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КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

Elements of Algebra and Analytic Geometry

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**

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Данные указания адресованы преподавателям и студентам технических ВУЗов для проведения аудиторных занятий и организации самостоятельной работы студентов при изучении материала из рассматриваемых разделов. Учебно-методическая разработка на английском языке «Elements of Algebra and Analytic Geometry» содержит необходимый материал по темам «Элементы линейной алгебры», «Элементы векторной алгебры», «Аналитическая геометрия на плоскости», «Аналитическая геометрия в пространстве» изучаемым студентами БрГТУ технических специальностей в курсе дисциплины «Математика». Теоретический материал сопровождается рассмотрением достаточного количества примеров и задач, при необходимости приводятся соответствующие иллюстрации.

Составители: Дворниченко А.В., м.э.н., старший преподаватель
Крагель Е.А., старший преподаватель
Лебедь С.Ф., к.ф.-м.н., доцент
Бань О.В., старший преподаватель кафедры иностранных языков
Гладкий И.И., старший преподаватель

Рецензент: Мирская Е.И., доцент кафедры алгебры, геометрии и математического моделирования УО «Брестский государственный университет им. А.С. Пушкина», к.ф.-м.н., доцент.

CONTENTS

I ELEMENTS OF ALGEBRA

1.1 The Algebra of Matrices

Definition A ***matrix*** is a rectangular array of numbers. Each number in the matrix is called an ***element*** of the matrix.

Definition A matrix of m rows and n columns is said to be of ***order*** $m \times n$ or ***dimension*** $m \times n$**.**

We will use the notation a_{ij} to refer to the element of a matrix in the i-th row and j-th column. The elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ form the ***main diagonal*** of a matrix. A matrix is denoted by either a capital letter or by surrounding the corresponding lower-case letter with brackets. For example, a matrix could be denoted as A or $\left[a_{ij}\right]$.

Caution Remember that $\left[a_{ij}\right]$ is a matrix and a_{ij} is the element in the i-th row and j-th column of the matrix.

Definition The $m \times n$ ***zero matrix***, denoted 0 is the matrix whose elements are all zeros.

Definition The matrix of order $1 \times n$ is called the ***row matrix***. For example, $A = \begin{bmatrix} 1 & 7 & -3 & 0 \end{bmatrix}$ is a row matrix of order 1×4.

Definition The matrix of order $n \times 1$ is called the ***column matrix***.

Definition The square matrix that has a 1 for each element on the main diagonal and zeros elsewhere is called the identity matrix.

The identity matrix has properties similar to the real number 1. If matrix A is a square matrix of order $n \times n$ and I_n is the identity matrix of order by n, then $AI_n = I_n A = A$.

Definition ***(Equality of Two Matrices)*** Two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

Remark The definition of equality implies that the two matrices have the same order.

Operations on matrices

Definition ***(Matrix Addition)*** If A and B are matrices of order $m \times n$ then the sum of the matrices is the $m \times n$ matrix given by $A + B = \left[a_{ij} + b_{ij}\right]$.

Definition ***(Additive Inverse of a Matrix)*** Given the matrix $A = \left[a_{ij}\right]$ the additive inverse of A is $-A = \left[-a_{ij}\right]$.

Definition ***(Subtraction on Matrices)*** Given two matrices A and B of order $m \times n$

the subtraction of the two matrices $A-B$ is $A-B=A+(-B)$.

Definition *(Product of a Real Number and a Matrix)* Given the $m\times n$ matrix $A=\left[a_{ij}\right]$ and a real number c, then $cA=\left[ca_{ij}\right]$.

Remark The product of a real number and a matrix is referred to as ***scalar multiplication.***

Definition *(Product of Two Matrices)* Let $A=\left[a_{ij}\right]$ be a matrix of order $m\times n$ and $B=\left[b_{ij}\right]$ be a matrix of order $n\times p$. Then the product AB is the matrix of order $m\times p$ given by $AB=\left[c_{ij}\right]$ where each element c_{ij} is

$$c_{ij}=\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in}\end{bmatrix}\begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj}\end{bmatrix}=a_{i1}\cdot b_{1j}+a_{i2}\cdot b_{2j}+a_{i3}\cdot b_{3j}+\ldots+a_{in}\cdot b_{nj}.$$

Remark This definition may appear complicated, but basically, to multiply two matrices one must multiply each row vector of the first matrix by each column vector of the second matrix.

For the product of two matrices to be possible, the number of columns of the first matrix must be equal to the number of rows of the second matrix $A_{m\times n}\cdot B_{n\times p}=C_{m\times p}$. The product matrix has as many rows as the first matrix and as many columns as the second matrix.

Example 1 Find the following product $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -1\end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 3 & 5\end{bmatrix}$.

Solution

If A has order 2×3 and B has order 3×2, then AB has order 2×2.

$$AB=\begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -1\end{bmatrix}_{2\times 3}\cdot\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 3 & 5\end{bmatrix}_{3\times 2}=$$

$$=\begin{bmatrix} 2\cdot 1+(-3)\cdot 4+0\cdot 3 & 2\cdot 0+(-3)\cdot(-2)+0\cdot 5 \\ 1\cdot 1+4\cdot 4+(-1)\cdot 3 & 1\cdot 0+4\cdot(-2)+(-1)\cdot 5\end{bmatrix}_{2\times 2}=\begin{bmatrix} -10 & 6 \\ 14 & -13\end{bmatrix}$$

A system of equations can be expressed as the product of matrices. Consider the matrix equation

$$\begin{bmatrix} 2 & -3 & 4 \\ 3 & 0 & 1 \\ 1 & -2 & -5 \end{bmatrix}_{3\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}_{3\times 1}.$$

Multiplying the two matrices on the left side of the equation, we have

$$\begin{bmatrix} 2x-3y+4z \\ 3x \quad\quad + z \\ x-2y-5z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}_{3\times 1}.$$

Now using the definition of matrix equality, we have that the given matrix equation is equivalent to the following system of equations:

$$\begin{cases} 2x-3y+4z=9 \\ 3x \quad\quad +z=4 \\ x-2y-5z=-2 \end{cases}.$$

Exercise Set 1

In Exercises 1 to 6 find

a) $A+B$; b) $A-B$; c) $2B$; d) $2A-3B$

1. $A=\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

2. $A=\begin{bmatrix} 0 & -2 \\ 2 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 5 & -1 \\ 3 & 0 \end{bmatrix}$

3. $A=\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$, $B=\begin{bmatrix} -3 & 1 & 2 \\ 2 & 5 & -3 \end{bmatrix}$

4. $A=\begin{bmatrix} 2 & -2 & 4 \\ 0 & -3 & -4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -5 & 6 \\ 4 & -2 & -3 \end{bmatrix}$

5. $A=\begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

6. $A=\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$

In Exercises 7 to 12 find AB and BA.

7. $A=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -2 & 4 \\ 2 & -3 \end{bmatrix}$

8. $A=\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, $B=\begin{bmatrix} -1 & -1 \\ 0 & 4 \end{bmatrix}$

9. $A=\begin{bmatrix}3 & -1\\ 2 & 3\end{bmatrix}$, $B=\begin{bmatrix}4 & 1\\ 2 & -3\end{bmatrix}$

10. $A=\begin{bmatrix}-3 & 2\\ 2 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & 2\\ -2 & 4\end{bmatrix}$

11. $A=\begin{bmatrix}2 & -1 & 3\\ 0 & 2 & -1\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 0 & 0\\ 1 & -1 & 0\\ 2 & -1 & -2\end{bmatrix}$

12. $A=\begin{bmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{bmatrix}$

In Exercises 13 to 18 find AB, if possible.

13. $A=\begin{bmatrix}1 & -2 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 2 & -1\\ 1 & 2\end{bmatrix}$

14. $A=\begin{bmatrix}-2 & 3\\ 1 & -2\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}3\\ -2\end{bmatrix}$

15. $A=\begin{bmatrix}2 & -1\\ 3 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -2\\ 3 & 1\\ 0 & -2\end{bmatrix}$

16. $A=\begin{bmatrix}2 & 0 & -1\\ 3 & 4 & -3\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 0\\ 2 & 4 & 5\end{bmatrix}$

17. $A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$, $B=\begin{bmatrix}3 & 6\\ -2 & -4\end{bmatrix}$

18. $A=\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 3 & 2\\ 2 & -1 & 0\\ 3 & 1 & 2\end{bmatrix}$

In Exercises 19 to 22 given the matrices $A=\begin{bmatrix}-1 & 3\\ 2 & -1\\ 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -2\\ 1 & 3\\ 4 & -3\end{bmatrix}$, find the 3×2 matrix X that is a solution of the equation.

19. $3X+A=B$.

20. $2X-A=X+B$.

21. $2A-3X=5B$.

22. $3X+2B=X-2A$.

In Exercises 23 to 26, use the matrices $A=\begin{bmatrix}2 & -3\\ 1 & -1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1 & 0\\ 2 & -2 & -1\\ 1 & 0 & 2\end{bmatrix}$, find

23. A^2 **24.** B^2 **25.** A^3 **26.** B^3

In Exercises 27 to 30 find the system of equations that is equivalent to the given matrix equation.

27. $\begin{bmatrix} 3 & -8 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}.$

28. $\begin{bmatrix} 2 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix}.$

29. $\begin{bmatrix} 1 & -3 & -2 \\ 3 & 1 & 0 \\ 2 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}.$

30. $\begin{bmatrix} 2 & 0 & 5 \\ 3 & -5 & 1 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 14 \end{bmatrix}.$

Individual Tasks 1

1. Find $3A + 2B - \left(A^T\right)^T$.

2. Find AB and BA.

3. Find AB, if possible.

4. Find the system of equations that is equivalent to the given matrix equation.

I.	**II.**
1. $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ 2. $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$ 3. $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & -2 & 1 & -3 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 4 & -2 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & -1 & 0 & 2 \\ 4 & 1 & 2 & -3 \\ 6 & 0 & 1 & -2 \\ 5 & 2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 10 \\ 8 \end{bmatrix}.$	1. $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$ 2. $A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ -3 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & -4 \end{bmatrix}$ 3. $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 1 & -2 \\ 1 & -1 & 0 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 5 & -1 & 2 & -3 \\ 4 & 0 & 2 & 0 \\ 2 & -2 & 5 & -4 \\ 3 & 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$
III.	**IV.**

1. $A=\begin{bmatrix}0 & -1\\ -1 & 7\\ 3 & 3\end{bmatrix}, B=\begin{bmatrix}-2 & 1\\ -2 & -1\\ -1 & 2\end{bmatrix}$	1. $A=\begin{bmatrix}-1 & 4 & 1\\ 0 & 2 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 0\\ -2 & -3 & 1\end{bmatrix}$
2. $A=\begin{bmatrix}2 & 4\\ 1 & -3\\ 1 & -2\end{bmatrix}, B=\begin{bmatrix}1 & -1 & 3\\ 2 & 0 & 2\end{bmatrix}$	2. $A=\begin{bmatrix}0 & 7\\ 3 & 4\\ 1 & 0\end{bmatrix}, B=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 1\end{bmatrix}$
3. $A=\begin{bmatrix}1 & 2 & 3\\ 0 & -2 & -3\end{bmatrix}, B=\begin{bmatrix}-2 & 0\\ 4 & -2\end{bmatrix}$	3. $A=\begin{bmatrix}5 & 3 & -2\\ -1 & -2 & 1\end{bmatrix}, B=\begin{bmatrix}-2 & 0\\ 4 & -2\end{bmatrix}$
4. $\begin{bmatrix}1 & -1 & 0 & 2\\ 4 & 3 & 2 & -3\\ 6 & 0 & 4 & -2\\ 2 & 2 & -1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}5\\ 5\\ 12\\ -3\end{bmatrix}.$	4. $\begin{bmatrix}-1 & -1 & 0 & 2\\ 2 & 1 & 0 & -3\\ 0 & 0 & 3 & -2\\ -5 & 2 & -1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ 5\\ 6\end{bmatrix}.$

1.2 Determinants

Associated with each square matrix A is a number called the ***determinant*** of A. We will denote the determinant of the matrix A by $\det A$ or by $|A|$.

Definition The determinant of the matrix $A=\left[a_{ij}\right]$ of order 2 is a number calculated by

$$|A|=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}.$$

Caution Be careful not to confuse the notation for a matrix and that for a determinant. The symbol $[\ \ ]$ (brackets) is used for a matrix; the symbol $|\ \ |$ (vertical bars) is used for the determinant of the matrix.

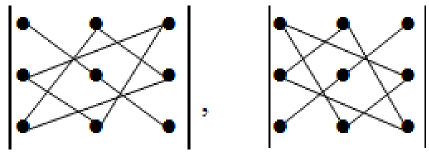
An easy way to remember the formula for the determinant of a 2×2 matrix is to recognize that the determinant is the difference in the products of the diagonal elements.

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

Definition The determinant of the matrix $A=\left[a_{ij}\right]$ of order 3 is a number calculated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

An easy way to remember the formula for the determinant of a 3×3 matrix is to recognize that the determinant contains three terms with a sign "plus" and three terms with a sign "minus" (see the picture):



"+"   "–"

Definition The ***minor*** M_{ij} of the element a_{ij} of a square matrix A of order $n\geq3$ is the determinant of the matrix of order $n-1$ obtained by deleting the *i*-th row and *j*-th column of A.

Definition The ***cofactor*** C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}.

The definition of minors and cofactors are used to define the determinant of a matrix of order 3 or greater.

Statement If the square matrix A has order 3 or greater, then the determinant of A is the sum of the products of the elements of any row or column and their cofactors. For instance, for the *r*-th row of A is

$$|A| = a_{r1}C_{r1} + a_{r2}C_{r2} + a_{r3}C_{r3} + \ldots + a_{rn}C_{rn}.$$

For the *k*-th column of A, the determinant of A is

$$|A| = a_{1k}C_{1k} + a_{2k}C_{2k} + a_{3k}C_{3k} + \ldots + a_{nk}C_{nk}.$$

Remark Whatever row or column is used, the expanding by cofactors gives the same value for the determinant. When you are evaluating determinants, choose the most convenient row or column, which is usually a row or column containing the most zeros.

Definition A matrix which determinant is zero is called a ***singular matrix***. In some cases it is possible to recognize when the determinant of a matrix is zero.

If A is a square matrix, then $|A|=0$, when any one of the following is true:

1. A row (column) consists entirely of zeros.

2. Two rows (columns) are identical.

3. One row (column) is a constant multiple of the second row (column).

The easiest determinants to evaluate have many zeros in a row or column. It is possible to transform a determinant into the one that has many zeros by using elementary row operations.

If A is a square matrix of order n, then the following ***elementary row operations*** produce the indicated change in the determinant of A:

a. *Interchanging* any two rows of A changes the sign of $|A|$.

b. *Multiplying* a row of A by a constant k multiplies the determinant of A by k.

c. *Adding* a multiple of a row of A to another row does not change the value of the determinant of A.

Remark The properties of determinants just stated remain true if the word "row" is replaced by "column". In that case, we would have elementary column operations.

These elementary row operations are used to rewrite a matrix in a diagonal form. A matrix is in ***a diagonal form*** if all elements below (or above) the main diagonal are zero.

Statement Let A be a square matrix of order n in diagonal form. The determinant of A is the product of the elements on the main diagonal.

Example 1 Evaluate the determinant by rewriting in diagonal form

$$\begin{vmatrix} 2 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 5 & 4 & -3 \\ 2 & 2 & 7 & -3 \end{vmatrix}.$$

Solution Rewrite the matrix in diagonal form by using elementary row operations.

$$\begin{vmatrix} 2 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 5 & 4 & -3 \\ 2 & 2 & 7 & -3 \end{vmatrix} \overset{\substack{-1R_1+R_2 \\ -2R_1+R_3 \\ -1R_1+R_4}}{=} \begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 6 & -9 \\ 0 & 1 & 8 & -6 \end{vmatrix} \overset{\substack{Factor\,3, \\ from\,row\,3}}{=} 3\begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 8 & -6 \end{vmatrix} \overset{\substack{-1R_2+R_3 \\ -1R_3+R_4}}{=} 3\begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 7 & -4 \end{vmatrix} =$$

$$\overset{-7R_3+R_4}{=} 3\begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 3\cdot(2)\cdot(1)\cdot(1)\cdot(3) = 18.$$

Remark The last example used only elementary row operations to reduce the matrix to diagonal form. Elementary column operations could also have been used, or a combination of row and column operations could have been used.

