives

Velocity is the rate at which distance changes. The rate at which velocity changes is called acceleration. Thus if $y = f(t)$ denotes position on a line at time t, then the derivative $\frac{dy}{dt} = y' = v(t)$ equals the velocity, and the derivative of the derivative, that is

$$\frac{d}{dt}\left(\frac{dy}{dt}\right)$$

equals the acceleration.

Definition The derivative of the derivative of a function $y = f(x)$ is called the second derivative of the function. It is denoted

$$y'' = \left(y'\right)'$$

$$\frac{d^2y}{dx^2}, y'', f''(x), y^{(2)}, f^{(2)}(x).$$

If $y = f(t)$ where t denotes time, the second derivative $\frac{d^2y}{dt^2}$ is also denoted y''.

For instance, if $y = x^3$, $y' = \left(x^3\right)' = 3x^2$ and $y'' = \left(3x^2\right)' = 6x$.

Definition The derivative of the second derivative

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = y'''$$

is called the third derivative and is denoted many ways, such as

$$\frac{d^3y}{dx^3}, y''', f'''(x), y^{(3)}, f^{(3)}(x).$$

The fourth derivative $f^{(4)}(x)$ is defined as the derivative of the third derivative and is represented by similar notations. Similarly, $f^{(n)}(x)$ is defined for $n = 5, 6, \ldots$

Definition The derivatives $f^{(n)}(x)$ for $n \geq 2$ are called the higher derivatives of $y = f(x)$. (The first derivative is also denoted $f^{(1)}(x)$)

Example 1. Compute $y^{(n)}$ if $y = x^3 - 5x^2 + 9x - 7$ if n is a positive integer.

Solution.

$$y' = 3x^2 - 10x + 9$$

$$y'' = \left(3x^2 - 10x + 9\right)' = 6x - 10$$

$$y''' = (6x - 10)' = 6$$

Since $y^{(4)}$ is constant, its derivative, $y^{(5)}$ is equal to 0 for all x. Similarly, $y^{(6)} = 0$, $y^{(7)} = 0$ and so on.

As Example 1 may suggest, for any polynomial $P(x)$ of degree at most 3, $y^{(n)} = 0$ for all integers $n \geq 4$. The next example is quite different.

Example 2. Compute $y^{(n)}$ if $y = \sin x$.

Solution.

$$y' = (\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$y'' = (\cos x)' = -\sin x = \sin(x + 2\cdot\frac{\pi}{2})$$

$$y''' = (-\sin x)' = -\cos x = \sin(x + 3\cdot\frac{\pi}{2})$$

$$y^{(4)} = (-\cos x)' = \sin x = \sin(x + 4\cdot\frac{\pi}{2})$$

$$y^{(5)} = (\sin x)' = \cos x = \sin(x + 5\cdot\frac{\pi}{2}).$$

Note that $y^{(4)} = y$, $y^{(5)} = y'$ and so on. The higher derivatives repeat every four steps.

$$y^{(n)} = (\sin x)^{(n)} = \sin(x + n\cdot\frac{\pi}{2}).$$

The following example shows how to find the second derivative of a function that is defined implicitly.

Example 3 Find y'', if $x^4 + y^4 = 16$.

Solution. Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3 \cdot y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3}.$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \left(-\frac{x^3}{y^3}\right)'_x = -\frac{(x^3)'\cdot y^3 - (y^3)'\cdot x^3}{(y^3)^2} = -\frac{3x^2\cdot y^3 - 3y^2\cdot y'\cdot x^3}{y^6}.$$

If we now substitute last equation into this expression, we get

$$y'' = -\frac{3x^2\cdot y^3 - 3y^2\cdot\left(-\frac{x^3}{y^3}\right)\cdot x^3}{y^6} = -\frac{3x^2\cdot y^4 + 3\cdot x^6}{y^7} = -\frac{3x^2\cdot(y^4 + x^4)}{y^7}.$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2 \cdot 16}{y^7} = -48 \cdot \frac{x^2}{y^7}.$$

Note 1 If some curves defined by parametric equations $x = x(t)$, $y = y(t)$ and $y'_x = \frac{y'_t(t)}{x'_t(t)}$, the second derivative of the function y is differentiated by formula:

$$y'' = \left(\frac{y'_t}{x'_t}\right)'_x = \frac{(y'_x)'_t}{x'_t}.$$

Example 4 Find y'',if $\begin{cases} x = \sqrt{1-t^2}, \\ y = \frac{1}{t}. \end{cases}$

Solution. Using formula $y'_x = \frac{y'_t(t)}{x'_t(t)}$, we get

$$x'_t = \left(\sqrt{1-t^2}\right)' = \frac{1}{2\sqrt{1-t^2}} \cdot (-2t)$$

$$y'_t = \left(\frac{1}{t}\right)'_t = -\frac{1}{t^2}$$

$$y'_x = \frac{y'_t(t)}{x'_t(t)} = \frac{-\frac{1}{t^2}}{\frac{1}{2\sqrt{1-t^2}} \cdot (-2t)} = \frac{\sqrt{1-t^2}}{t^3}.$$

Using formula $y'' = \left(\frac{y'_t}{x'_t}\right)'_x = \frac{(y'_x)'_t}{x'_t}$, we get

$$(y'_x)'_t = \left(\frac{\sqrt{1-t^2}}{t^3}\right)'_t = \frac{\left(\sqrt{1-t^2}\right)' \cdot t^3 - \sqrt{1-t^2} \cdot \left(t^3\right)'}{\left(t^3\right)^2} = \frac{\frac{-2t}{2\sqrt{1-t^2}} \cdot t^3 - \sqrt{1-t^2} \cdot 3t^2}{t^6} =$$

$$= -\frac{t^2 + 3\left(\sqrt{1-t^2}\right)^2}{t^4} = -\frac{3-2t^2}{t^4}$$

$$y'' = \frac{(y'_x)'_t}{x'_t} = \frac{-\frac{3-2t^2}{t^4}}{\frac{1}{2\sqrt{1-t^2}} \cdot (-2t)} = \frac{\left(3-2t^2\right) \cdot \sqrt{1-t^2}}{t^5}.$$