Exercise Set 2

In Exercises 1 to 10 evaluate the determinants.

1. $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ ***2.*** $\begin{vmatrix} 2 & 9 \\ -6 & 2 \end{vmatrix}$ ***3.*** $\begin{vmatrix} 5 & 0 \\ 2 & -3 \end{vmatrix}$ ***4.*** $\begin{vmatrix} 0 & -8 \\ 3 & 4 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix}$ ***6.*** $\begin{vmatrix} -3 & 6 \\ 4 & -8 \end{vmatrix}$ ***7.*** $\begin{vmatrix} 0 & 9 \\ 0 & -2 \end{vmatrix}$ ***8.*** $\begin{vmatrix} -3 & 9 \\ 0 & 0 \end{vmatrix}$

9. $\begin{vmatrix} a+1 & b-c \\ a^2+a & ab-ac \end{vmatrix}$ ***10.*** $\begin{vmatrix} \cos^2\varphi - \sin^2\varphi & 2\cos\varphi\sin\varphi \\ -2\cos\varphi\sin\varphi & \cos^2\varphi - \sin^2\varphi \end{vmatrix}$

In Exercises 11 to 14 evaluate the indicated minor and a cofactor for the determinant $\begin{vmatrix} 5 & -2 & -3 \\ 2 & 4 & -1 \\ 4 & -5 & 6 \end{vmatrix}$.

11. M_{11}, C_{11}. ***12.*** M_{21}, C_{21}. ***13.*** M_{32}, C_{32}. ***14.*** M_{33}, C_{33}.

In Exercises 15 to 19 evaluate the determinant by expanding by cofactors.

15. $\begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{vmatrix}$ ***16.*** $\begin{vmatrix} 3 & 1 & -2 \\ 2 & -5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ ***17.*** $\begin{vmatrix} -2 & 3 & 2 \\ 1 & 2 & -3 \\ -4 & -2 & 1 \end{vmatrix}$ ***18.*** $\begin{vmatrix} 3 & -2 & 0 \\ 2 & -3 & 2 \\ 8 & -2 & 5 \end{vmatrix}$ ***19.*** $\begin{vmatrix} 2 & -3 & 10 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix}$

In Exercises 20 to 33, without expanding, give a reason for each equality.

20. $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 0$ ***21.*** $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & 1 \\ -4 & 2 & -6 \end{vmatrix} = 0$ ***22.*** $\begin{vmatrix} 2 & 3 & 0 \\ 1 & -2 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 0$

23. $\begin{vmatrix} 2 & -4 & 5 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{vmatrix} = -12$ ***24.*** $\begin{vmatrix} 1 & 4 & -1 \\ 2 & 4 & 12 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$ ***25.*** $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{vmatrix} = -15$

26. $\begin{vmatrix}1&-3&4\\4&6&1\\0&-9&3\end{vmatrix}=-3\begin{vmatrix}1&1&4\\4&-2&1\\0&3&3\end{vmatrix}$ **27.** $\begin{vmatrix}1&1&-3\\2&2&5\\1&-2&4\end{vmatrix}=\begin{vmatrix}1&1&-3\\2&2&5\\0&-3&7\end{vmatrix}$ **28.** $\begin{vmatrix}1&5&-2\\2&-1&4\\3&0&-2\end{vmatrix}=\begin{vmatrix}1&5&-2\\0&-11&8\\3&0&-2\end{vmatrix}$

29. $\begin{vmatrix}6&0&-2\\2&-1&-3\\1&5&-7\end{vmatrix}=-\begin{vmatrix}0&6&-2\\-1&2&-3\\5&1&-7\end{vmatrix}$ **30.** $\begin{vmatrix}3&5&-2\\2&1&0\\9&-2&-3\end{vmatrix}=-\begin{vmatrix}9&-2&-3\\2&1&0\\3&5&-2\end{vmatrix}$ **31.** $a^3\begin{vmatrix}1&1&1\\a&a&a\\a^2&a^2&a^2\end{vmatrix}=\begin{vmatrix}a&a&a\\a^2&a^2&a^2\\a^3&a^3&a^3\end{vmatrix}$

32. $\begin{vmatrix}4&-3&2\\6&2&1\\-2&2&4\end{vmatrix}=2\begin{vmatrix}2&-3&2\\3&2&1\\-1&2&4\end{vmatrix}$ **33.** $\begin{vmatrix}1&1&1\\2&2&2\\3&3&3\end{vmatrix}=0$

In Exercises 34 to 42 evaluate the determinant by rewriting the determinant in diagonal form by using elementary row or column operations.

34. $\begin{vmatrix}2&4&1\\1&2&-1\\1&2&2\end{vmatrix}$ **35.** $\begin{vmatrix}3&-2&-1\\1&2&4\\2&-2&3\end{vmatrix}$ **36.** $\begin{vmatrix}1&2&-1\\2&3&1\\3&4&3\end{vmatrix}$

37. $\begin{vmatrix}1&2&5\\-1&1&-2\\3&1&10\end{vmatrix}$ **38.** $\begin{vmatrix}0&-1&1\\1&0&-2\\2&2&0\end{vmatrix}$ **39.** $\begin{vmatrix}2&-1&3\\1&1&1\\3&-4&5\end{vmatrix}$

40. $\begin{vmatrix}1&-2&5&9\\1&-1&7&4\\1&3&3&4\\1&2&3&4\end{vmatrix}$ **41.** $\begin{vmatrix}1&2&3&4\\0&2&5&9\\0&0&3&7\\-2&-4&-6&0\end{vmatrix}$ **42.** $\begin{vmatrix}1&2&-1&2\\1&-2&0&3\\3&0&1&5\\-2&-4&1&6\end{vmatrix}$

In Exercises 43 to 46 evaluate the determinants by using elementary row or column operations.

43. $\begin{vmatrix}2&6&4\\1&2&1\\3&8&6\end{vmatrix}$ **44.** $\begin{vmatrix}3&0&10\\3&-2&7\\2&-1&5\end{vmatrix}$ **45.** $\begin{vmatrix}4&9&-11\\2&6&-3\\3&7&-8\end{vmatrix}$ **46.** $\begin{vmatrix}1&-1&2&1\\2&-1&6&3\\3&-1&8&7\\3&0&9&9\end{vmatrix}$

47. Solve the inequality.

(a) $\begin{vmatrix} 3 & -2 & 1 \\ 1 & x & -2 \\ -1 & 2 & -1 \end{vmatrix} < 1;$ (b) $\begin{vmatrix} 2 & x+2 & -1 \\ 1 & 1 & -2 \\ 5 & -3 & x \end{vmatrix} > 0$

48. Solve the equality.

(a) $\begin{vmatrix} x^2-4 & -1 \\ x-2 & x+2 \end{vmatrix} = 0;$ (b) $\begin{vmatrix} 2 & -1 & 2 \\ 3 & 5 & 3 \\ 1 & 6 & x+5 \end{vmatrix} = 0$

In Exercises 49 to 54 evaluate the determinants

49. $\begin{vmatrix} 12\,314 & 16\,536 & 20\,537 \\ 6157 & 8268 & 10\,268 \\ 513 & 689 & 126 \end{vmatrix}$ ***50.*** $\begin{vmatrix} 15325 & 15323 & 37527 \\ 23735 & 23735 & 17417 \\ 23737 & 23737 & 17418 \end{vmatrix}$ ***51.*** $\begin{vmatrix} 2 & 4 & -1 & 2 \\ -1 & 2 & 3 & 1 \\ 2 & 5 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{vmatrix}$

52. $\begin{vmatrix} 7 & 8 & 5 & 5 & 3 \\ 10 & 11 & 6 & 7 & 5 \\ 5 & 3 & 6 & 2 & 4 \\ 6 & 7 & 5 & 4 & 2 \\ 7 & 10 & 7 & 5 & 0 \end{vmatrix}$ ***53.*** $\begin{vmatrix} 7 & 3 & 2 & 6 \\ 8 & -9 & 4 & 9 \\ 7 & -2 & 7 & 3 \\ 5 & -3 & 3 & 4 \end{vmatrix}$ ***54.*** $\begin{vmatrix} 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & -4 \\ 1 & 1 & 5 & 1 \end{vmatrix}$

Individual Tasks 2

1. Evaluate the determinant by expanding by cofactors.

2. Evaluate the determinants by using elementary row or column operations.

3. Solve inequality

I. 1. $\begin{vmatrix} 0 & -2 & 4 \\ 1 & 0 & -7 \\ 5 & -6 & 0 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 4 & 12 \\ 1 & -1 & 0 & 8 \end{vmatrix}$ 3. $\begin{vmatrix} 2 & 6 & 4 \\ 1 & x+1 & 1 \\ 3 & 8 & 6 \end{vmatrix} < 0$	**II.** 1. $\begin{vmatrix} 5 & -8 & 0 \\ 2 & 0 & -7 \\ 0 & -2 & -1 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 6 & 5 & 9 & 8 \\ 2 & 4 & 12 & -1 \\ 1 & 2 & 6 & -1 \end{vmatrix}$ 3. $\begin{vmatrix} 1 & 2 & 5 \\ x-1 & 1 & -2 \\ 3 & 1 & x \end{vmatrix} > 0$
III.	**IV.**

1. $\begin{vmatrix} 4 & -3 & 3 \\ 2 & 1 & -4 \\ 6 & -2 & -1 \end{vmatrix}$	1. $\begin{vmatrix} -2 & 3 & 9 \\ 4 & -2 & -6 \\ 0 & -8 & -24 \end{vmatrix}$
2. $\begin{vmatrix} 1 & 1 & -2 & 0 \\ 3 & 6 & -2 & 5 \\ 1 & 0 & 6 & 4 \\ 2 & 3 & 5 & -1 \end{vmatrix}$	2. $\begin{vmatrix} 2 & 0 & -1 & 3 \\ 6 & 3 & -9 & 0 \\ 0 & 2 & -1 & 3 \\ 4 & 2 & 0 & 6 \end{vmatrix}$
3. $\begin{vmatrix} 1 & 2 & 4 \\ 1 & x-2 & 1 \\ 3 & 4 & 6 \end{vmatrix} \le 0$	3. $\begin{vmatrix} 1 & x & 5 \\ x-1 & 1 & -2 \\ 3 & 0 & 2 \end{vmatrix} \ge 0$

1.3 The Inverse of a Matrix

Definition *(Multiplicative Inverse of a Matrix)* If A is a square matrix of order n, then the inverse of matrix A, denoted by A^{-1}, has the property that

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n,$$

where I_n is the identity matrix of order n.

Definition The square matrix A is called ***nonsingular matrix*** if $\det A \neq 0$. If $\det A = 0$, then matrix A is called ***singular matrix.***

Theorem *(Existence of the Inverse of a Square Matrix)* If A is a square matrix of order n, then A has a multiplicative inverse if and only if $|A| \neq 0$.

Statement Inverse matrix A^{-1} is found by formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix},$$

where C_{ij} is the cofactor of a_{ij}.

You should verify that this matrix satisfies the condition of an inverse matrix. That is, show that $A^{-1} \cdot A = A \cdot A^{-1} = I_3$.

Properties of the Inverse Matrix

1. $\det A^{-1} = \dfrac{1}{\det A}$. ***2***. $\left(A^{-1}\right)^{-1} = A$ ***3***. $\left(AB\right)^{-1} = B^{-1}A^{-1}$

Exercise Set 3

In Exercises 1 to 12 find the inverse of the given matrix.

1. $\begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$ ***2.*** $\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$ ***3.*** $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -2 \\ -1 & -5 & 6 \\ 2 & 6 & -3 \end{bmatrix}$ ***5.*** $\begin{bmatrix} 1 & 4 \\ 2 & 10 \end{bmatrix}$ ***6.*** $\begin{bmatrix} -2 & 3 \\ -6 & -8 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 1 \\ 3 & 6 & -4 \end{bmatrix}$ ***8.*** $\begin{bmatrix} 2 & 1 & -1 \\ 6 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$ ***9.*** $\begin{bmatrix} 2 & 4 & -4 \\ 1 & 3 & -4 \\ 2 & 4 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 3 & -6 & 6 \end{bmatrix}$ ***11.*** $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 5 & 1 \\ 3 & -3 & 7 & 5 \\ -2 & 3 & -4 & -1 \end{bmatrix}$ ***12.*** $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & -1 & 5 \\ 2 & 2 & -1 & 5 \\ 4 & 4 & -4 & 7 \end{bmatrix}$

In Exercises 13 to 18 find the inverse, if it exists, of the given matrix.

13. $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$ ***14.*** $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ ***15.*** $\begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 7 \\ 2 & -3 & 6 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ 3 & 8 & 6 \end{bmatrix}$ ***17.*** $\begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 7 & -3 & 1 \\ 2 & 7 & 4 & 3 \\ 1 & 4 & 2 & 4 \end{bmatrix}$ ***18.*** $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 6 & 5 \\ 3 & -1 & 9 & 6 \\ 2 & 2 & 4 & 7 \end{bmatrix}$

In Exercises 19 to 20 solve the matrix equation:

19. $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \cdot X = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$

20. $XA - 2B = I$, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 9 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & -1 & -4 \end{bmatrix}$

Individual Tasks 3

1. Find the inverse of the given matrix.

2. For which value $r \in R$ the given matrix is nonsingular? Find A^{-1} for every such value

r.

3. Solve the matrix equation.

4. Whether two matrices are inverse matrix?

I.	**II.**
1. $\begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & -2 & 7 \\ 2 & -1 & r \\ 3 & 0 & 10 \end{bmatrix}$ 3. $\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \cdot X = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$ 4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A^{-1} = \frac{1}{3}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	1. $\begin{bmatrix} -2 & 3 \\ 2 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} r & 9 & -11 \\ 3 & 7 & -8 \\ 2 & 6 & -3 \end{bmatrix}$ 3. $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \cdot X = \begin{bmatrix} 5 & 4 \\ -2 & 0 \end{bmatrix}$ 4. $A = \begin{bmatrix} 0 & -2 & 1 \\ -4 & 5 & -2 \\ 5 & -3 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 8 \end{bmatrix}$
III.	**IV.**
1. $\begin{bmatrix} -1 & -4 \\ 7 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 1 & -5 \\ 1 & 2 & r \\ 3 & 2 & -1 \end{bmatrix}$ 3. $X \cdot \begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$ 4. $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 5 \\ 2 & 2 & 2 \end{bmatrix}, A^{-1} = \frac{1}{8}\begin{bmatrix} -6 & -4 & 13 \\ 4 & 0 & -2 \\ 2 & 4 & -7 \end{bmatrix}$	1. $\begin{bmatrix} 7 & -4 \\ 5 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 5 & -2 \\ 1 & -3 & 2 \\ r & 7 & -3 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \cdot X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ 4. $A = \begin{bmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{bmatrix}, A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 \\ -38 & 41 & 34 \\ 27 & 29 & -24 \end{bmatrix}$

1.4 System of Linear Equations

Recall that an equation of the form $Ax + By = C$ is a linear equation in two variables. A solution of a linear equation in two variables is an ordered pair $(x; y)$, which makes the equation a true statement. The graph of a linear equation, a straight line, is the set of points whose ordered pairs satisfy the equation.

A system of equations is two or more equations considered together. The solution of a system of equations is an ordered pair that is the solution of each equation.

Definition The system of equations is called a ***consistent*** system of equations when the system has at least one solution.

The system of equations is called a ***dependent*** system of equations when the system has an infinite set of solutions.

Definition The system of equations is called an ***inconsistent*** system of equations when the system has no solution.