Exercise Set 13

In Exercises 1 to 18 find y''

1. $y = 3x^3 + 4x^2 - x - 2$	2. $y = x - \ln 2 + e^x$	3. $y = \sin\sqrt{3} + 3\cos 6x$
4. $y = \dfrac{tgx - 3ctgx}{\sqrt{2}}$	5. $y = \dfrac{(2x^2 - 1)}{3x^3}$	6. $y = e^{2x} \cdot \sin x$
7. $y = x^5 \cdot \cos x$	8. $y = \left(3x^2 - 6x + 8\right)^{21}$	9. $y = \sin^6 7x$
10. $y = 2\sqrt{e^x + 1}$	11. $x^3 + 4y^2 = 5xy$	12. $e^{x+y} = 5x - 6y$
13. $x^2 + 2y^2 = \dfrac{x}{y}$	14. $e^{x+y} = x^4 + 2y$	15. $\begin{cases} x = t + \sin t, \\ y = 2 - \cos t. \end{cases}$
16. $\begin{cases} x = \dfrac{1}{t}, \\ y = \dfrac{1}{t^2 + 1}. \end{cases}$	17. $\begin{cases} x = \sqrt{t}, \\ y = \dfrac{1}{\sqrt{1 - t}}. \end{cases}$	18. $\begin{cases} x = tgt, \\ y = \dfrac{1}{\sin 2t}. \end{cases}$

In Exercises 19 to 24 find $y^{(n)}$

19. $y = e^x$	20. $y = \cos x$	21. $y = \sin 3x$
22. $y = \dfrac{1}{x}$	23. $y = \ln 3x$	24. $y = \sin^2 x$

3.6 L'Hopital's Rule

The problem of finding a limit has arisen in graphing a curve and will appear often in later chapters. Fortunately, there are some general techniques for computing a wide variety of limits. This section discusses one of the most important of these methods - l'Hopital's rule, which concerns the limit of a quotient of two functions.

Theorem 1 describes a general technique for dealing with the troublesome quotient

$$\frac{f(x)}{g(x)}$$

when $f(x) \to 0$ and $g(x) \to 0$ (it is known as the zero-over-zero case of l'Hopital's rule).

Theorem 1 (*L'Hopital's rule*). Let a be a number and let $f(x)$ and $g(x)$ be differentiable over some open interval (a,b). Assume also that $g'(x)$ is not 0 for any x in that interval. If

$$\lim_{x \to a^+} f(x) = 0\ ,\ \lim_{x \to a^+} g(x) = 0 \text{ and } \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L\ .$$

Example 1 Find $\lim\limits_{x \to 1^+} \dfrac{x^5 - 1}{x^3 - 1}$.

Solution. In this case,

$$a = 1,\quad f(x) = x^5 - 1,\quad \text{and}\quad \lim_{x \to 1^+} \left(x^3 - 1\right) = 0$$

According to l'Hopital's rule,

$$\lim_{x\to1^+}\frac{x^5-1}{x^3-1}=\lim_{x\to1^+}\frac{\left(x^5-1\right)'}{\left(x^3-1\right)'}$$

if the latter limit exists. Now

$$\lim_{x\to1^+}\frac{\left(x^5-1\right)'}{\left(x^3-1\right)'}=\lim_{x\to1^+}\frac{5x^4}{3x^2}=\lim_{x\to1^+}\frac{5}{3}x^2=\frac{5}{3}.$$

Thus $\lim\limits_{x\to1^+}\dfrac{x^5-1}{x^3-1}=\dfrac{5}{3}.$

Example 2 Find $\lim\limits_{x\to0}\dfrac{\sin x-x}{x^3}$

Solution. As $x\to0,$ both numerator and denominator approach 0. By l'Hopital's rule,

$$\lim_{x\to0}\frac{\sin x-x}{x^3}=\lim_{x\to0}\frac{\cos x-1}{3x^2}$$

But as $x\to0,$ both $\cos x-1\to0$. So use l'Hopital's rule again:

$$\lim_{x\to0}\frac{\cos x-1}{3x^2}=\lim_{x\to0}\frac{-\sin x}{6x}.$$

Both $\sin x$ and $6x$ approach 0 as $x\to0.$ Use l'Hopital's rule yet another time:

$$\lim_{x\to0}\frac{-\sin x}{6x}=\lim_{x\to0}\frac{-\cos x}{6}=-\frac{1}{6}.$$

The next theorem presents a form of l'Hopital's rule that covers the case in which $f(x)\to\infty$ and $g(x)\to\infty$. It is called the infinity-over-infinity case of l'Hopital's rule.

Theorem 2 *L'Hopital's rule (infinity-over-infinity case).* Let $f(x)$ and $g(x)$ be defined and differentiable for all x larger than some fixed number. Then, if

$$\lim_{x\to\infty}f(x)=\infty,\ \lim_{x\to\infty}g(x)=\infty \text{ and } \lim_{x\to\infty}\frac{f'(x)}{g'(x)}=L$$

It follows that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L.$$

Example 3 Use l'Hopital's rule to find $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$.

Solution. By Theorem 2,

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{x'}{\left(e^x\right)'}=\lim_{x\to\infty}\frac{1}{e^x}=0.$$

Example 4 Find $\lim\limits_{x\to\infty}\dfrac{x^3}{2^x}$

Solution. Since both numerator and denominator approach ∞ as $x\to\infty,$ l'Hopital's rule may be applied. It asserts that

$$\lim_{x\to\infty}\frac{x^3}{2^x}=\lim_{x\to\infty}\frac{3x^2}{2^x\ln 2}=\lim_{x\to\infty}\frac{6x}{2^x(\ln 2)^2}=\lim_{x\to\infty}\frac{6}{2^x(\ln 2)^3}.$$

Thus as $x\to\infty, 2^x$ grows much faster than x^3.

Transforming some limits so that l'Hopital's rule applies to many limit problems can be transformed to limits to which l'Hopital's rule applies. For instance, the problem of finding

$$\lim_{x\to 0^+} x\ln x$$

does not seen to be related to l'Hopital's rule, since it does not involve the quotient of two functions. As $x\to 0^+$, one factor, x, approaches 0 and the other factor, $\ln x$, approaches $-\infty$. It is not obvious how their product, $x\ln x$, behaves as $x\to 0^+$. But a little algebraic manipulation transforms it into a problem to which l'Hopital's rule applies, as the next example shows.

Example 5 Find $\lim\limits_{x\to 0+} x\ln x$

Solution .Rewrite $x\ln x$ as a quotient $\dfrac{\ln x}{\frac{1}{x}}$. Let $f(x)=\ln x$ and $g(x)=\dfrac{1}{x}$. Note that

$$\lim_{x\to 0^+}\ln x=-\infty \text{ and } \lim_{x\to 0^+}\frac{1}{x}=\infty.$$

A case of Theorem 2, with $x\to 0^+$, asserts that

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0}(-x)=0.$$

Thus $\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x}=0.$

From which it follows that $\lim\limits_{x\to 0^+} x\ln x=0.$ (The factor x, which approaches 0, dominates the factor $\ln x$, which gets arbitrarily large in absolute value).

Exercise Set 14

In Exercises 1 to 46 check that l'Hopital's rule applies and use it to find the limits.

1. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}$	2. $\lim\limits_{x\to 1}\dfrac{x^7-1}{x^3-1}$	3. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 2x}$
4. $\lim\limits_{x\to 0}\dfrac{\sin x^2}{(\sin x)^2}$	5. $\lim\limits_{x\to\infty}\dfrac{x^3}{e^x}$	6. $\lim\limits_{x\to\infty}\dfrac{x^5}{3^x}$
7. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$	8. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{(\sin x)^3}$	9. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\ln(1+x)}$
10. $\lim\limits_{x\to 1}\dfrac{\cos(\pi/2)}{\ln x}$	11. $\lim\limits_{x\to\infty}\dfrac{(\ln x)^2}{x}$	12. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{e^{2x}-1}$
13. $\lim\limits_{x\to 0}(1-2x)^{1/x}$	14. $\lim\limits_{x\to 0}(1+\sin 2x)^{ctgx}$	15. $\lim\limits_{x\to 0^+}(\sin x)^{(e^x-1)}$

16. $\lim\limits_{x\to 0^+} x^2 \ln x$	17. $\lim\limits_{x\to 0^+} (tgx)^{tg2x}$	18. $\lim\limits_{x\to 0^+} \left(e^x - 1\right)\ln x$
19. $\lim\limits_{x\to\infty} \dfrac{2^x}{3^x}$	20. $\lim\limits_{x\to\infty} \dfrac{2^x + x}{3^x}$	21. $\lim\limits_{x\to\infty} \dfrac{\log_2 x}{\log_3 x}$
22. $\lim\limits_{x\to 1} \dfrac{\log_2 x}{\log_3 x}$	23. $\lim\limits_{x\to\infty} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$	24. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3} - \sqrt{x^2+4x}\right)$.
25. $\lim\limits_{x\to\infty} \dfrac{x^2 + 3\cos 5x}{x^2 - 2\sin 4x}$	26. $\lim\limits_{x\to\infty} \dfrac{e^x - 1/x}{e^x + 1/x}$	27. $\lim\limits_{x\to 0} \dfrac{3x^2 + x^2 - x}{5x^3 + x^2 + x}$
28. $\lim\limits_{x\to\infty} \dfrac{3x^3 + x^2 - x}{5x^3 + x^2 + x}$	29. $\lim\limits_{x\to\infty} \dfrac{\sin x}{4 + \sin x}$	30. $\lim\limits_{x\to\infty} 5\sin 3x$
31. $\lim\limits_{x\to 1^+} (x-1)\ln(x-1)$	32. $\lim\limits_{x\to\pi/2} \dfrac{tgx}{x - (\pi/2)}$	33. $\lim\limits_{x\to 0} (\cos x)^{1/x}$
34. $\lim\limits_{x\to 0^+} x^{1/x}$	35. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{\sin 3x}$	36. $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x^3 - 1}$
37. $\lim\limits_{x\to 0} \dfrac{xe^x (1+x)^3}{e^x - 1}$	38. $\lim\limits_{x\to 0} \dfrac{xe^x \cos^2 6x}{e^{2x} - 1}$	39. $\lim\limits_{x\to 0} \left(ctgx - \dfrac{1}{\sin x}\right)$
40. $\lim\limits_{x\to 0} \dfrac{\sin 7x - \sin 3x}{\sin x}$	41. $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{\sin x}$	42. $\lim\limits_{x\to 0} \dfrac{tg^5 x - tg^3 x}{1 - \cos x}$
43. $\lim\limits_{x\to 2} \dfrac{x^3 + 8}{x^2 + 5}$	44. $\lim\limits_{x\to\pi/2} \dfrac{\sin 5x}{\sin 3x}$	45. $\lim\limits_{x\to 0} \left(\dfrac{1}{1-\cos x} - \dfrac{2}{x^2}\right)$
46. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{\tan^{-1} 2x}$		

3.7 The Hyperbolic Functions And Their Inverses

Certain combinations of the exponential functions e^x and e^{-x} occur often enough in differential equations and engineering – for instance, in the study of electric transmission and suspension cables – to be given names. This section defines these so-called hyperbolic functions and obtains their basic properties. Since the letter x will be needed later for another purpose, we will use the letter t when writing the two preceding exponentials, namely, e^t and e^{-t}.