1.4.1 Gauss-Jordan Elimination Method

A matrix can be created from a system of linear equations. There are three matrixes, which can describe a system of linear equations.

The matrix formed by the coefficients and constants of this system is called the ***augmented matrix*** of system of equations. The matrix formed by the coefficients of the system is the ***coefficient matrix***. The matrix formed from the constants is the ***constant matrix*** for the system.

Remark When a term is a missing one of the equations of a system (as in the second equation), the coefficient of that term is 0 and 0 is entered in the matrix.

An augmented matrix is in the ***echelon form*** if all the following conditions are satisfied:

1. The first nonzero number in any row is 1.

2. Rows are arranged so that the column containing the first nonzero number is to the left of the column containing the first nonzero number of the next row.

3. All rows consisting entirely of zeros appear at the bottom of the matrix.

We can write an augmented matrix in the echelon form by using the so-called ***elementary row operations***. These operations are a rewording, using matrix terminology of the operations that produce equivalent equations.

Given the augmented matrix for a system of linear equations, each of the following elementary row operations produces a matrix of an equivalent system of equations. ERO are:

1. Interchanging two rows (Interchange the i-th and j-th rows: $R_i \leftrightarrow R_j$).

2. Multiplying all the elements in a row by the same nonzero number (Multiply the i-th row by k, a nonzero constant: kR_i).

3. Replacing a row by the sum of that row and a nonzero multiple of any other row (Replace the j-th row by the sum of that row a nonzero multiple of the i-th row: $kR_i + R_j$).

The ***Gauss-Jordan elimination method*** is an algorithm that uses elementary row operations to solve a system of linear equations. The goal of this method is to rewrite an augmented matrix in the echelon form.

To conserve space, we will occasionally perform more than one elementary row operation in one step. For example, the notation $\xrightarrow[-2R_1+R_3]{3R_1+R_2}$ means that two elementary row operations were performed. First, multiply row 1 by 3 and add to row 2. Replace row 2. Now multiply row 1 by –2 and add to row 3. Replace row 3.

Example 1 Solve the system of equations using the Gauss-Jordan method

$$\begin{cases} x-3y+z=5 \\ 3x-7y+2z=12 \\ 2x-4y+z=3 \end{cases}.$$

Solution Write the augmented matrix and then use elementary row operations to write the matrix in the echelon form.

$$A/B=\left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 3 & -7 & 2 & 12 \\ 2 & -4 & 1 & 3 \end{array}\right] \xrightarrow[-2R_1+R_3]{3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -3 \\ 0 & 2 & -1 & -7 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 2 & -1 & -7 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 0 & -4 \end{array}\right].$$

The equivalent system is $\begin{cases} x-3y+z=5 \\ y-\dfrac{1}{2}z=-\dfrac{3}{2} \\ 0z=-4 \end{cases}.$

Because the equation 0z = -4 has no solution, the system also has no solution.

Example 2 Solve a system of equations using the Gauss-Jordan method

$$\begin{cases} x_1-2x_2-3x_3-2x_4=1 \\ 2x_1-3x_2-4x_3-2x_4=3 \\ x_1+x_2+x_3-7x_4=-7 \end{cases}.$$

Solution Write the augmented matrix and use elementary row operations to reduce the matrix to the echelon form.

$$A/B=\left[\begin{array}{cccc|c} 1 & -2 & -3 & -2 & 1 \\ 2 & -3 & -4 & -2 & 3 \\ 1 & 1 & 1 & -7 & -7 \end{array}\right] \xrightarrow[-1R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 3 & 4 & -5 & -8 \end{array}\right] \xrightarrow{-3R_2-\frac{1}{2}R_3}$$

$$\xrightarrow{-3R_2-\frac{1}{2}R_3}\left[\begin{array}{cccc|c} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & -11/2 & -11/2 \end{array}\right].$$

The equivalent system is $\begin{cases} x_1 - 2x_2 - 3x_3 - 2x_4 = 1 \\ x_2 + 2x_3 + 2x_4 = 1 \\ x_3 + \dfrac{11}{2}x_4 = \dfrac{11}{2} \end{cases}$.

We now express each of the variables in terms of x_4. Solve the third equation for x_3.

$$x_3 = -\frac{11}{2}x_4 + \frac{11}{2}$$

Substitute this value into the second equation and solve for x_2.

$$x_2 + 2\left(-\frac{11}{2}x_4 + \frac{11}{2}\right) + 2x_4 = 1$$

Simplifying, we have $x_2 = 9x_4 - 10$. Substitute the values for x_2 and x_3 into the first equation and solve for x_1:

$$x_1 - 2(9x_4 - 10) - 3\left(-\frac{11}{2}x_4 + \frac{11}{2}\right) - 2x_4 = 1.$$

Simplifying, we have $x_1 = \frac{7}{2}x_4 - \frac{5}{2}$. If x_4 is any real number c, the solution is of the form $\left(\frac{7}{2}c - \frac{5}{2}, 9c - 10, -\frac{11}{2}c + \frac{11}{2}, c\right)$.

1.4.2 Method of Inverse Matrix

Systems of linear equations can be solved by finding the inverse of the coefficient matrix. Consider the system of linear equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \qquad (1)$$

Using matrix multiplication and the concept of equality of matrices, this system can be written as a matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then the equation (1) can be written as following

$$AX = B \tag{3}$$

The inverse of the coefficient matrix A is A^{-1}. To solve the system of equations, multiply each side of the equation (3) by the inverse A^{-1}.

$$A^{-1}AX = A^{-1}B \overset{A^{-1}A=I}{\Leftrightarrow} X = A^{-1}B \tag{4}$$

Thus, we can find the solution of the system (1) using the formula (4).

Example 3 Find the solution of the system of equations

$$\begin{cases} x_1 + \quad\quad 7x_3 = 20 \\ 2x_1 + x_2 - x_3 = -3 \\ 7x_1 + 3x_2 + x_3 = 2 \end{cases}$$

by using the inverse of the coefficient matrix.

Solution Write the system as a matrix equation

$$\begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & -1 \\ 7 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -3 \\ 2 \end{bmatrix}.$$

The inverse of the coefficient matrix is $A^{-1} = \begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix}$.

Multiplying each side of the matrix equation by the inverse, we have

$$\begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & -1 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 20 \\ -3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Thus $x_1 = -1$, $x_2 = 2$ and $x_3 = 3$. The answer is $(-1;2;3)$.

1.4.3 Cramer's Rule

Let $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ be the system of equations for which the determinant of the coefficient matrix is not zero. The solution of the system of equations is the ordered pair which coordinates are

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \text{ and } x_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Note that the denominator Δ - is the determinant of the coefficient matrix of the variables. The denominator Δ_1 is formed by replacing column 1 of the coefficient determinant with the constants b_1 and b_2. The determinant Δ_2 is formed by replacing column 2 of the coefficient determinant by the constants b_1 and b_2.

Example 4 Solve the following system of equations using Cramer′s Rule:

$$\begin{cases} 5x_1 - 3x_2 = 6 \\ 2x_1 + 4x_2 = -7 \end{cases}.$$

Solution

$$x_1 = \frac{\begin{vmatrix} 6 & -3 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix}} = \frac{3}{26}, \quad x_2 = \frac{\begin{vmatrix} 5 & 6 \\ 2 & -7 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix}} = -\frac{47}{26}.$$

The answer is $\left(\frac{3}{26}, -\frac{47}{26}\right)$.

Cramer′s Rule can be extended to a system of n linear equations in n variables.

Let

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n \end{cases}$$

be a system of n equations in n variables. The solution of the system is given by $(x_1, x_2, x_3, \ldots, x_n)$ where

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ x_3 = \frac{\Delta_3}{\Delta}, \ldots,\ x_n = \frac{\Delta_n}{\Delta}$$

where Δ is *the determinant of the coefficient matrix* and $\Delta \neq 0$. Δ_i is *the determinant formed by replacing the i-th column of the coefficient matrix with the column of constants* $b_1, b_2, b_3, \ldots, b_n$.

Because the determinant of the coefficient matrix must be nonzero to use Cramer′s Rule, this method is not appropriate for the systems of linear equations with no solution or infinitely many solutions. In fact, the only time a system of linear equations will have a unique solution is when the coefficient determinant is not zero, the fact summarized in the following theorem.

Theorem ***(Systems of Linear Equations with Unique Solutions)*** A system of n linear equations in n variables has a unique solution if and only if the determinant of the coefficient matrix is not zero.

Cramer′s Rule is also useful when we want to determine only the value of a single variable in a system of equations.

Example 5 Find x_3 for the system of equations $\begin{cases} 4x_1 + 3x_3 - 2x_4 = 2 \\ 3x_1 + x_2 + 2x_3 - x_4 = 4 \\ x_1 - 6x_2 - 2x_3 + 2x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = -1 \end{cases}$.

Solution Find Δ and Δ_3: $\Delta = \begin{vmatrix} 4 & 0 & 3 & -2 \\ 3 & 1 & 2 & -1 \\ 1 & -6 & -2 & 2 \\ 2 & 2 & 0 & -1 \end{vmatrix} = 39$, $\Delta_3 = \begin{vmatrix} 4 & 0 & 2 & -2 \\ 3 & 1 & 4 & -1 \\ 1 & -6 & 0 & 2 \\ 2 & 2 & -1 & -1 \end{vmatrix} = 96$.

Thus, $x_3 = \dfrac{96}{39} = \dfrac{32}{13}$.

Exercise Set 4

In Exercises 1 to 4 write the augmented matrix, the coefficient matrix, and the constant matrix.

1. $\begin{cases} 2x-3y+z=1 \\ 3x-2y+3z=0 \\ x \quad + \quad 5z \quad =4 \end{cases}$ **2.** $\begin{cases} -3y+2z=3 \\ 2x-y \qquad =-1 \\ 3x-2y+3z=4 \end{cases}$

3. $\begin{cases} 2x-3y-4z+\omega=2 \\ \qquad 2y+z \qquad =2 \\ x-y+2z \qquad =4 \\ 3x-3y-2z \qquad =1 \end{cases}$ **4.** $\begin{cases} x-y+2z+3\omega=-2 \\ 2x \qquad +z-2\omega=1 \\ 3x \qquad\quad -2\omega=3 \\ -x+3y-z \qquad =3 \end{cases}$

In Exercises 5 to 10 use elementary row operations to write the matrix in the echelon form.

5. $\begin{bmatrix} 2 & -1 & 3 & -2 \\ 1 & -1 & 2 & 2 \\ 3 & 2 & -1 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 7 & 3 \\ 3 & 6 & 8 & -1 \end{bmatrix}$ **7.** $\begin{bmatrix} 4 & -5 & -1 & 2 \\ 3 & -4 & 1 & -2 \\ 1 & -2 & -1 & 3 \end{bmatrix}$

8. $\begin{bmatrix} -2 & 1 & -1 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 1 & -1 & 2 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 3 & -6 & 10 & -14 \\ 5 & -8 & 19 & -21 \\ 2 & -4 & 7 & -10 \end{bmatrix}$ **10.** $\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & 5 & -2 & 2 \\ 4 & 3 & 1 & 8 \end{bmatrix}$

In Exercises 11 to 31 solve the system of equations by the Gauss-Jordan method.

11. $\begin{cases} x+2y-2z=-2 \\ 5x+9y-4z=-3 \\ 3x+4y-5z=-3 \end{cases}$ **12.** $\begin{cases} x-3y+z=8 \\ 2x-5y-3z=2 \\ x+4y+z=1 \end{cases}$ **13.** $\begin{cases} 3x+7y-7z=-4 \\ x+2y-3z=0 \\ 5x+6y+z=-8 \end{cases}$

14. $\begin{cases} 2x-3y+2z=13 \\ 3x-4y-3z=1 \\ 3x+y-z=2 \end{cases}$ **15.** $\begin{cases} x+2y-2z=3 \\ 5x+8y-6z=14 \\ 3x+4y-2z=8 \end{cases}$ **16.** $\begin{cases} 3x-5y+2z=4 \\ x-3y+2z=4 \\ 5x-11y+6z=12 \end{cases}$

17. $\begin{cases} 3x+2y-z=1 \\ 2x+3y-z=1 \\ x-y+2z=3 \end{cases}$ **18.** $\begin{cases} 2x+5y+2z=-1 \\ x+2y-3z=5 \\ 5x+12y+z=10 \end{cases}$ **19.** $\begin{cases} x-3y+2z=0 \\ 2x-5y-2z=0 \\ 4x-11y+2z=0 \end{cases}$

20. $\begin{cases} x+y-2z=0 \\ 3x+4y-z=0 \\ 5x+6y-5z=0 \end{cases}$ **21.** $\begin{cases} 2x+y-3z=4 \\ 3x+2y+z=2 \end{cases}$ **22.** $\begin{cases} 3x-6y+2z=2 \\ 2x+5y-3z=2 \end{cases}$

23. $\begin{cases} x-4y+3z=4 \\ 3x-10y+3z=4 \\ 5x-18y+9z=10 \\ 2x+2y-3z=-11 \end{cases}$

24. $\begin{cases} t+2a-3v+w=-7 \\ 3t+5u-8v+5w=-8 \\ 2t+3a-7v+3w=-11 \\ 4t+8a-10v+7w=-10 \end{cases}$

25. $\begin{cases} t+4u+2v-3w=11 \\ 2t+10u+3v-5w=17 \\ 4t+16u+7v-9w=34 \\ t+4u+v-w=4 \end{cases}$

26. $\begin{cases} 2t-u+3v+2w=2 \\ t-u+2v+w=2 \\ 3t \quad -2v-3w=13 \\ 2t+2u \quad -2w=6 \end{cases}$

27. $\begin{cases} 4t+7u-10v+3w=-29 \\ 3t+5u-7v+2w=-20 \\ t+2u-3v+w=-9 \\ 2t-u+2v-4w=15 \end{cases}$

28. $\begin{cases} 3t+10u+7v-6w=7 \\ 2t+8u+6v-5w=-5 \\ t+4u+2v-3w=2 \\ 4t+14u+9v-8w=8 \end{cases}$

29. $\begin{cases} 3t-4u+v \quad =2 \\ t+u-2v+3w=1 \end{cases}$

30. $\begin{cases} t-3u+2v+4w=13 \\ 3t-8u+4v+13w=35 \\ 2t-7u+8v+5w=28 \\ 4t-11u+6v+17w=56 \end{cases}$

31. $\begin{cases} 2t \quad +3v-4w=2 \\ t+2u-4v+w=-3 \end{cases}$

In Exercises 32 to 37 solve the system of equations by using inverse matrix methods.

32. $\begin{cases} x+4y=6 \\ 2x+7y=11 \end{cases}$

33. $\begin{cases} 2x+3y=5 \\ x+2y=4 \end{cases}$

34. $\begin{cases} x-2y=8 \\ 3x+2y=-1 \end{cases}$

35. $\begin{cases} x+2y-z=5 \\ 2x+3y-z=8 \\ 3x+6y-2z=14 \end{cases}$

36. $\begin{cases} x+y+2z=4 \\ 2x+3y+3z=5 \\ 3x+3y+7z=14 \end{cases}$

37. $\begin{cases} w+2x \quad +z=6 \\ 2w+5x+y+2z=10 \\ 2w+4x+y+4z=8 \\ 3w+6x \quad +z=16 \end{cases}$

In Exercises 38 to 52 solve each system of equations by using Cramer′s Rule.