Definition *(The hyperbolic cosine*). Let t be a real number. The hyperbolic cosine of t, denoted cht, is given by the formula

$$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$$

Definition *(The hyperbolic sine).* Let t be a real number. The hyperbolic sine of t, denoted sht, is given by the formula

$$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}$$

The four other hyperbolic functions, namely, the hyperbolic tangent, the hyperbolic secant, the hyperbolic cotangent, and the hyperbolic cosecant, are defined as follows:

$$\mathrm{th}\,t=\frac{\mathrm{sh}\,t}{\mathrm{ch}\,t}\quad \sec ht=\frac{1}{\mathrm{ch}\,t}\quad \mathrm{cth}\,t=\frac{\mathrm{ch}\,t}{\mathrm{sh}\,t},\csc ht=\frac{1}{\mathrm{sht}}$$

Each can be expressed in terms of exponentials. For instance,

$$\mathrm{th}\,t=\frac{\left(e^{t}-e^{-t}\right)/2}{\left(e^{t}+e^{-1}\right)/2}=\frac{e^{t}-e^{-1}}{e^{t}+e^{-t}}$$

As $t\to\infty, e^{t\to\infty}$ and $e^{-t}\to 0$. Thus $\lim\limits_{t\to\infty}\mathrm{th}\,t=1$. Similarly, $\lim\limits_{t\to-\infty}\mathrm{th}\,t=-1$.

The derivatives of the four hyperbolic functions can be computed directly.

$$\left(\mathrm{ch}\,t\right)'=\left(\frac{e^{t}+e^{-t}}{2}\right)'=\frac{e^{t}-e^{-t}}{2}=sht$$

$$\left(s\mathrm{h}\,t\right)'=\left(\frac{e^{t}-e^{-t}}{2}\right)'=\frac{e^{t}+e^{-t}}{2}=cht$$

$$\left(\mathrm{th}\,t\right)'=\left(\frac{\mathrm{sh}\,t}{\mathrm{ch}\,t}\right)'=\frac{1}{ch^{2}t}$$

$$\left(ctht\right)'=\left(\frac{cht}{sht}\right)'=-\frac{1}{sh^{2}t}.$$

4 USING THE DERIVATIVE AND LIMITS WHEN GRAPHING A FUNCTION

4.1 Using The First Derivative When Graphing a Function

The primitive and inefficient way to graph a function is to make a table of values, plot many points, and draw a curve through the points (hoping that the chosen points adequately represent the function). Chapter 2 refined the technique somewhat. The x and y intercepts are of aid in graphing, for they tell where the graph meets the x and y axes. Furthermore, horizontal and vertical asymptotes were discussed; they can be of use in sketching the graph for large $|x|$ and also near a number where the function becomes infinite (usually because a denominator is 0). For instance, the line $x=1$ is a vertical asymptote of $y=\frac{1}{x-1}$, the line $y=0$ is a horizontal asymptote of the same curve. The line $x=\frac{\pi}{2}$ is a vertical asymptote of the curve $y=tgx$.

This section shows how to use the derivative and limits to help graph a function.

Definition *(Critical number and critical point).* A number c, at which $f'(c)=0$ is called a critical number for the function f. The corresponding point $\left(c,f(c)\right)$ on the graph of f is a critical point on that graph.

Definition *(Relative maximum (local maximum)).* The function f has a relative maximum (or local maximum) at the number c if there is an open interval (a,b) around c such that $f(c) \geq f(x)$ for all x in (a,b) that lie in the domain of f. A local or relative minimum is defined analogously.

Definition *(Global maximum).* The function f has a global maximum (or absolute maximum) at the number c if $f(c) \geq f(x)$ for all x in the domain of f. A global minimum is defined analogously.

Note that a global maximum is necessarily a local maximum as well. A local maximum is like the summit of a single mountain; a global maximum corresponds to Mount Everest.

Fig. 4.1 illustrates the notions of critical points b, c, d, e, local maximum $f(b)$, global maximum $f(d)$, local minimum $f(c)$, and global minimum $f(a)$ in the graph of a hypothetical function. Any given function may have none of these, or some, or all.

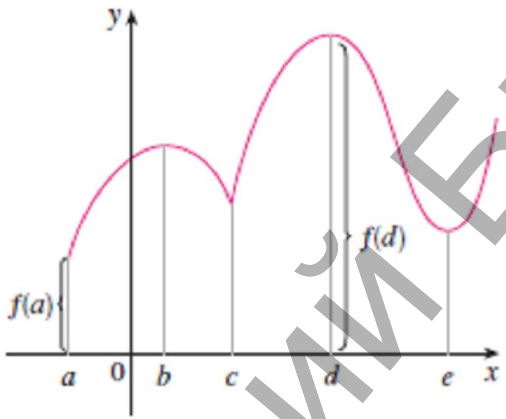


Fig. 4.1

The following test for local maximum or local minimum is an immediate consequence of the fact that when the derivative is positive the function increases and when it is negative it decreases.

First-Derivative Test For Local Maximum (Minimum)

Let f be a function and let c be a number in its domain. Assume that numbers a and b exist such that $a < c < b$ and

1. f is continuous on the open interval (a,b).
2. f is differentiable on the open interval (a,b), except possibly at c.
3. $f'(x)$ is positive for all $x < c$ in the interval and is negative for all $x < c$ in the interval.

Then f has a local maximum at c.

A similar test, with "positive" and "negative" interchanged, holds for a local minimum (see Fig.4.2)

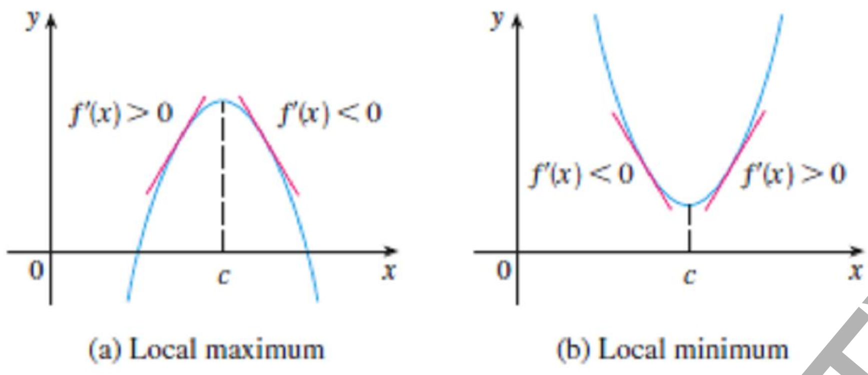


Fig.4.2

Informally, the derivative test says, "if the derivative changes sign at c, then the function has either a local minimum or a local maximum”.