38. $\begin{cases} 3x_1+4x_2=8 \\ 4x_1-5x_2=1 \end{cases}$

39. $\begin{cases} x_1-3x_2=9 \\ 2x_1-4x_2=-3 \end{cases}$

40. $\begin{cases} 5x_1+4x_2=-1 \\ 3x_1-6x_2=5 \end{cases}$

41. $\begin{cases} 2x_1+5x_2=9 \\ 5x_1+7x_2=8 \end{cases}$

42. $\begin{cases} 7x_1+2x_2=0 \\ 2x_1+x_2=-3 \end{cases}$

43. $\begin{cases} 3x_1-8x_2=1 \\ 4x_1+5x_2=-2 \end{cases}$

44. $\begin{cases} 3x_1-7x_2=0 \\ 2x_1+4x_2=0 \end{cases}$

45. $\begin{cases} 5x_1+4x_2=-3 \\ 2x_1-4x_2=0 \end{cases}$

46. $\begin{cases} 1{,}2x_1+0{,}3x_2=2{,}1 \\ 0{,}8x_1-1{,}4x_2=-1{,}6 \end{cases}$

47. $\begin{cases} 3{,}2x_1-4{,}2x_2=1{,}1 \\ 0{,}7x_1+3{,}2x_2=-3{,}4 \end{cases}$

48. $\begin{cases} 3x_1-4x_2+2x_3=1 \\ x_1-x_2+2x_3=-2 \\ 2x_1+2x_2+3x_3=-3 \end{cases}$

49. $\begin{cases} 5x_1-2x_2+3x_3=-2 \\ 3x_1+x_2-2x_3=3 \\ x_1-2x_2+3x_3=-1 \end{cases}$

50. $\begin{cases} x_1+4x_2-2x_3=0 \\ 3x_1-2x_2+3x_3=4 \\ 2x_1+x_2-3x_3=-1 \end{cases}$

51. $\begin{cases} 4x_1-x_2+2x_3=6 \\ x_1+3x_2-x_3=-1 \\ 2x_1+3x_2-2x_3=5 \end{cases}$

52. $\begin{cases} \quad 2x_2-3x_3=1 \\ 3x_1-5x_2+x_3=0 \\ 4x_1 \quad +2x_3=-3 \end{cases}$

In Exercises 53 to 56 solve for the indicated variable.

53. $\begin{cases} 2x_1-3x_2+4x_3-x_4=1 \\ x_1+2x_2+2x_4=-1 \\ 3x_1+x_2-2x_4=2 \\ x_1-3x_2+2x_3-x_4=3 \end{cases}$

Solve for x_2.

54. $\begin{cases} 3x_1+x_2+3x_4=4 \\ 2x_1-3x_2=-2 \\ x_1+x_2+2x_4=3 \\ 2x_1+3x_3-2x_4=4 \end{cases}$

Solve for x_4.

55. $\begin{cases} x_1-3x_2+2x_3+4x_4=0 \\ 3x_1+5x_2-6x_3+2x_4=-2 \\ 2x_1-x_2+9x_3+8x_4=0 \\ x_1+x_2+x_3-8x_4=-3 \end{cases}$

Solve for x_1.

56. $\begin{cases} 2x_1+5x_2-5x_3-3x_4=-3 \\ x_1+7x_2+8x_3-x_4=4 \\ 4x_1+x_3+x_4=3 \\ 3x_1+2x_2-x_3=0 \end{cases}$

Solve for x_3.

Individual Tasks 4

1. Use elementary row operations to write the matrix in the echelon form, find the rang of the matrix.

2. Solve the system of equations by the Gauss-Jordan method.

3. Solve the system of equations by using inverse matrix methods and by using Cramer′s Rule.

I.	**II.**
1. $\begin{bmatrix} 1 & -3 & 4 & 2 & 1 \\ 2 & -3 & 5 & -2 & -1 \\ -1 & 2 & -3 & 1 & 3 \end{bmatrix}$	1. $\begin{bmatrix} 2 & -1 & 3 & 2 & 2 \\ 1 & -2 & 2 & 1 & -1 \\ 3 & -5 & -1 & -2 & 3 \end{bmatrix}$
2. $\begin{cases} 2x+2y-4z=4 \\ 2x+3y-5z=4 \\ 4x+5y-9z=8 \end{cases}$;	2. $\begin{cases} 3x-10y+2z=34 \\ x-4y+z=13 \\ 5x-2y+7z=31 \end{cases}$;
3. $\begin{cases} x+2y+2z=5 \\ -2x-5y-2z=8 \\ 2x+4y+7z=19 \end{cases}$	3. $\begin{cases} x-y+3z=5 \\ 3x-y+10z=16 \\ 2x-2y+5z=9 \end{cases}$

III.	**IV.**
1. $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & -1 & 4 & -2 \\ 5 & 3 & 10 & 8 \end{bmatrix}$	1. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 0 & 1 \\ 2 & 4 & 1 & 8 \\ 0 & 10 & 1 & 10 \\ 1 & 7 & 6 & 9 \end{bmatrix}$
2. $\begin{cases} t-u+3v-5w=10 \\ 2t-3u+4v+w=7 \\ 3t+u-2v-2w=6 \end{cases}$	2. $\begin{cases} t-u+2v-3w=9 \\ 4t \quad +11v-10w=46 \\ 3t-u+8v-6w=27 \end{cases}$
3. $\begin{cases} x-2y+z=0 \\ -2x-y-2z=-15 \\ 2x+4y+z=21 \end{cases}$	3. $\begin{cases} x-y+z=2 \\ 3x-y+z=6 \\ 2x-y+5z=12 \end{cases}$

1.5 Eigenvalues and eigenvectors of a matrix

Consider a square matrix A_{nn} and a vector-column $X_{n\times 1} \neq \vec{0}$.

Definition The vector $X_{n\times 1}$ is called its ***eigenvectors*** of the matrix $A_{n\times n}$, if there is such a real number $\lambda \neq 0$, which satisfies the equality

$$AX = \lambda X \qquad (1)$$

Number λ is called the ***eigenvalues*** of a matrix. For finding the eigenvalues of the matrix the following characteristic equation must be solved

$$|A-\lambda E| = 0 \qquad (2)$$

The substituting values found in equation (1) are eigenvectors of the matrix $A_{n\times n}$.

Example 1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 8 & 5 & 3 \\ 0 & 2 & -6 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution

1. Form a square matrix $A-\lambda E$:

$$A-\lambda E=\begin{pmatrix}8 & 5 & 3\\ 0 & 2 & -6\\ 0 & -1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}8-\lambda & 5 & 3\\ 0 & 2-\lambda & -6\\ 0 & -1 & 1-\lambda\end{pmatrix}.$$

2. Compose the characteristic equation

$$\begin{vmatrix}8-\lambda & 5 & 3\\ 0 & 2-\lambda & -6\\ 0 & -1 & 1-\lambda\end{vmatrix}=(8-\lambda)\cdot\left((2-\lambda)\cdot(1-\lambda)-6\right)=(8-\lambda)\cdot\left(\lambda^2-3\lambda-4\right)=0.$$

The solutions of getting equation are the eigenvalues of the matrix .In this example the eigenvalues of the matrix are $\lambda_1=8,\ \lambda_2=-1,\ \lambda_3=4$.

3. For each of the eigenvalues we will find the eigenvectors of a matrix A.

a) If $\lambda=8$, then $A-\lambda E=\begin{pmatrix}8-8 & 5 & 3\\ 0 & 2-8 & -6\\ 0 & -1 & 1-8\end{pmatrix}=\begin{pmatrix}0 & 5 & 3\\ 0 & -6 & -6\\ 0 & -1 & -7\end{pmatrix}$

and a matrix equation is $\begin{pmatrix}0 & 5 & 3\\ 0 & -6 & -6\\ 0 & -1 & -7\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.

This equation corresponds to a homogeneous system of linear equations

$$\begin{cases}5x_2+3x_3=0\\ -6x_2-6x_3=0\\ -x_2-7x_3=0\end{cases}$$

From the second equation $x_2=-x_3$, then

$$\begin{cases}-5x_3+3x_3=0;\\ x_3-7x_3=0;\end{cases}\Rightarrow\begin{cases}-2x_3=0;\\ -6x_3=0;\end{cases}\Rightarrow\ x_3=0,\ x_2=0,\ x_1=m,\ m\in R,\ m\neq 0$$

and vector $X_1=\begin{pmatrix}m\\ 0\\ 0\end{pmatrix}$, $m\in R,\ m\neq 0$ is the eigenvector of a matrix A.

b) If $\lambda=-1$, then a homogeneous system of linear equations is

$$\begin{cases}9x_1+5x_2+3x_3=0;\\ \quad 3x_2-6x_3=0;\\ \quad -x_2+2x_3=0;\end{cases}\Rightarrow\begin{cases}9x_1+5x_2+3x_3=0;\\ \quad -x_2+2x_3=0;\end{cases}\Rightarrow\begin{cases}x_1=-\dfrac{13}{9}x_3;\\ x_2=2x_3;\end{cases}$$

$$\Rightarrow\ x_3=9k,\ x_2=18k,\ x_1=-13k,\ k\in R,\ k\neq 0.$$

and vector $X_2 = \begin{pmatrix} -13k \\ 18k \\ 9k \end{pmatrix}, k \in R, k \neq 0$ is the eigenvector of a matrix A.

c) If $\lambda = 4$, then a homogeneous system of linear equations is

$$\begin{cases} 4x_1 + 5x_2 + 3x_3 = 0; \\ \quad -2x_2 - 6x_3 = 0; \\ \quad -x_2 - 3x_3 = 0; \end{cases} \Rightarrow \begin{cases} 4x_1 + 5x_2 + 3x_3 = 0; \\ \quad -x_2 - 3x_3 = 0; \end{cases} \Rightarrow \begin{cases} x_1 = 3x_3; \\ x_2 = -3x_3; \end{cases}$$

$$\Rightarrow \; x_3 = t,\; x_2 = -3t,\; x_1 = 3t,\; t \in R,\; t \neq 0.$$

and vector $X_3 = \begin{pmatrix} 3t \\ -3t \\ t \end{pmatrix}, t \in R, t \neq 0$ is the eigenvector of a matrix A.

Exercise Set 5

In exercises 1 to 8 find the eigenvalues and eigenvectors of a matrix.

1. $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$ ***2.*** $A = \begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}$ ***3.*** $A = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 3 \\ 3 & 8 \end{pmatrix}$ ***5.*** $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ ***6.*** $A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$

7. $A = \begin{pmatrix} 11 & -6 & 2 \\ -6 & 10 & -4 \\ 2 & -4 & 6 \end{pmatrix}$ ***8.*** $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

Individual Tasks 5

1-2. Find the eigenvalues and eigenvectors of a matrix.
3. Find the eigenvectors of A matrix among the vectors X_1, X_2, X_3.

I.	**II.**
1. $A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$	1. $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$	2. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$

3. $A=\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $X_1=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, X_2=\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, X_3=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.	3. $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$, $X_1=\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, X_2=\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, X_3=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
III. 1. $A=\begin{pmatrix} 5 & 1 \\ 1 & 8 \end{pmatrix}$ 2. $A=\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$ 3. $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$, $X_1=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, X_2=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, X_3=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.	**IV.** 1. $A=\begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}$ 2. $A=\begin{pmatrix} 5 & 6 & 3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ 3. $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$, $X_1=\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, X_2=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, X_3=\begin{pmatrix} -2 \\ 0 \\ 10 \end{pmatrix}$.

II ANALYTIC GEOMETRY

2.1 Algebraic Operations on Vectors

2.1.1 The Algebra of Vectors

The term ***vector*** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

Definition A ***vector*** is a directed line segment $\overrightarrow{AB}$ from A to B. A is the “tail” and B is the “head” of the vector.

It is useful to introduce a ***zero vector***. It has length 0 and no direction. In print letters $\vec{a}, \vec{b}, \vec{u}, \vec{v}\ldots$ or $\overrightarrow{AB}, \overrightarrow{CD},\ldots$ are used to denote vectors. The length of $\vec{a}$ is denoted by $|\vec{a}|$ and also is called the ***magnitude of*** $\vec{a}$. Any vector of length 1 is called a ***unit vector*.**

If the vector $\vec{a}$ has the same length and the same direction as $\vec{b}$ even though it is in a different position, then we say that $\vec{a}$ and $\vec{b}$ are ***equivalent*** (or ***equal***) and we write $\vec{a}=\vec{b}$.

If the origin of a rectangular coordinate system is at the tail of $\vec{a}$, then the head of $\vec{a}$ has coordinates $(a_1;a_2)$ or $(a_1;a_2;a_3)$, depending on whether coordinate system is two- or three-dimensional. The numbers a_1, a_2, a_3 are called the scalar ***components of*** $\vec{a}$ relative to the coordinate system.

If $A(x_1;y_1;z_1)$ and $B(x_2;y_2;z_2)$ are the points of three dimension space, then the vector $\overrightarrow{AB}$ has the following ***scalar components*:**

$$\overrightarrow{AB}=(x_2-x_1;y_2-y_1;z_2-z_1)$$

The magnitude of $\vec{a}=(x;y;z)$ can be calculated by the formula $|\vec{a}|=\sqrt{x^2+y^2+z^2}$.

Adding and Subtracting Vectors

1. Addition of vectors*.* The sum of two vectors $\vec{u}$ and $\vec{v}$ is defined as follows. Place $\vec{v}$ in such a way that its tail is at the head of $\vec{u}$. Then the vector $\vec{c}=\vec{u}+\vec{v}$ goes from the tail of $\vec{u}$ to the head of $\vec{v}$. Observe, that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$, since both sums lie on the diagonal of the parallelogram, as shown in Fig. 1.1.

If $\vec{a}=(x_a;y_a;z_a)$ and $\vec{b}=(x_b;y_b;z_b)$ are the vectors of three dimension space, then the scalar components of $\vec{c}$ can be calculated by the formula $\vec{c}=\vec{a}+\vec{b}=(x_a+x_b;y_a+y_b;z_a+z_b)$.

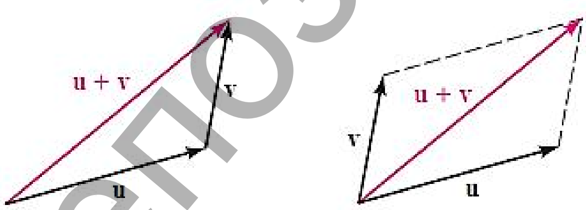


Figure 1.1

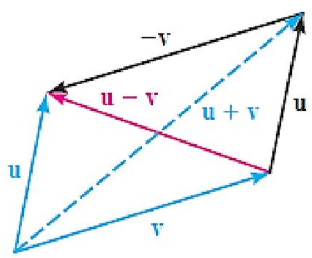


Figure 1.2

2. Subtraction of vectors. Let $\vec{u}$ and $\vec{v}$ be vectors. The vector $\vec{c}$ such that $\vec{c}+\vec{v}=\vec{u}$ is called the ***difference*** of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{c}=\vec{u}-\vec{v}$. Thus, if $\vec{a}=(x_a;y_a;z_a)$ and $\vec{b}=(x_b;y_b;z_b)$ then the scalar components of $\vec{c}$ can be calculated by the formula $\vec{c}=\vec{a}-\vec{b}=(x_a-x_b;y_a-y_b;z_a-z_b)$ (see Fig 1.2).

3. Scalar multiplication The ***negative*** of the vector $\vec{a}$ is defined as the vector having the same magnitude as $\vec{a}$ but the opposite direction. It is denoted by $-\vec{a}$. If $\vec{a}=\overrightarrow{AB}$, then $-\vec{a}=\overrightarrow{BA}$. Observe that $\vec{a}+\left(\overrightarrow{-a}\right)=\vec{0}$, just as with scalars.

Definition If c is a scalar and $\vec{a}$ is a vector, the product $c\vec{a}$ is the vector which length is $|c|$ times the length of $\vec{a}$ and which direction is the same as that of $\vec{a}$ if c is positive and opposite to that of $\vec{a}$ if c is negative.

Observe that $0\vec{a}$ has length 0 and thus is the zero vector $\vec{0}$ to which no direction is assigned. The vector $c\vec{a}$ is called a ***scalar multiple*** of the vector $\vec{a}$. If $\vec{a}=(x;y;z)$, then the scalar components of $\vec{a}$ can be calculated by the formula $c\vec{a}=(cx;cy;cz)$.

Three-dimensional coordinate systems

To locate a point in a plane, two numbers are necessary. We know that any point in the plane can be represented as an ordered pair $(a;b)$ of real numbers, where number a is the $x-$coordinate and b is the $y-$coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple $(a;b;c)$ of real numbers.