To decide which it is, just make a crude sketch of the graph near $(c, f(c))$ to show on which side of c the function is increasing and on which side it is decreasing.

Example 1 The graph of the function $f(x)=3x^4-16x^3+18x^2$; $-1\le x\le 4$ is shown in Fig. 4.3. You can see that $f(1)=5$ is a local maximum, whereas the absolute maximum is $f(-1)=37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0)=0$ is a local minimum and $f(3)=-27$ is both a local and an absolute minimum. Note that $f(x)$ has neither a local nor an absolute maximum at $x=4$.

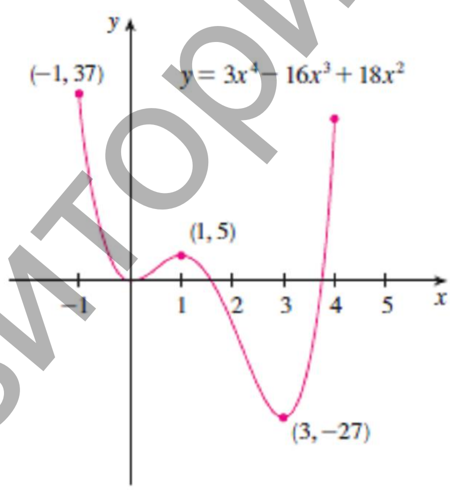


Fig. 4.3

Example 2 Graph $f(x)=3x^4-4x^3$. Discuss relative maxima and minima.

Solution. To find the intercepts note that $f(0)=0$ and $3x^4-4x^3=0$ when $x^3\cdot(3x-4)=0$ that is, when $x=0$ or $x=\dfrac{4}{3}$. The derivative is

$$f'(x)=12x^3-12x^2=12x^2\cdot(x-1).$$

The critical numbers are the solutions of the equation

$$12x^2\cdot(x-1)=0.$$

Namely, $x=0$ and $x=1$.

How does the sign of $f'(x) = 12x^3 - 12x^2 = 12x^2 \cdot (x-1)$ behave when x is near 0?

$$f'(x) > 0 \Rightarrow 12x^2 \cdot (x-1) > 0 \Rightarrow x > 1$$

$$f'(x) < 0 \Rightarrow 12x^2 \cdot (x-1) < 0 \Rightarrow x < 1.$$

Thus the sign of $f'(x)$ does not change as x passes through 0. In fact, since $f'(x)$ remains negative (except at 0), the function f is decreasing for $x \le 1$. Thus there is no relative maximum or minimum at $x = 0$.

How does the sign of $f'(x) = 12x^2 \cdot (x-1)$ behave when x is near 1?

The factor $12x^2$ remains positive, but $(x-1)$ changes sign from negative to positive. Hence at $x = 1$ the function has a local minimum.

Writing $f(x) = 3x^4 - 4x^3 = x^4 \cdot \left(3 - \frac{4}{x}\right)$ shows that when $|x|$ is large, $f(x)$ behaves like $3x^4$, since $\frac{4}{x}$ is near 0. Since $3x^4$ becomes arbitrarily large when x is large, the function has no global maximum. The graph in Fig. 4.4 shows the x intercepts and the critical points. Note that when $x = 1$ a global minimum occurs.

In many applied problems we are interested in the behavior of a differentiable function just over some closed interval $[a, b]$. Such a function will have a global maximum for that interval by the maximum-value theorem of Sec.2.6. That maximum can occur either at an endpoint - a or b -or else at some number c in the open interval (a, b). In the latter case, c must be a critical number, for $f'(c) = 0$ by the interior-maximum theorem of Sec. 4.1.

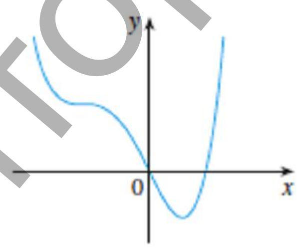


Fig. 4.4

Fig. 4.5 shows some of the ways in which a relative or global maximum or minimum can occur for a function considered only on a closed interval $[a, b]$.

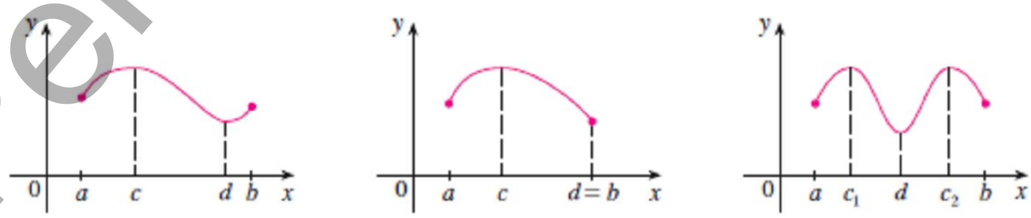


Fig. 4.5

The major point to keep in mind is that the maximum value of a function f that is differentiable on a closed interval occurs

1. At an endpoint of the interval, or

2. At a critical number [where $f'(x) = 0$].

Example 3 Find the maximum value of $f(x) = x^3 - 3x^2 + 3x$ for x in $[0,2]$.

Solution. First compute f at the ends of the interval, 0 and 2:

$$f(0) = 0 \text{ and } f(2) = 2 .$$

Next, compute $f'(x)$, which is $f'(x) = 3x^2 - 6x + 3$. When is $f'(x) = 0$? When

$$3x^2 - 6x + 3 = 0$$
$$3(x^2 - 2x + 1) = 0$$
$$3(x-1)^2 = 0$$

Thus 1 is the only critical number, and it lies in the interval $[0,2]$.

The maximum of f must therefore occur either at an endpoint of the interval (at 0 or 2) or at the only critical number, 1. It is necessary to calculate $f(1)$ to determine where the maximum occurs:

$$f(1) = 1^3 - 3 \cdot 1^2 + 3 \cdot 1 = 1$$

Since $f(0) = 0$, $f(2) = 2$ and $f(1) = 1$, the maximum value is 2, occurring at the end-point 2.