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called ***the coordinate axes*** and labeled the $x-$axis, $y-$axis, and $z-$axis.

The three coordinate axes determine the three ***coordinate planes***. These three coordinate planes divide space into eight parts, called ***octants***. The ***first octant***, in the foreground, is determined by the positive axes.

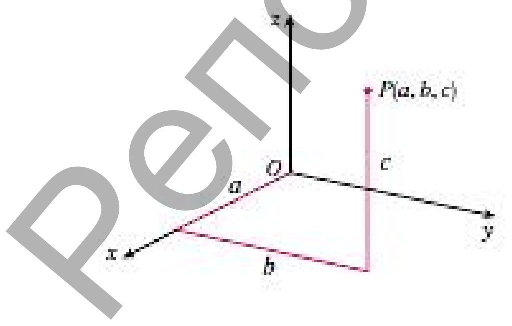


Figure 1.3

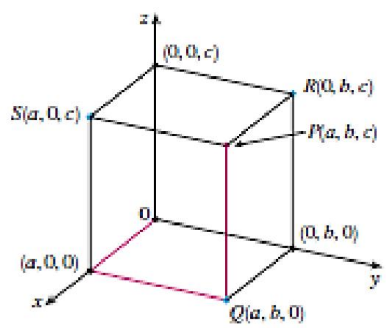


Figure 1.4

If P is any point in space, let a be the (directed) distance from the yz-plane to P let b be the distance from the xz-plane to P and let c be the distance from the xy-plane to P. The point P can be represented by the ordered triple $(a;b;c)$ of real numbers and we call a, b, and c the coordinates of P. Thus, to locate the point $(a;b;c)$, we can start at the origin O and move a units along the $x-$axis, then b units parallel to the $y-$axis, and then c units parallel to the $z-$axis as in Figure 1.3.

The point $P(a;b;c)$ determines a rectangular box as in Figure 1.4.

Basis

A set of vectors in space is called a ***basis***, or a set of ***basis vectors***, if the vectors are linearly independent and every vector in the vector space is a linear combination of this set. In more general terms, a basis is linearly independent.

Given a basis of a vector space V, every element of V can be expressed uniquely as a linear combination of basis vectors, whose coefficients are referred to as vector ***coordinates*** or ***components***. A vector space can have several distinct sets of basis vectors; however each such set has the same number of elements, with this number being the dimension of the vector space. The same vector can be represented in two different bases.

Definition A ***basis*** B of a vector space V is a linearly independent subset of V that spans V.

In more detail, suppose that $B=\{v_1,v_2,...,v_n\}$ is a finite subset of a vector space V. Then B is a basis if it satisfies the following conditions:

1. the *linear independence* property, for all $\alpha_1,\ldots,\alpha_n \in R$, if $\alpha_1 v_1+\ldots+\alpha_n v_n=0$, then necessarily $\alpha_1=\ldots=\alpha_n=0$;
2. the *spanning* property, for every x in V it is possible to choose $\alpha_1,\ldots,\alpha_n \in R$ such that $x=\alpha_1 v_1+\ldots+\alpha_n v_n$.

The numbers α_i are called the ***coordinates of the vector*** x with respect to the basis B, and by the first property they are uniquely determined.

A vector space that has a finite basis is called ***finite-dimensional***.

Definition The ***orthonormal basis*** for space V with a finite dimension is a basis which vectors are orthonormal, that is, they are all unit vectors and orthogonal to each other.

The set of vectors $\{\vec{i}=(1;0;0),\vec{j}=(0;1;0),\vec{k}=(0;0;1)\}$ (the standard basis) forms an orthonormal basis of R^3. All vectors $(x;y;z)$ in R^3 can be expressed as a sum of the basis vectors scaled $(x;y;z)=x\vec{i}+y\vec{j}+z\vec{k}$.

Example 1 Let the vectors $\vec{a}=(2;-1;8)$, $\vec{e_1}=(1;2;3)$, $\vec{e_2}=(1;-1;-2)$, $\vec{e_3}=(1;-6;0)$ be given in basis $\{\vec{i},\vec{j},\vec{k}\}$. Prove, that $\vec{e_1},\vec{e_2},\vec{e_3}$ form the basis of R^3 and express vector $\vec{a}$ via the new basis.

Solution The set of vectors $\vec{e_1},\vec{e_2},\vec{e_3}$ forms the basis of the three-dimensional coordinate system, if $\alpha_1\vec{e_1}+\alpha_2\vec{e_2}+\alpha_3\vec{e_3}=0$, then necessarily $\alpha_1=\alpha_2=\alpha_3=0$. This means, that corresponding scalar components are disproportionate or, accordingly properties of a determinant, the determinant of the 3×3 matrix, having the three vectors as its rows do not equal 0.

$\vec{e_1}\vec{e_2}\vec{e_3}=\begin{vmatrix}1 & 2 & 3\\ 1 & -1 & -2\\ 1 & -6 & 0\end{vmatrix}=-18-4+3-12=-31\neq 0$, it means that vectors $\vec{e_1},\vec{e_2},\vec{e_3}$ are linearly independent, hence form the basis of R^3.

Now let us express vector $\vec{a}$ via the new basis. We can write down the following representation $\vec{a}=x\vec{e_1}+y\vec{e_2}+z\vec{e_3}$ where numbers $(x;y;z)$ are the coordinates of the vector $\vec{a}$ with respect to the basis $\vec{e_1},\vec{e_2},\vec{e_3}$. We have:

$$\vec{a}=x\left(\vec{i}+2\vec{j}+3\vec{k}\right)+y\left(\vec{i}-\vec{j}-2\vec{k}\right)+z\left(\vec{i}-6\vec{j}\right)=$$

$$=\vec{i}(x+y+z)+\vec{j}(2x-y-6z)+\vec{k}(3x-2y)=2\vec{i}-\vec{j}+8\vec{k}$$

Using this representation we have following system of linear equations:

$$\begin{cases}x+y+z=2\\ 2x-y-6z=-1\\ 3x-2y=8\end{cases}\Rightarrow\begin{cases}x=2\\ y=-1\\ z=1\end{cases}\Rightarrow\ \vec{a}=2\vec{e_1}-y\vec{e_2}+z\vec{e_3}.$$

Exercise Set 6

1. Draw the vector $(2;3)$, placing its tail at:

a) $(0;0)$ b) $(-1;2)$ c) $(1;1)$

2. Find $|\vec{a}|$ if $\vec{a}$ is:

a) $\vec{a}=(4;0)$ b) $\vec{a}=(-5;12)$ c) $\vec{a}=(2;1;-5)$ d) $\vec{a}=(1;3;-2)$

3. Find the magnitude of $\overrightarrow{AB}$ if:

a) $A=(2;1)$, $B=(1;4)$ b) $A=(1;-3;-2)$, $B=(-2;4;5)$

4. Find $\vec{a}+\vec{b}$ if:

a) $\vec{a}=10\left(\frac{7}{2};\frac{13}{2}\right), \vec{b}=(2;0)$ b) $\vec{a}=(2;2;-3), \vec{b}=(1;-1;4)$

5. Find $3\vec{a}-2\vec{b}$ if:

a) $\vec{a}=(4;3), \vec{b}=(2;0)$ b) $\vec{a}=(3;4), \vec{b}=(5;1)$ c) $\vec{a}=(2;3;4), \vec{b}=(1;5;0)$

6. Find the distance between the points:

a) $(1;4;2)$ and $(2;1;5)$ b) $(-3;2;1)$ and $(4;0;-2)$

7. Find $\left|3\vec{a}+2\vec{b}\right|$ if:

a) $\vec{a}=(1;2;0), \vec{b}=(2;3;5)$ b) $\vec{a}=(3;-2;1), \vec{b}=(-4;3;2)$

In Exercises 8 to 11 find the scalar components of the vector $\vec{a}$ in the basis $\vec{e_1},\vec{e_2},\vec{e_3}$.

8. $\vec{e_1}=(5;4;1), \vec{e_2}=(-3;5;2), \vec{e_3}=(2;1;-3), \vec{a}=(7;23;4)$

9. $\vec{e_1}=(1;-3;1), \vec{e_2}=(-2;-4;3), \vec{e_3}=(0;-2;3), \vec{a}=(-8;-10;13)$

10. $\vec{e_1}=(3;1;-2), \vec{e_2}=(-2;4;1), \vec{e_3}=(4;-5;-1), \vec{a}=(-5;11;1)$

11. $\vec{e_1}=(3;1;2), \vec{e_2}=(-4;3;-1), \vec{e_3}=(2;3;4), \vec{a}=(14;14;20)$

Individual Tasks 6

1. Find the magnitude of $\overrightarrow{AB}$.

2. Find $2\vec{a}+5\vec{b}$.

3. Find the distance between the given points.

4. Find the scalar components of the vector $\vec{a}$ in the basis $\vec{e_1},\vec{e_2},\vec{e_3}$.

I.	**II.**
1. $A=(0;-2;-2), B=(5;1;-5)$	1. $A=(1;0;8), B=(-4;4;0)$
2. $\vec{a}=(-2;1;3), \vec{b}=(0;-3;2)$	2. $\vec{a}=(2;-1;-4), \vec{b}=(3;-5;-1)$
3. $A(1;2;-5), B(1;0;1)$	3. $P(1;-3;-3); Q(2;0;-4)$

4. $\vec{e_1}=(5;7;-2)$, $\vec{e_2}=(-3;1;3)$, $\vec{e_3}=(1;-4;6)$, $\vec{a}=(14;9;-1)$	4. $\vec{e_1}=(-1;4;3)$, $\vec{e_2}=(3;2;-4)$, $\vec{e_3}=(-2;-7;1)$, $\vec{a}=(6;20;-3)$
III. 1. $A=(0;-3;-3)$, $B=(6;1;-6)$ 2. $\vec{a}=(-1;2;3)$, $\vec{b}=(2;-3;0)$ 3. $A(1;0;-3), B(2;0;2)$ 4. $\vec{e_1}=(3;-1;2)$, $\vec{e_2}=(-2;3;1)$, $\vec{e_3}=(4;-5;-3)$, $\vec{a}=(-3;2;-3)$	**IV.** 1. $A=(8;0;1)$, $B=(-1;4;1)$ 2. $\vec{a}=(4;-1;-2)$, $\vec{b}=(1;-5;-3)$ 3. $P(1;-4;-4);Q(3;0;-2)$ 4. $\vec{e_1}=(5;3;1)$, $\vec{e_2}=(-1;2;-3)$, $\vec{e_3}=(3;-4;2)$, $\vec{a}=(-9;34;-20)$

2.1.2 The Dot Product of Two Vectors

Definition Let $\vec{a}$ and $\vec{b}$ be two nonparallel and nonzero vectors. They determine a triangle and an angle φ, shown in Figure 1.5. The angle between $\vec{a}$ and $\vec{b}$ is φ. Note that $0<\varphi<\pi$.

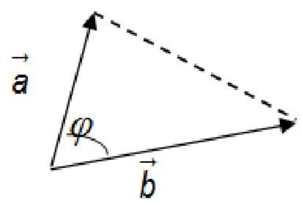


Figure 1.5

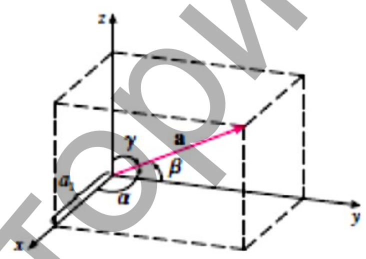


Figure 1.6

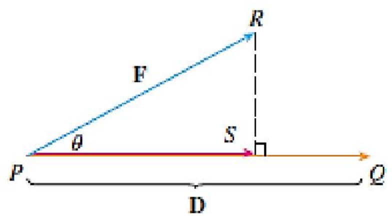


Figure 1.7

If $\vec{a}$ and $\vec{b}$ are parallel, the angle between them is 0 (if they have the same direction) or π (if they have opposite directions). The angle between $\vec{0}$ and any other vector is not defined.

Definition Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Their ***dot product*** is the number

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\varphi \qquad (1)$$

where φ is the angle between $\vec{a}$ and $\vec{b}$. If $\vec{a}$ and $\vec{b}$ is $\vec{0}$, their dot product is 0. The dot product is a scalar and is also called the ***scalar product*** of $\vec{a}$ and $\vec{b}$.

Properties of the dot product

1. For any two vectors $\vec{a}$ and $\vec{b}$: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.

2. For any vector $\vec{a}$: $\vec{a}\cdot\vec{a}=\left|\vec{a}\right|^2$

3. If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}\cdot\vec{b}=0$. *Consequently, the vanishing of the dot product is a test for perpendicularity.*

4. *Formula for the dot product in scalar components.* To find the dot product just add the products of corresponding components: if $\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ and $\vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$, then

$$\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2 \tag{2}$$

Application of dot product

1. The angle between two vectors $\vec{a}$ and $\vec{b}$ can be determined by the formula:

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|} \tag{3}$$

2. Figure 1.8 shows the representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors $\vec{a}$ and $\vec{b}$ with the same initial point P . If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the ***vector projection*** of $\vec{b}$ onto $\vec{a}$ and denoted by $proj_{\vec{a}}\,\vec{b}$.

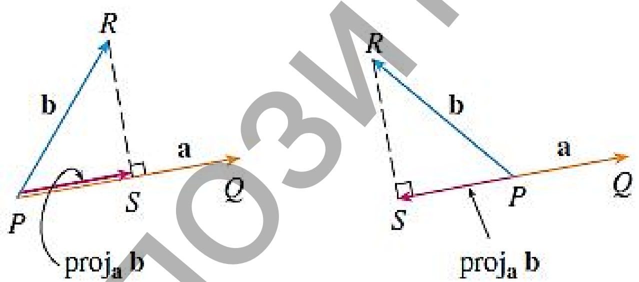


Figure 1.8

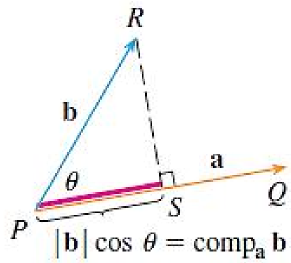


Figure 1.9

The scalar projection of $\vec{b}$ onto $\vec{a}$ (also called the ***component of*** $\vec{b}$ ***along*** $\vec{a}$) is defined to be the signed magnitude of the vector projection, which is the number $\left|\vec{b}\right|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ (see Figure 1.9). This is denoted by $comp_{\vec{a}}\,\vec{b}$.

Let $\vec{a}$ be a vector and let $\vec{u}$ be a unit vector. Then $\vec{a}$ can be expressed as the sum of two vectors, $\vec{a}=\overrightarrow{a_1}+\overrightarrow{a_2}$, where $\overrightarrow{a_1}\parallel\vec{u}$, $\overrightarrow{a_2}\perp\vec{u}$, as shown in Figure 1.10.

The dot product provides the tool for finding $\overrightarrow{a_1}, \overrightarrow{a_2}$:

$$\vec{a} = \overrightarrow{a_1} + \overrightarrow{a_2} = \left(\vec{a} \cdot \vec{u}\right)\vec{u} + \left[\vec{a} - \left(\vec{a} \cdot \vec{u}\right)\vec{u}\right] \tag{4}$$

The number $\vec{a} \cdot \vec{u}$ is called the ***scalar component*** of $\vec{a}$ along $\vec{u}$. It is positive if $0 \le \theta < \dfrac{\pi}{2}$, negative if $\dfrac{\pi}{2} < \theta \le \pi$, and 0 if $\theta = \dfrac{\pi}{2}$.