Exercise Set 15

In each of Exercises 1 to 9 find all critical numbers of the given function and use the first-derivative test to determine whether a local maximum, a local minimum, or neither occurs there.

1. x^5	2. x^6	3. $(x-1)^3$
4. $(x-1)^4$	5. $3x^4 + x^3$	6. $2x^3 + 3x^2$
7. $x\sin x + \cos x$	8. $x\cos x + \sin x$	9. $x^5 + 5x^3$

In Exercises 10 to 27 graph the given functions, showing any intercepts, asymptotes, critical points, or local or global extrema.

10. $3x^4 - 4x^3$	11. $2x^3 - 3x^2$	12. $x^3 - 3x^2 - 9x$
13. $x^3 + 6x^2 - 15x$	14. $x^4 - 4x^3 - 20x^2$	15. $x^3 + x^2 - 5x$
16. $x^4 + 4x^3$	17. $3x^4 + 8x^3$	18. $\frac{3x+1}{3x-1}$
19. $\frac{x}{x-1}$	20. $\frac{x}{x^2-1}$	21. $\frac{x}{x^2+1}$
22. $\frac{1}{2x^2-x}$	23. $\frac{1}{x^2-3x+2}$	24. $\frac{x^2+3}{x^2-1}$
25. $\frac{\sqrt{x^2+1}}{x}$	26. $\frac{1}{x^2-4x+4}$	27. $\frac{x-2}{x^2-3x+2}$

Exercises 28 to 33 concern functions whose domains are restricted to closed intervals. In each case find the maximum and the minimum value for the given function over the given interval.

28. $x^2 - x^4$, [0,1]	29. $4x^2 - 5x^3$, [0,4]	30. $4x - x^2$, [0,2]
31. $2x^2 - 5x$, [−1,1]	32. $x^3 - 3x^2 - 9x$, [0,2]	33. $\frac{x}{x^2+1}$, [0,3]

4.2 Concavity And The Second Derivative

Whether the first derivative is positive, negative, or zero tells a good deal about a function and its graph. This section will explore the geometric significance of the second derivative being positive, negative, or zero. The following section will show how the second derivative is used in the study of motion.

Concave Upward and Concave Downward

Assume that $f''(x)$ is positive for all x in the open interval (a,b). Since $f''(x)$ is the derivative of $f'(x)$, it follows that $f'(x)$ is an increasing function throughout the interval (a,b). In other words, as x increases, the slope of the graph of $y = f(x)$ increases as we move from left to right on that part of the graph corresponding to the interval (a,b).

Definition *(Concave upward).* A function f whose first derivative is increasing throughout the open interval (a,b) is called concave upward in that interval.

Note that when a function is concave upward, it is shaped like part of a cup. It can be proved that where a curve is concave upward it lies above its tangent lines and below its chords, as shown in Fig. 4.6(a).

As was observed, in an interval where $f'(x)$ is positive, the function f is increasing, and so the function f is concave upward. However, if a function is concave upward, $f'(x)$ is not necessarily positive. For instance, $y = x^4$ is concave upward over any interval, since the derivative $4x^3$ is increasing. The second derivative $12x^2$ is not always positive; at $x = 0$ it is 0.

If, on the other hand, $f''(x)$ is negative throughout (a,b) then $f'(x)$ is a decreasing function and the graph of f looks like part of the curve in Fig. 4.6(b).

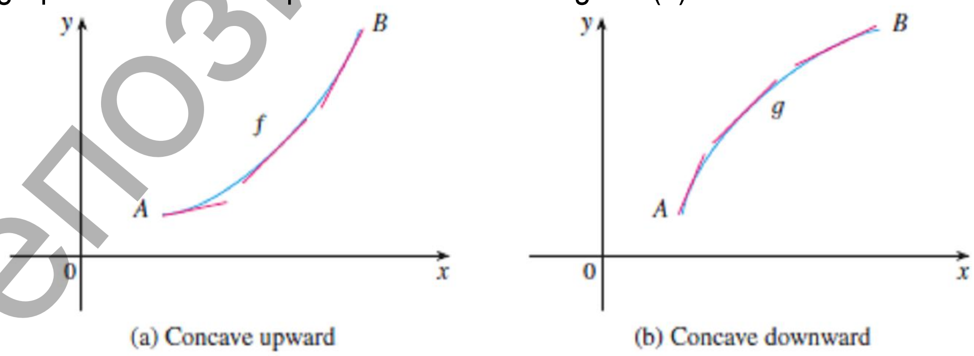


Fig. 4.6

Definition (*Concave downward*). A function f whose first derivative is decreasing throughout an open interval (a,b) is called concave downward in that interval.

Example 1. Where is the graph of $f(x) = x^3$ concave upward? Concave downward?

Solution. First compute the second derivative. Since $f'(x) = 3x^2$, $f''(x) = 6x$.

Clearly $6x$ is positive for all positive x and negative for all negative x. The graph, shown in Fig. 4.7, is concave upward if $x > 0$ and concave downward if $x < 0$. Note that the sense of concavity changes at $x = 0$. When you drive along this curve from left to right, your car turns to the right until you pass through $(0,0)$. Then it starts turning to the left.

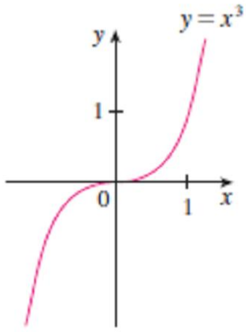


Fig. 4.7

Example 2. Consider the function $f(x) = \sin x$ for x in $[0, 2\pi]$. Where is the graph concave upward? Concave downward?

Solution.

$$y' = (\sin x)' = \cos x$$

$$y'' = (\cos x)' = -\sin x$$

The second derivative, $-\sin x$, is negative for $0 < x < \pi$. It is positive for $\pi < x < 2\pi$ Therefore, the graph is concave downward for x in $(0, \pi)$ and concave upward for x in $(\pi, 2\pi)$.

The sense of concavity is a useful tool in sketching the graph of a function. Of special interest in Examples 1 and 2 is the presence of a point on the graph where the sense of concavity changes. Such a point is called an inflection point.

Definition (*Inflection point and inflection number*). Let f be a function and let a be a number. Assume that there are numbers b and c such that $b < a < c$ and

1. f is continuous on the open interval (b, c)
2. f is concave upward in the interval (b, a) and concave downward in the interval (a, c) or vice versa.

Then the point $(a, f(a))$ is called **an inflection point** or **point inflection**. The number a is called an **inflection number**.

Observe that if the second derivative changes sign at the number a, then a is an inflection number.

If the second derivative exists at an inflection point, it must be 0. But there can be an inflection point even if $f''(x)$ is not defined there, as shown by the next example, which is closely related to Example 1.

The Second Derivative and Local Extrema

The second derivative is also useful in testing whether at a critical number there is a relative maximum or relative minimum. For instance, let a be a critical number for the function f and assume that $f''(a)$ happens to be negative. If $f'(x)$ is continuous in some open interval that contains a, then $f''(a)$ remains negative for a suitably small open interval that contains a. This means that the graph of f is concave downward near $(a, f(a))$ hence lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point $(a, f(a))$. Thus the function has a relative maximum at the critical number a. This observation suggests the following test for a relative maximum or minimum.

The Second Derivative Test

Suppose $f''(x)$ is continuous near a.

(a) If $f'(a)=0$ and $f''(a)>0$, then f has a local minimum at a.

(b) If $f'(a)=0$ and $f''(a)<0$, then f has a local maximum at a.

Example 3. Discuss the curve $f(x)=x^4-4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution Since f is differentiable throughout its domain, any local extremum can occur only at a critical number. So begin by finding the critical numbers, as follows:

$$f'(x)=\left(x^4-4x^3\right)'=4x^3-12x^2=4x^2(x-3).$$

Setting $f'(x)=0$ gives $x^2=0$ or $x-3=0$. The critical numbers are therefore

$$x=0 \text{ and } x=3.$$

Now use the second derivative to determine whether either of these corresponds to a local extremum.

The second derivative is

$$f''(x)=\left(4x^3-12x^2\right)'=12x^2-24x=12x(x-2)$$

At $x=3$ we have

$$f''(3)=12\cdot 3^2-24\cdot 3=36$$

Since $f'(3)=0$ and $f''(3)>0$, f has a local minimum at $x=3$.

How about the other critical number, $x=0$? In this case,

$$f''(0)=0.$$

Since $f''(0)=0$, the second-derivative test tells us nothing about the critical number 0. Instead, we must resort to the first-derivative test and examine the sign of $f'(x)=\left(x^4-4x^3\right)'=4x^3-12x^2=4x^2(x-3)$ for x near 0. For x sufficiently near 0, whether to the right of 0 or to the left, x^2 is positive and $x-3$ is negative. Thus $f'(x)$ is negative for x near 0. Since f is a decreasing function near 0, it has neither a local maximum nor a local minimum at 0.

Since $f''(x)=0$ when $x=0$ or $x=2$, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x)=12x(x-2)$	Concavity
$(-\infty,0)$	+	upward
$(0,2)$	-	downward
$(2,+\infty)$	+	upward

The point $(0,0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also, $(2,-16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Fig. 4.8.

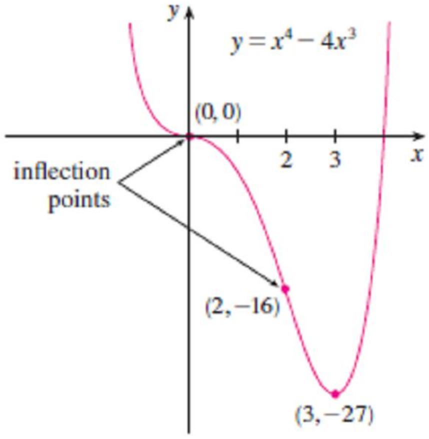


Fig. 4.8

Exercise Set 16

In Exercises 1 to 18 graph the functions, showing any relative maxima, relative minima, and inflection points.

1. $2x^4 - 5x^3$	2. $7x^3 - 4x^2$	3. $x^3 - 3x^2 - 9x$
4. $x^3 + 6x^2 - 15x$	5. $x^4 - 4x^3 - 20x^2$	6. $x^3 + x^2 - 5x$
7. $2x^4 - 7x^3$	8. $5x^4 + 8x^3$	9. $\dfrac{2x+1}{2x-1}$
10. $\dfrac{x}{2x-1}$	11. $\dfrac{x}{x^2-1}$	12. $\dfrac{x}{x^2+1}$
13. $\dfrac{1}{2x^2-3}$	14. $\dfrac{1}{x^2-5x+4}$	15. $\dfrac{x^2+3}{x^2-1}$
16. $\dfrac{\sqrt{x^2+1}}{x}$	17. $\dfrac{1}{x^2-6x+9}$	18. $\dfrac{x-1}{x^2-3x+2}$

4.3 Guidelines For Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $y = f(x)$ is defined.

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect it about the y-axis to obtain the complete curve.

(ii) If $f(-x) = -f(x)$ for all x in D, then it is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$. [Rotate 180° about the origin.]

(iii) If $f(x+p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.3 that if either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim\limits_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.3 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x\to a^-} f(x) = -\infty,$$

$$\lim_{x\to a^+} f(x) = -\infty,$$

$$\lim_{x\to a^-} f(x) = \infty,$$

$$\lim_{x\to a^+} f(x) = \infty. \quad (1)$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true.

(iii) *Slant Asymptotes.* Some curves have asymptotes that are *oblique,* that is, neither horizontal nor vertical. If

$$\lim_{x\to\pm\infty}\left[f(x) - (kx + b)\right] = 0$$

then the line $y = kx + b$ is called a **slant asymptote,** because the vertical distance between the curve $y = f(x)$ and the line $y = kx + b$ approaches 0. For rational functions, slant asymptotes occur when the degree of the numerator is more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division.