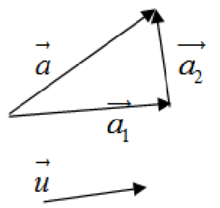


Figure 1.10

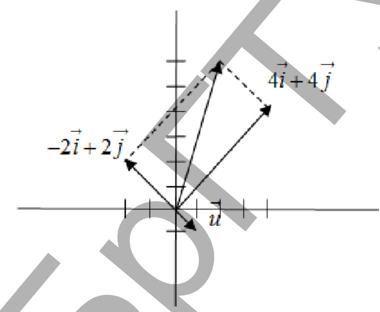


Figure 1.11

The vector $\left(\vec{a} \cdot \vec{u}\right) \cdot \vec{u}$ is called the ***vector component,*** or ***projection*** of $\vec{a}$ along $\vec{u}$. The vector $\vec{a} - \left(\vec{a} \cdot \vec{u}\right)\vec{u}$ is the ***vector component*** of $\vec{a}$ perpendicular to $\vec{u}$.

Example 1 Express $\vec{a} = 2\vec{i} + 6\vec{j}$ as a sum of the vector parallel to $\vec{u} = \dfrac{\sqrt{2}}{2}\vec{i} - \dfrac{\sqrt{2}}{2}\vec{j}$ and the vector perpendicular to $\vec{u}$.

Solution By property 6 $\vec{a} = \overrightarrow{a_1} + \overrightarrow{a_2} = \left(\vec{a} \cdot \vec{u}\right)\vec{u} + \left[\vec{a} - \left(\vec{a} \cdot \vec{u}\right)\vec{u}\right]$.

The scalar component is

$$\vec{a} \cdot \vec{u} = \left(2\vec{i} + 6\vec{j}\right) \cdot \left(\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}\right) = 2 \cdot \frac{\sqrt{2}}{2} + 6 \cdot \left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

The vector component of $\vec{a}$ along $\vec{u}$ is

$$\overrightarrow{a_1} = \left(\vec{a} \cdot \vec{u}\right)\vec{u} = -2\sqrt{2} \cdot \left(\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}\right) = -2\vec{i} + 2\vec{j}$$

and $\overrightarrow{a_2} = \vec{a} - \left(\vec{a} \cdot \vec{u}\right)\vec{u} = \left(2\vec{i} + 6\vec{j}\right) - \left(-2\vec{i} + 2\vec{j}\right) = 4\vec{i} + 4\vec{j}$.

So $\vec{a} = \left(-2\vec{i} + 2\vec{j}\right) + \left(4\vec{i} + 4\vec{j}\right)$. These vectors are shown in Figure 1.11.

3. Direction angles and direction cosines. The ***direction angles*** of a nonzero vector $\vec{a}$ are the angles α, β and γ in the interval $[0;\pi]$, whose $\vec{a}$ makes with the positive x-, y-, and z-axes (see Figure 1.8).

The cosines of these direction angles $\cos\alpha, \cos\beta$ and $\cos\gamma$, are called the ***direction cosines*** of the vector $\vec{a} = (x_a; y_a; z_a)$. Using the property of the dot product, we obtain

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{x_a}{|\vec{a}|}, \qquad \cos\beta = \frac{y_a}{|\vec{a}|}, \qquad \cos\gamma = \frac{z_a}{|\vec{a}|} \tag{5}$$

Theorem 1 If α, β and γ are the direction angles of the vector $\vec{a}$, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 2 Prove, that the set of vectors $\{\vec{i} = (1;0;0), \vec{j} = (0;1;0), \vec{k} = (0;0;1)\}$ (the standard basis) forms an orthonormal basis of R^3.

Solution A straightforward computation shows that the dot products of these vectors equal zero $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{i}\cdot\vec{k} = 0$ and that each of their magnitudes equals one $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$. This means that $\{\vec{i}, \vec{j}, \vec{k}\}$ is an orthonormal set.

4. Suppose that the constant force is a vector $\vec{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 1.7. If the force moves the object from P to Q, then the ***displacement vector*** is $\vec{D} = \overrightarrow{PQ}$. The ***work*** done by this force is defined as a product of the component of the force along the distance $\vec{D}$: $A = (\vec{F}\cos\theta)\cdot|\vec{D}|$. Thus the work done by a constant force $\vec{F}$ is the dot product $\vec{F}\cdot\vec{D}$, where $\vec{D}$ is the displacement vector.

$$A = \vec{F}\cdot\vec{D} \tag{6}$$

Exercise Set 7

In Exercises 1 to 6 compute $\vec{a}\cdot\vec{b}$, if:

1. Vector $\vec{a}$ has length 3, $\vec{b}$ has length 4, the angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$.

2. Vector $\vec{a}$ has length 2, $\vec{b}$ has length 2, the angle between $\vec{a}$ and $\vec{b}$ is $3\pi/4$.

3. Vector $\vec{a}$ has length 5, $\vec{b}$ has length 1/2, the angle between $\vec{a}$ and $\vec{b}$ is $\pi/2$.

4. Vector $\vec{a}$ is the zero vector, $\vec{b}$ has length 5.

5. $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} - \vec{j} - \vec{k}$.

6. $\vec{a} = \overrightarrow{PQ}$ and $\vec{b} = \overrightarrow{PR}$, where $P(1;0;2), Q(1;1;-1), R(2;3;5)$.

7. Find: $\vec{a}\cdot\vec{b}$, $\vec{a}^2$, $\vec{b}^2$, $(\vec{a}+\vec{b})^2$, $(\vec{a}-\vec{b})^2$, $(-4\vec{a}+3\vec{b})\cdot(\vec{a}-2\vec{b})$ if $\vec{a}=(3;1;-4)$ and $\vec{b}=(2;-1;1)$.

8. Find $(\vec{a}+\vec{b})^2, (\vec{a}-\vec{b})^2$, $(2\vec{a}-4\vec{b})\cdot(\vec{a}+3\vec{b})$ if $|\vec{a}|=3, |\vec{b}|=5, \varphi=(\widehat{\vec{a};\vec{b}})=\frac{\pi}{4}$.

9. a) Draw the vectors $7\vec{i}+12\vec{j}$ and $9\vec{i}-5\vec{j}$.

b) Do they seem to be perpendicular?

c) Determine whether they are perpendicular by examining their dot product.

10. a) Draw the vectors $\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$.

b) Do they seem to be perpendicular?

c) Determine whether they are perpendicular by examining their dot product.

11. a) Estimate the angle between $\vec{a}=3\vec{i}+4\vec{j}$ and $\vec{b}=5\vec{i}+12\vec{j}$ by drawing them.

b) Find the angle between $\vec{a}$ and $\vec{b}$.

12. Find the cosine of the angle between $2\vec{i}-4\vec{j}+6\vec{k}$ and $\vec{i}+2\vec{j}+3\vec{k}$.

In Exercises 13 to 16 find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

13. $\vec{a}=(3;-4);\vec{b}=(5;0)$

14. $\vec{a}=(3;6;-2);\vec{b}=(1;2;3)$

15. $\vec{a}=(1;2);\vec{b}=(-4;1)$

16. $\vec{a}=(-2;3;-6);\vec{b}=(5;-1;4)$

17. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A(1;3), B(7;4), C(2;8)$ and $D(1;-5)$.

18. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A(1;2;-5), B(1;0;1), C(0;-1;3)$ and $D(2;1;4)$.

In Exercises 19 to 22 find the vector components of $\vec{a}$ parallel and perpendicular to $\vec{b}$.

19. $\vec{a}=2\vec{i}+3\vec{j}+4\vec{k}$ and $\vec{b}=\vec{i}+\vec{j}+\vec{k}$.

20. $\vec{a}=\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}-\vec{j}$.

21. $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{b}=2\vec{j}+3\vec{k}$.

22. $\vec{a}=\vec{j}$ and $\vec{b}=2\vec{i}+3\vec{j}-\vec{k}$.

23. Find the vector $\vec{d}$ such that $\vec{d}\cdot\vec{a}=-5; \vec{d}\cdot\vec{b}=-11; \vec{d}\cdot\vec{c}=20$, if $\vec{a}=(2;-1;3)$, $\vec{b}=(1;-3;2)$, $\vec{c}=(3;2;-4)$.

24. Find $\vec{c}$ which is perpendicular to the axes OZ, perpendicular to the vector $\vec{a}=(8;-15;3)$, forms the acute angle with Ox and $|\vec{c}|=51$.

25. Prove that vectors $\vec{a}=(7;6;-6)$ and $\vec{b}=(6;2;9)$ can be regarded as the edges of the cube and find the third edge.

In Exercises 26 to 29 find the direction cosines and direction angles of the vector. Give the direction angles correct to the nearest degree.

26. $\vec{a}=(3;4;5)$

27. $\vec{a}=(1;-2;-1)$

28. $\vec{a}=2\vec{i}+3\vec{j}-6\vec{k}$

29. $\vec{a}=2\vec{i}-\vec{j}+2\vec{k}$

30. If a vector has direction angles $\alpha=\pi/4, \beta=\pi/3$, find the third direction angle γ.

In Exercises 31 to 36 determine whether the given vectors are orthogonal, parallel, or neither:

31. $\vec{a}=(-5;3;7);\vec{b}=(6;-8;2)$

32. $\vec{a}=(4;6);\vec{b}=(-3;2)$

33. $\vec{a}=-\vec{i}+2\vec{j}+5\vec{k};\vec{b}=3\vec{i}+4\vec{j}-\vec{k}$

34. $\vec{a}=2\vec{i}+6\vec{j}-4\vec{k};\vec{b}=-3\vec{i}-9\vec{j}+6\vec{k}$

35. $\vec{a}=(-3;9;6);\vec{b}=(4;-12;-8)$

36. $\vec{a}=\vec{i}-\vec{j}+2\vec{k};\vec{b}=2\vec{i}-\vec{j}+\vec{k}$

37. Use vectors to decide whether the triangle with vertices $P(1;-3;-3);Q(2;0;-4)$ and $R(6;-2;-5)$ is right-angled.

38. A wagon is pulled a distance of 100m along a horizontal path by a constant force of 70N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

39. A force is given by a vector $\overrightarrow{F}=(3;4;5)$ and moves a particle from the point $P(2;1;0)$ to the point $Q(4;6;2)$. Find the work done.

40. A boat sails south with the help of a wind blowing in the direction $S\ 36^\circ E$ with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

41. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

42. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck pulling the car 1 km?

Individual Tasks 7

1. Find: $\vec{a}\cdot\vec{b}$, $\vec{a}^{\,2}$, $\vec{b}^{\,2}$, $\left(\vec{a}+\vec{b}\right)^2$, $\left(\vec{a}-\vec{b}\right)^2$, $\left(3\vec{a}-2\vec{b}\right)\cdot\left(\vec{a}+2\vec{b}\right)$.

2. Find $\left(\vec{a}+\vec{b}\right)^2$, $\left(\vec{a}-\vec{b}\right)^2$, $\left(2\vec{a}-3\vec{b}\right)\cdot\left(-\vec{a}+2\vec{b}\right)$.

3. Find the third direction angle γ of the vector.

4. Find the scalar and the vector projection

I.	**II.**
1. $\vec{a}=-3\vec{i}+\vec{j}+5\vec{k}$, $\vec{b}=-\vec{i}-\vec{j}+\vec{k}$	1. $\vec{a}=-2\vec{i}-\vec{j}+4\vec{k}$, $\vec{b}=-2\vec{i}+\vec{j}+5\vec{k}$
2. $\lvert\vec{a}\rvert=3, \lvert\vec{b}\rvert=4$, $\varphi=\left(\widehat{\vec{a};\vec{b}}\right)=2\pi/3$	2. $\lvert\vec{a}\rvert=6$, $\lvert\vec{b}\rvert=8$, $\varphi=\left(\widehat{\vec{a};\vec{b}}\right)=\pi/6$
3. $\alpha=-\pi/4,\ \beta=\pi/6$	3. $\alpha=\pi/2,\ \beta=-\pi/3$
4. $\vec{a}=(-5;3;7)$, $\vec{b}=(6;-8;2)$	4. $\vec{a}=(-3;9;6)$, $\vec{b}=(4;-12;-8)$
5. Find the angle between the diagonal of the cube and one of its edges.	5. Find the angle between the diagonal of the cube and a diagonal of one of its faces.
III.	**IV.**
1. $\vec{a}=(-2;1;3)$, $\vec{b}=(0;-3;2)$	1. $\vec{a}=(2;-1;-4)$, $\vec{b}=(3;-5;-1)$
2. $\lvert\vec{a}\rvert=2, \lvert\vec{b}\rvert=8$, $\varphi=\left(\widehat{\vec{a};\vec{b}}\right)=\pi/3$	2. $\lvert\vec{a}\rvert=3$, $\lvert\vec{b}\rvert=4$, $\varphi=\left(\widehat{\vec{a};\vec{b}}\right)=5\pi/6$
3. $\alpha=\pi/4,\ \beta=-\pi/6$	3. $\alpha=-\pi/2,\ \beta=\pi/3$
4. $\vec{a}=(-5;3;4)$, $\vec{b}=(2;-2;2)$	4. $\vec{a}=(-1;3;2)$, $\vec{b}=(2;-6;-4)$
5. Find the acute angle between two diagonals of the cube.	5. Find the angle between the diagonal of the cube and one of its bases.

2.1.3 The Cross Product of Two Vectors

It is frequently necessary in applications of vectors in space to construct a nonzero vector perpendicular to two given vectors $\vec{a}$ and $\vec{b}$.

If $\vec{a}$ and $\vec{b}$ are not parallel and drawn with their tails at a single point, they determine a plane. Any vector $\vec{c}$ perpendicular to this plane is perpendicular to both $\vec{a}$ and $\vec{b}$. There are many such vectors, all parallel to each other and having various lengths.

Definition ***Cross product (vector product)*** Let $\vec{a} = x_a\vec{i} + y_a\vec{j} + z_a\vec{k}$ and $\vec{b} = x_b\vec{i} + y_b\vec{j} + z_b\vec{k}$. The vector

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \vec{i}\begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} - \vec{j}\begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} + \vec{k}\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} =$$

$$= \left(y_a z_b - z_a y_b\right)\vec{i} - \left(x_a z_b - z_a x_b\right)\vec{j} + \left(x_a y_b - y_a x_b\right)\vec{k}$$

is called the ***cross product*** (or ***vector product***) of $\vec{a}$ and $\vec{b}$. It is denoted $\vec{a}\times\vec{b}$.

Properties of the cross product

1. $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

2. The order of the factors in the vector product is critical $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$. This property corresponds to the fact that when two rows of a matrix are interchanged, its determinant changes a sign.

3. If $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a}\times\vec{b} = \vec{0}$. This corresponds to the fact that if two rows of a matrix are identical, then its determinant is 0.

4. For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$: $\vec{a}\times\left(\vec{b}+\vec{c}\right) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$. This distributive law can be established by a straightforward computation.

5. A scalar can be factored out of a cross product: $\left(\alpha\vec{a}\right)\times\vec{b} = \alpha\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\left(\alpha\vec{b}\right)$.

6. The magnitude of $\vec{a}\times\vec{b}$ is equal to the ***area of the parallelogram*** spanned by $\vec{a}$ and $\vec{b}$: $\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\sin\varphi$, where $\varphi = \left(\widehat{\vec{a};\vec{b}}\right)$ is the angle between $\vec{a}$ and $\vec{b}$.

Example 1 Find a vector perpendicular to the plane determined by the three points $A(1;3;2), B(4;-1;1), C(3;0;2)$.

Solution The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in a plane. The vector $\vec{c} = \overrightarrow{AB}\times\overrightarrow{AC}$, being perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, is perpendicular to the plane. Now, $\overrightarrow{AB} = 3\vec{i} - 4\vec{j} - \vec{k}$ and $\overrightarrow{AC} = 2\vec{i} - 3\vec{j} + 0\vec{k}$. Thus $\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & -1 \\ 2 & -3 & 0 \end{vmatrix} = -3\vec{i} - 2\vec{j} - \vec{k} = (-3;-2;-1)$.

Application of the cross product

The idea of a cross product often occurs in physics. In particular, we consider a force

$\vec{F}$, acting on a rigid body at a point given by a position vector $\vec{r}$. (For instance, if we tighten a bolt applying a force to a wrench as in Figure 1.10, we produce a turning effect). The ***torque*** (relative to the origin) is defined as a cross product of the position and force vectors

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (1)$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. The magnitude of the torque vector is

$$\left|\vec{\tau}\right| = \left|\vec{r} \times \vec{F}\right| = |\vec{r}| \cdot |\vec{F}| \cdot \sin\theta \qquad (2)$$

where θ is the angle between the position and force vectors. Observe that the only component of $\vec{F}$ that can cause a rotation is the one perpendicular to $\vec{r}$, that is, $|\vec{F}| \cdot \sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by $\vec{r}$ and $\vec{F}$.

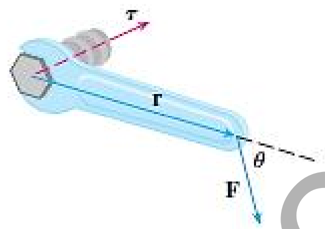


Figure 1.10

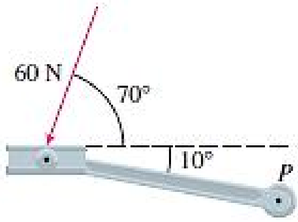


Figure 1.11

Exercise Set 8

In Exercises 1 to 4 compute $\vec{a} \times \vec{b}$:

1. $\vec{a} = \vec{k};\ \vec{b} = \vec{j}$.

2. $\vec{a} = \vec{i} + \vec{j};\ \vec{b} = \vec{i} - \vec{j}$

3. $\vec{a} = \vec{i} + \vec{j} + \vec{k};\ \vec{b} = \vec{i} + \vec{j}$

4. $\vec{a} = \vec{k};\ \vec{b} = \vec{i} + \vec{j}$

In Exercises 5 and 6 compute $\vec{a} \times \vec{b}$ and check that it is perpendicular to both $\vec{a}$ and to $\vec{b}$:

5. $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k};\ \vec{b} = \vec{i} + \vec{j} + 2\vec{k}$

6. $\vec{a} = \vec{i} - \vec{j};\ \vec{b} = \vec{j} + 4\vec{k}$

7. Calculate $\left|\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right)\right|$ and $\left|\left(3\vec{a} - \vec{b}\right) \times \left(\vec{a} - 2\vec{b}\right)\right|$, if $\left|\vec{a}\right| = 3$ and $\left|\vec{b}\right| = 4$.

In Exercises 8 to 11 determine the scalar components of vectors $\left(2\vec{a}+3\vec{b}\right)\times\left(\vec{a}-4\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)\times\left(3\vec{a}+\vec{b}\right)$, if

8. $\vec{a}=2\vec{i}-3\vec{j}+\vec{k},\ \vec{b}=\vec{j}+4\vec{k}$

9. $\vec{a}=3\vec{i}+4\vec{j}+\vec{k},\ \vec{b}=\vec{i}-2\vec{j}+7\vec{k}$

10. $\vec{a}=2\vec{i}-4\vec{j}-2\vec{k},\ \vec{b}=7\vec{i}+3\vec{j}-2\vec{k}$

11. $\vec{a}=-7\vec{i}+\vec{j}+2\vec{k},\ \vec{b}=2\vec{i}-6\vec{j}+4\vec{k}$

12. Find the area of a parallelogram three of its vertices are

a) $A(0;0;0),\ B(1;5;4),\ C(2;-1;3)$

b) $A(1;2;-1),\ B(2;1;4),\ C(3;5;2)$

13. Find a vector perpendicular to the plane determined by the three points $A(1;2;1), B(2;1;-3), C(0;1;5)$.

14. Find a vector perpendicular to the line through $A(1;2;1)$ and $B(4;1;0)$ also to the line passing through points $C(3;5;2)$ and $D(2;6;-3)$.

15. Find the area of the triangle which vertices are:

a) $A(0;0),\ B(3;5),\ C(2;-1)$

b) $A(1;4),\ B(3;0),\ C(-1;2)$

c) $A(1;1;1),\ B(2;0;1),\ C(3;-1;4)$

In Exercises 16 to 19 find:

a) a nonzero vector orthogonal to the plane through the point P, Q, R;

b) the area of ΔPQR.

16. $P(1;0;0)$, $Q(0;2;0)$, $R(0;0;3)$.

17. $P(2;1;5)$, $Q(-1;3;4)$, $R(3;0;6)$.

18. $P(0;-2;0)$, $Q(4;1;-2)$, $R(5;3;1)$.

19. $P(-1;3;1), Q(0;5;2), R(4;3;-1)$.

20. Find $\left|\vec{a}\times\vec{b}\right|$, if $|\vec{a}|=10$, $|\vec{b}|=2$, $\vec{a}\cdot\vec{b}=12$.

21. Find $\vec{a}\cdot\vec{b}$, if $|\vec{a}|=3, |\vec{b}|=26$, $\left|\vec{a}\times\vec{b}\right|=72$.

22. Prove that points $A(3;-1;2), B(1;2;-1), C(-1;1;-3), D(3;-5;3)$ are the vertices of trapezoid.

23. A bicycle pedal is pushed by a foot with a 60-N force as shown in Figure 1.11. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.

24. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\vec{a}=(0;3;-4)$ at the end of the wrench. Find the magnitude of the force needed to supply 100 $N\cdot m$ of torque to the bolt.

Individual Tasks 8

1. Determine the scalar components of vectors $\left(\vec{a}+2\vec{b}\right)\times\left(3\vec{a}-4\vec{b}\right)$ and calculate $\left|\left(\vec{a}-\vec{b}\right)\times\left(-3\vec{a}+2\vec{b}\right)\right|$.

2. Find the area of the triangle which vertices are given.

3. Solve the given problem.

I.	**II.**
1. $\vec{\text{a}}=-3\vec{i}+\vec{j}+5\vec{k}$, $\vec{b}=-\vec{i}-\vec{j}+\vec{k}$ 2. $A(1;-1-;1)$, $B(2;3;1)$, $C(0;-1;2)$ 3. Find $\left\|\vec{a}\times\vec{b}\right\|$, if $\|\vec{a}\|=10$, $\|\vec{b}\|=2$, $\vec{a}\cdot\vec{b}=12$.	1. $\vec{\text{a}}=-2\vec{i}-\vec{j}+4\vec{k}$, $\vec{b}=-2\vec{i}+\vec{j}+5\vec{k}$ 2. $A(1;0;-2)$, $B(3;0;-1)$, $C(0;-1;5)$ 3. Find $\vec{a}\cdot\vec{b}$, if $\|\vec{a}\|=3$, $\|\vec{b}\|=26$, $\left\|\vec{a}\times\vec{b}\right\|=72$.
III.	**IV.**
1. $\vec{a}=(-2;1;3)$, $\vec{b}=(0;-3;2)$ 2. $A(-2;-5-;-1)$, $B(-6;-7;9)$, $C(4;-5;1)$ 3. Find $\left\|\vec{a}\times\vec{b}\right\|$, if $\|\vec{a}\|=5$, $\|\vec{b}\|=2$, $\vec{a}\cdot\vec{b}=8$.	1. $\vec{a}=(3;-3;0)$, $\vec{b}=(2;-1;4)$ 2. $A(1;3;2)$, $B(5;2;-1)$, $C(2;2;4)$ 3. Find $\vec{a}\cdot\vec{b}$, if $\|\vec{a}\|=13$, $\|\vec{b}\|=1$, $\left\|\vec{a}\times\vec{b}\right\|=5$.

2.1.4 Triple Products

The triple product is a product of three 3-dimensional vectors, usually Euclidean vectors.

Definition The ***scalar triple product*** (also called the ***mixed product, box product***) is defined as the dot product of one of the vectors with the cross product of the other two:

$$\vec{a}\,\vec{b}\,\vec{c}=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) \qquad (1)$$

Geometric interpretation Geometrically, the scalar triple product is the (signed) volume of the parallelepiped defined by the three vectors given (see Figure 1.12). Here, the parentheses may be omitted without causing ambiguity, since the dot product cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined.

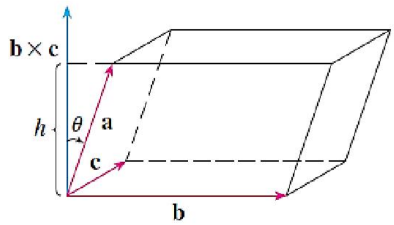


Figure 1.12

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = \left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right| = \left|\vec{a}\vec{b}\vec{c}\right| \tag{2}$$

Properties of triple products

1. The scalar triple product is invariant under a circular shift of its three operands $\vec{a},\vec{b},\vec{c}$:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{a}\times\vec{c}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right).$$

2. Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a}$$

3. Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = -\vec{a}\cdot\left(\vec{c}\times\vec{b}\right) = -\vec{b}\cdot\left(\vec{a}\times\vec{c}\right) = -\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)$$

4. The scalar triple product can also be understood as the determinant of the 3×3 matrix (thus also its inverse) having the three vectors either as its rows or its columns (a matrix has the same determinant as its transpose):

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{3}$$

5. If the scalar triple product is equal to zero, then the three vectors $\vec{a},\vec{b},\vec{c}$ are ***coplanar*** (lie in the same plane), since the parallelepiped defined by them would be flat and have no volume.

6. If any two vectors in the triple scalar product are equal, then its value is zero:

$$\vec{a}\cdot\left(\vec{a}\times\vec{b}\right)=\vec{a}\cdot\left(\vec{b}\times\vec{a}\right)=\vec{a}\cdot\left(\vec{b}\times\vec{b}\right)=\vec{a}\cdot\left(\vec{a}\times\vec{a}\right)=0.$$

Exercise Set 9

In Exercises 1 and 2 find the volume of the parallelepiped determined by the vectors $\vec{a},\vec{b},\vec{c}$.

1. $\vec{a}=(6;3;-1)$; $\vec{b}=(0;1;2)$; $\vec{c}=(4;-2;5)$.

2. $\vec{a}=\vec{i}+\vec{j}-\vec{k}$; $\vec{b}=\vec{i}-\vec{j}+\vec{k}$; $\vec{c}=-\vec{i}+\vec{j}+\vec{k}$.

In Exercises 3 and 4 find the volume and height of the parallelepiped with adjacent to edges PQ, PR, and PS.

3. $P(2;0;-1), Q(4;1;0), R(3;-1;1), S(2;-2;2)$.

4. $P(3;0;1), Q(-1;2;5), R(5;1;-1), S(0;4;2)$.

In Exercises 5 to 8 show that the given vectors are coplanar (noncoplanar).

5. $\vec{e_1}=(0;4;1), \vec{e_2}=(3;5;-2), \vec{e_3}=(2;0;-3)$

6. $\vec{e_1}=(1;4;1), \vec{e_2}=(0;0;3), \vec{e_3}=(0;-1;3)$

7. $\vec{e_1}=(3;1;0), \vec{e_2}=(-2;0;1), \vec{e_3}=(0;-5;-1)$

8. $\vec{e_1}=(3;-1;-2), \vec{e_2}=(3;3;-1), \vec{e_3}=(2;3;0)$

In Exercises 9 to 14 prove that $\vec{p},\vec{q},\vec{r}$ form the basis of R^3 and express the vector $\vec{a}$ via the new basis.

9. $\vec{p}=(4;5;1)$, $\vec{q}=(3;4;1)$, $\vec{r}=(2;3;2)$, $\vec{a}=(6;3;4)$

10. $\vec{p}=(-1;4;3)$, $\vec{q}=(3;2;-4)$, $\vec{r}=(-2;-7;1)$, $\vec{a}=(6;20;-3)$

11. $\vec{p}=(5;7;-2)$, $\vec{q}=(-3;1;3)$, $\vec{r}=(1;-4;6)$, $\vec{a}=(14;9;-1)$

12. $\vec{p}=(1;-3;1)$, $\vec{q}=(-2;-4;3)$, $\vec{r}=(9;-2;3)$, $\vec{a}=(-8;-10;13)$

13. $\vec{p}=(4;5;2)$, $\vec{q}=(3;0;1)$, $\vec{r}=(-1;4;2)$ $\vec{a}=(5;7;8)$

14. Prove that $\left(\vec{a}+\vec{b}\right)\left(\vec{b}+\vec{c}\right)\left(\vec{c}+\vec{a}\right)=2\left(\vec{a}\vec{b}\vec{c}\right)$.

15. Find vector $\vec{d}$ such that $\vec{d}\cdot\vec{a}=-5$, $\vec{d}\cdot\vec{b}=-11$, $\vec{d}\cdot\vec{c}=20$, if $\vec{a}=(2;-1;3)$, $\vec{b}=(1;-3;2)$, $\vec{c}=(3;2;-4)$.

16. Find the vector $\vec{c}$, if it is perpendicular to the vector $\vec{a}=(8;-15;3)$, perpendicular to the $z-axis$, forms acute angle with $x-axis$ and $\left|\vec{c}\right|=51$.

17. Prove that the vectors $\vec{a}=(7;6;-6)$ and $\vec{b}=(6;2;9)$ can be regarded as cube edges, find the third edge.

18. Vectors $\vec{a}=(8;4;1)$, $\vec{b}=(2;-2;1)$, $\vec{c}=(-6;9;2)$ are given. Find the unit vector $\vec{d}$, such that $\vec{d}\perp\vec{a}$, $\vec{d}\perp\vec{b}$ and ordered triples of vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{a},\vec{b},\vec{d}$ have the same orientation.

19. Three vertexes of the trapeze $A(-3;-2;-1)$, $B(1;2;3)$, $C(9;6;4)$ are given. Find the fourth vertex.

Individual Tasks 9

1. Show that the given vectors are coplanar (noncoplanar).

2. Find the volume and height of the parallelepiped with adjacent to edges PQ, PR, and PS.

3. Prove that $\vec{p},\vec{q},\vec{r}$ form the basis of R^3 and express vector $\vec{a}$ via the new basis.

I. 1. $\vec{e_1}=(5;7;-2)$, $\vec{e_2}=(-3;1;3)$, $\vec{e_3}=(1;-4;6)$ 2. $P(1;0;-2)$, $Q(3;-1;0)$, $R(2;-1;5)$, $S(0;-2;2)$ 3. $\vec{p}=(5;4;1)$, $\vec{q}=(-3;5;2)$, $\vec{r}=(2;1;-3)$, $\vec{a}=(7;23;4)$	**II.** 1. $\vec{e_1}=(-1;4;3)$, $\vec{e_2}=(3;2;-4)$, $\vec{e_3}=(-2;-7;1)$ 2. $P(2;1;0)$, $Q(-1;3;4)$, $R(0;-1;5)$, $S(2;0;0)$ 3. $\vec{p}=(1;-3;1)$, $\vec{q}=(-2;-4;3)$, $\vec{r}=(0;-2;3)$, $\vec{a}=(-8;-10;13)$
III. 1. $\vec{e_1}=(2;3;-1)$, $\vec{e_2}=(1;-1;3)$, $\vec{e_3}=(1;9;-11)$ 2. $P(1;3;2)$, $Q(5;2;-1)$, $R(2;2;4)$, $S(5;5;6)$ 3. $\vec{p}=(4;2;3)$, $\vec{q}=(-3;1;-8)$, $\vec{r}=(2;-4;5)$, $\vec{a}=(-12;14;-31)$	**IV.** 1. $\vec{e_1}=(-2;1;3)$, $\vec{e_2}=(3;-6;2)$, $\vec{e_3}=(-5;-3;-1)$ 2. $P(1;0;-2)$, $Q(3;-1;0)$, $R(2;-1;5)$, $S(0;-2;2)$ 3. $\vec{p}=(1;3;6)$, $\vec{q}=(-3;4;-5)$, $\vec{r}=(1;-7;2)$, $\vec{a}=(8;47;65)$

2.2 Line, Parabola, Ellipse, Hyperbola

2.2.1 Lines in the plane

A line (i.e., a straight line) is a geometric object. When it is placed in a coordinate plane, the points (in the plane) through which the line passes, satisfy certain geometric conditions. For example, any two distinct points $M_1(x_1;y_1)$ and $M_2(x_2;y_2)$ on the line, determine it completely.