For finding $y = kx + b$ we use next formulas:

$$k = \lim_{x\to\pm\infty}\frac{f(x)}{x}$$

$$b = \lim_{x\to\infty}\left(f(x) - kx\right).$$

E. Intervals of Increase or Decrease Use the I /D Test. Compute $f'(x)$ and find the intervals on which it is positive (f is increasing) and the intervals on which $f'(x)$ is negative(f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$d oes not exist]. Then use the First Derivative Test. If $f'(x)$ changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Example 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

Solution

A. The domain is

$$D = \left\{ x \mid x^2 - 1 \neq 0 \right\} = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty).$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function is even. The curve is symmetric about the y-axis.

D.

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2x^2}{x^2} = 2.$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = +\infty, \ \lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \ \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty, \ \lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = +\infty.$$

Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes.

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} \div x = \lim_{x \to \pm\infty} \frac{2x^2}{x^3 - x} = \lim_{x \to \pm\infty} \frac{2x^2}{x^3} = \lim_{x \to \pm\infty} \frac{2}{x} = 0.$$

It's means that $y = f(x)$ hasn't a slant asymptote.

E.

$$f'(x) = \left(\frac{2x^2}{x^2 - 1} \right)' = \frac{4x(x^2 - 1) - 2x \cdot 2x^2}{\left(x^2 - 1\right)^2} = -\frac{4x}{\left(x^2 - 1\right)^2}.$$

Since $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, +\infty)$.

F. The only critical number is $x = 0$. Since $f'(x)$ changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G.

$$f''(x) = \left(-\frac{4x}{\left(x^2 - 1\right)^2} \right)' = -4 \cdot \frac{\left(x^2 - 1\right)^2 - x \cdot 2 \cdot \left(x^2 - 1\right) \cdot 2x}{\left(x^2 - 1\right)^4} = -4 \cdot \frac{x^2 - 1 - 16x^2}{\left(x^2 - 1\right)^3} =$$

$$= 4 \cdot \frac{3x^2 + 1}{\left(x^2 - 1\right)^3}$$

Since $3x^2 + 1 > 0$ for all x, we have

$$f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow |x| > 1$$

and

$$f''(x) < 0 \Leftrightarrow x^2 - 1 < 0 \Leftrightarrow |x| < 1.$$

Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, +\infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure 4.9.

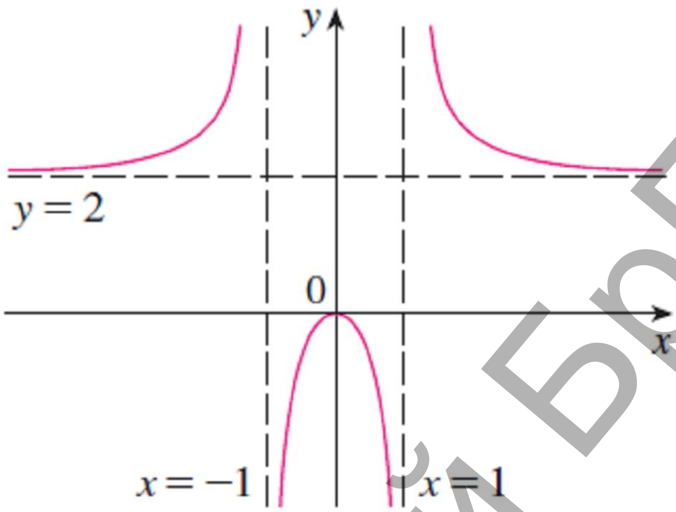


Fig. 4.9.

Exercise Set 17

Use the guidelines of this section to sketch the curve.

1. $y = \frac{1}{x^4 - 1}$	2. $y = -\left(\frac{x}{x+2}\right)^2$	3. $y = \frac{x^3 - 32}{x^2}$
4. $y = \frac{4(x+1)^2}{x^2 + 2x + 4}$	5. $y = \frac{3x - 2}{x^3}$	6. $y = \frac{x^2 - 6x + 9}{(x-1)^2}$
7. $y = \frac{x^3 - 27x + 54}{x^3}$	8. $y = \frac{4}{x^2 + 2x - 3}$	9. $y = \frac{4}{3 + 2x - x^2}$
10. $y = \frac{x^2 + 2x - 7}{x^2 + 2x - 3}$	11. $y = \frac{12 - 3x^2}{x^2 + 12}$	12. $y = \frac{-8x}{x^2 + 4}$
13. $y = \frac{3x^4 + 1}{x^3}$	14. $y = \frac{4x}{(x+1)^2}$	15. $y = \frac{8(x-1)}{(x+1)^2}$
16. $y = \frac{1 - 2x^3}{x^2}$	17. $y = \frac{x^3 + 4}{x^2}$	18. $y = \frac{12x}{9 + x^2}$

Literature

1 Stewart James Calculus Early Transcendental. 2008. pp. 1308.

Contents

УЧЕБНОЕ ИЗДАНИЕ

Составители:
Гладкий Иван Иванович
Дворниченко Александр Валерьевич
Каримова Татьяна Ивановна
Лебедь Светлана Федоровна
Шишко Татьяна Витальевна

FUNCTIONS
LIMITS
DERIVATIVES

for foreign first-year students

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**
для студентов 1-го курса

Ответственный за выпуск: Дворниченко А.В.
Редактор: Боровикова Е.А.
Компьютерная вёрстка: Соколюк А.П.
Корректор: Никитчик Е.В.

Подписано к печати 15.10.2014 г. Формат 60x84 $^{1}/_{16.}$ Гарнитура Arial Narrow.
Бумага «Performer». Усл. п. л. 4,4. Уч. изд. 4,75. Заказ № 864. Тираж 40 экз.
Отпечатано на ризографе учреждения образования «Брестский государственный технический университет». 224017, г. Брест, ул. Московская, 267.
224017, г. Брест, ул. Московская, 267.