Equations of a line

1. General linear equation

Let $\vec{n} = A\vec{i} + B\vec{j} = (A;B)$ be a nonzero vector and $(x_0;y_0)$ be a point in the OXY plane. There is a unique line passing through the point $(x_0;y_0)$ that is perpendicular to $\vec{n} = (A;B)$, as shown in Fig. 2.1. Vector $\vec{n} = (A;B)$ is called a ***normal*** to the line.

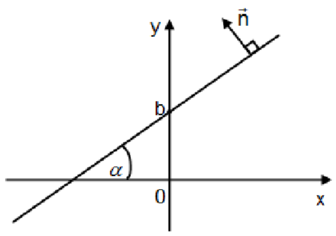


Figure 2.1

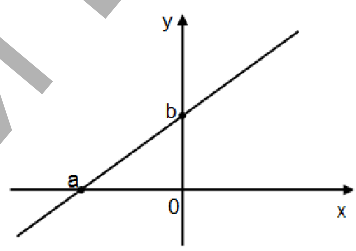


Figure 2.2

Theorem 1 An equation of the line (in the OXY plane) to the nonzero vector $\vec{n} = (A;B)$ is given by:

$$Ax + By + C = 0 \tag{1}$$

where A, B, C are constants, with the condition that both A and B are not zero simultaneously $(A^2 + B^2 \neq 0)$.

2. Slope form of the equation of a line

A line L can be described by the following equation

$$y = kx + b \tag{2}$$

where $k = tg\alpha$ is called the ***slope of the line***; α - the ***angle of inclination*** (or simply inclination) of a line (the smallest positive angle between the line and the $x-$axis); the line makes an intercept b on the $y-$axis (see Fig. 2.1).

3. Point-slope form of the equation of a line

A line L can be described by following equation

$$y - y_0 = k(x - x_0) \tag{3}$$

where the point $M_0(x_0; y_0)$ lies on the line.

4. Two - point equation of a line

A line L can be described by the following equation

$$\frac{y - y_2}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2} \tag{4}$$

where points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ are any two distinct fixed points on the line.

5. Equation of a nonvertical line in the «intercept form»

Let L be any nonvertical line, which makes an intercept «a» on the $x-$axis and an intercept «b» on the $y-$axis ($a \neq 0, b \neq 0$). Equation

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5}$$

is called the ***«intercept form»*** of the equation of a line (see Fig 2.2).

6. Equation of the line passing through the point $M_0(x_0; y_0)$ and perpendicular to $\vec{n} = (A; B)$ has the following form:

$$A(x - x_0) + B(y - y_0) = 0 \tag{6}$$

7. Parametric equation of a line

A line L can be described by the following equations

$$\begin{cases} x = x_0 + mt, \\ y = y_0 + nt. \end{cases} \tag{7}$$

These equations are called ***parametric equations*** of the line L through the point $M_0(x_0; y_0)$ and parallel to the vector $\vec{s} = (m; n)$. Each value of the parameter $t \in R$ gives a point on L.

8. Symmetric equation of a line

Another way of describing a line L is to eliminate the parameter $t \in R$ from Equations 7. If none of m or n is 0, we can solve each of these equations for t , equate the results, and obtain

$$\frac{x-x_0}{m}=\frac{y-y_0}{n} \tag{8}$$

These equations are called ***symmetric equations*** of L. Notice, that the numbers m and n that appear in the denominators of Equations 8 are ***direction numbers*** of L, that is, components of a vector parallel to L.

Relations between the lines

Definition The angle from L_1 to L_2 is the angle φ through which L_1 must be rotated counter clockwise about the point of intersection in order to coincide with L_2.

The following results can be obtained, depending on whether the lines are given by the general linear equation or the slope form of the equation.

	$L_1: A_1x+B_1y+C_1=0$ $L_2: A_2x+B_2y+C_2=0$	$L_1: y=k_1x+b_1$ $L_2: y=k_2x+b_2$
The angle from L_1 to L_2	$\cos\varphi=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}$	$tg\varphi=\frac{k_2-k_1}{1+k_1k_2}$
Perpendicular lines ($L_1 \perp L_2$)	$A_1A_2+B_1B_2=0$	$k_1k_2=-1$ *or* $k_1=-\frac{1}{k_2}$
Parallel lines ($L_1 \parallel L_2$)	$\frac{A_1}{A_2}=\frac{B_1}{B_2}\neq\frac{C_1}{C_2}$	$k_1=k_2$

Theorem 2 The distance from the point $P_1(x_1;y_1)$ to the line L which equation is $Ax+By+C=0$ can be calculated by

$$d(P;L)=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} \tag{9}$$

Exercise Set 10

1. Find the equation of the line passing through $(2;1)$ with the given slope, and sketch the line:

a) $k=0$; b) $k=-3$; c) $k=2/3$.

In Exercises 2 to 5 find the line through the given point and perpendicular to the given vector:

2. $M(2;3),\ \vec{n}=(4;5)$;

3. $M(1;0),\ \vec{n}=(2;-1)$;

4. $M(4;5),\ \vec{n}=(2;3)$;

5. $M(2;-1),\ \vec{n}=(1;3)$.

In Exercises 6 to 9 find a vector perpendicular to the given line:

6. $2x-3y+8=0$;

7. $\pi x-\sqrt{2}y=7$;

8. $y=3x+7$;

9. $2(x-1)+5(y+2)=0$.

10. Find the distance from the point $M(0;0)$ to the line $3x+4y-10=0$.

11. The line passes through the points $A(-1;3)$ and $B(4;5)$. Find the parametric equation of a line.

12. Find the equation of the line passing through the point $A(-2;3)$ and perpendicular to the line $2x-3y+8=0$.

13. Find the equation of the line passing through the point $A(4;-3)$ and forming the triangle with coordinate axes and its area is 3.

14. Find the equation of the line passing through the point $O(0;0)$ and forming the angle 45° with the line $y=2x+5$.

15. The point $A(2;-5)$ is a vertex of a square, one side of this square lies on the line $x-2y-7=0$. Find the area of the square.

16. Find the smallest angle between the lines $3x+4y-2=0$ and $8x+6y+5=0$. Prove that the point $A\left(\frac{13}{14};1\right)$ belongs to the bisector line of this angle.

Let A,B,C be the vertices of the triangle ΔABC. Find: a) the equation of the line AB; b) the equation of the height CH; c) the equation of the median AM; d) the point of intersection of the lines CH and AM; e) the equation of the line passing through the point C parallel to the line AB; f) the distance from the point C to the line AB; g) the equation of the bisector of the inner angle $\angle ABC$; h) the centroid of the triangle ΔABC; i) the area of the triangle ΔABC.

17. $A(2;5), B(-3;1), C(0;4)$

18. $A(-5;1), B(8;-2), C(1;4)$

19. $A(1;-3), B(0;7), C(-2;4)$

20. $A(7;0), B(1;4), C(-8;-4)$

Additional Problems

21. Let $A(3;-1)$ and $B(5;7)$ be the vertices of the triangle ΔABC and its heights intercept at the point $N(4;-1)$. Find the equation of the lines passing through the vertices of the triangle.

22. Find the equation of the bisector of the angle from $L_1: x-3y+5=0$; to $L_2: 3x-y-2=0$

23. The sides of the triangle ABC: AB: $2x+3y-6=0$, AC: $x+2y-5=0$ and the angle $\angle ABC=\dfrac{\pi}{4}$ are given. Find the equation of the height from the vertex A to the side BC.

24. Find the equation of the bisector of that angle between the lines $x-7y-1=0$ and $x+y+7=0$ to which the point $A(1;1)$ belongs.

25. Find the equations of the triangle sides if $A(-4;2)$ is one of its vertices and the lines $3x-2y+2=0$, $3x+5y-12=0$ are its medians.

26. The equations of the parallelogram diagonals $x-2y=0$, $x-y-1=0$ and the point of its intersection $M(3;-1)$ are given. Find the equations of the parallelogram sides.

27. Let $x+3y-8=0$ be the equation of the rhombus side and $2x+y+4=0$ be the equation of its diagonal. Find the equations of the rhombus sides, if the point $M(-9;-1)$ lies at the side parallel to the given.

28. Find the coordinates of the point Q symmetrical to the point $P(1;-9)$ with respect to the line $4x-3y-7=0$.

Individual Tasks 10

1. Let A,B,C be the vertices of the triangle ΔABC. Find: a) the equation of the line AB; b) the equation of the height CH; c) the equation of the median AM; d) the point of intersection of the lines CH and AM; e) the equation of the line passing through the point C parallel to the line AB; f) the distance from the point C to the line AB; g) the equation of the bisector of the inner angle $\angle ABC$; h) the centroid of the triangle ΔABC; i) the area of the triangle ΔABC.

2. Find the coordinates of the point Q symmetrical to the point $P_1(x_1;y_1)$ with respect to the line $Ax+By+C=0$.

I.	**II.**
1. $A(-1;7), B(3;-1), C(1;6)$	1. $A(3;9), B(-3;-1), C(-2;4)$

2. $P(3;-1)$, $8x-6y+5=0$	2. $P(2;-3)$, $5x+12y-3=0$
III. 1. $A(-2;4), B(3;1), C(10;7)$ 2. $P(3;-1)$, $2x-6y+1=0$	**IV.** 1. $A(-3;-2), B(14;4), C(6;8)$ 2. $P(-1;-3)$, $5x+2y-3=0$

2.2.2 Parabolas

Definition A ***parabola*** is the set of points in the plane that are equidistant from a fixed line (the ***directrix***) and a fixed point (the ***focus***) which are not on the directrix.

The line that passes through the focus and is perpendicular to the directrix is called the ***axis of symmetry*** of the parabola. The midpoint of the line segment between the focus and the directrix on the axis of symmetry is the ***vertex*** of the parabola (see Figure 2.3).

Using the definition of parabola, we can determine the equation of a parabola. Suppose that the coordinates of the vertex of a parabola are $(0;0)$ and the axis of symmetry is the x-axis. The equation of the directrix is $x=-p/2, p>0$. The focus lies on the axis of symmetry and is the same distance from the vertex as the vertex is from the directrix. Thus the coordinates of the focus are $(p/2;0)$.

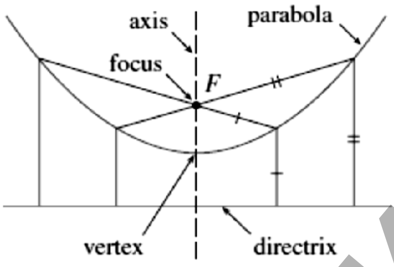


Figure 2.3

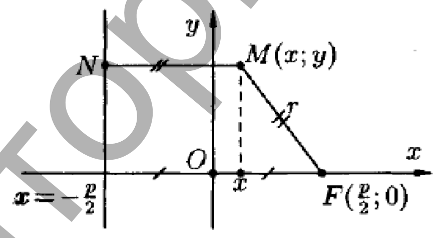


Figure 2.4

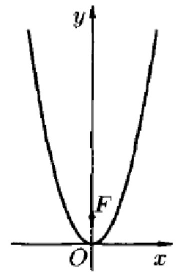


Figure 2.5

Let $M(x;y)$ be any point on the parabola. Then, using the distance formula and the fact, that the distance between any point on the parabola and the focus is equal to the distance from the point M to the directrix, we can write the equation $d(M,F)=d(M,N)$ (see Figure 2.4).

By the distance formula, $\sqrt{\left(x-\frac{p}{2}\right)^2+(y-0)^2}=x+\frac{p}{2}$.

Now squaring each side and simplifying,

$$y^2=2px \qquad (1)$$

This is ***an equation of a parabola with a vertex at the origin and a horizontal axis***

of symmetry. The equation of a parabola with a vertical axis of symmetry is derived in a similar manner (see Figure 2.5).

Standard Form of the Equation of a Parabola with a Vertex at $(a;b)$

1. Vertical Axis of Symmetry. The standard form of the equation of a parabola with a vertex $(a;b)$ and a vertical axis of symmetry is $(x-a)^2 = 2p(y-b)$. The focus is $\left(a;b+\dfrac{p}{2}\right)$, and the equation of the directrix is $y = b - p/2$.

2. Horizontal Axis of Symmetry. The standard form of the equation of a parabola with a vertex $(a;b)$ and a horizontal axis of symmetry is $(y-b)^2 = 2p(x-a)$. The focus is $\left(a+\dfrac{p}{2};b\right)$, and the equation of the directrix is $x = a - p/2$.

Example 1 Find the equation of the directrix and the coordinates of the vertex and focus of the parabola given by the equation $3x - 2y^2 + 8y - 4 = 0$.

Solution Rewrite the equation and then complete the square.

$$3x + 2y^2 + 8y - 4 = 0$$
$$2y^2 + 8y = -3x + 4$$
$$2\left(y^2 + 4y\right) = -3x + 4$$

$2\left(y^2 + 4y + 4\right) = -3x + 4 + 8$	Complete the square. Note, that 8 is added to each side.
$2(y+2)^2 = -3(x-4)$	
$(y+2)^2 = -\dfrac{3}{2}(x-4)$	Write the equation in a standard form.

Comparing this equation to $(y-b)^2 = 2p(x-a)$, we have a parabola that opens to the left with the vertex $(4;-2)$ and $2p = -3/2$. Thus, $p = -3/4$. The coordinates of the focus are $\left(4+\left(\dfrac{-3}{4}\right);-2\right) = \left(\dfrac{13}{4};-2\right)$. The equation of the directrix is $x = 4 - \left(\dfrac{-3}{4}\right) = \dfrac{19}{4}$.

Choosing some values for y and finding the corresponding values for x, we plot a few points. We use the fact that the line $y = -2$ is the axis of symmetry. Thus, for a point on one side of the axis of symmetry, there is a corresponding point on the other side. Two points are $(-2;1)$ and $(-2;-5)$.

Exercise Set 11

In Exercises 1 to 20, find the vertex, the focus and the directrix of the parabola given by each equation. Sketch the graph.

1.	$x^2 = -4y$	***2.***	$2y^2 = x$
3.	$(x-2)^2 = 8(y+3)$	***4.***	$(y+1)^2 = 6(x-1)$
5.	$(y+4)^2 = -4(x-2)$	***6.***	$(x-3)^2 = -(y+2)$
7.	$(y-1)^2 = 2x+8$	***8.***	$(x+2)^2 = 3y-6$
9.	$(2x-4)^2 = 8y-16$	***10.***	$x^2+8x-y+6=0$
11.	$(3x+6)^2 = 18y-36$	***12.***	$x^2-6x+y+10=0$
13.	$2x-y^2-6y+1=0$	***14.***	$3x+y^2+8y+4=0$
15.	$2x^2-8x-4y+3=0$	***16.***	$6x-3y^2-12y+4=0$
17.	$2x+4y^2+8y-5=0$	***18.***	$4x^2-12x+12y+7=0$
19.	$3x^2-6x-9y+4=0$	***20.***	$6x-3y^2+9y+5=0$

In Exercises 21 to 26 find an equation for the parabola that satisfies the given condition(s):

21. With the vertex at the origin and the focus $(0;-4)$.

22. With the vertex at the origin and the focus $(5;0)$.

23. With the vertex at $(-1;2)$ and the focus $(-1;3)$.

24. With the vertex at $(2;-3)$ and the focus $(0;-3)$.

25. With the vertex $(-4;1)$, the axis of symmetry parallel to the y-axis, and passing through the point $(-2;2)$.

26. With the vertex $(3;-5)$, the axis of symmetry parallel to the x-axis, and passing through the point $(4;3)$.

In Exercises 27 to 29, use the following definition of latus rectum: the line segment with endpoints on the parabola, through the focus of a parabola and perpendicular to the axis of symmetry is called the ***latus rectum*** of the parabola.

27. Find the length of the latus rectum for the parabola $x^2 = 4y$.

28. Find the length of the latus rectum for the parabola $y^2 = -8x$.

29. Find the length of the latus rectum for any parabola in terms of $|p|$, the distance from the vertex of the parabola to the focus.

30. Show that the point on the parabola closest to the focus is the vertex. (Hint: Consider the parabola $x^2 = 2py$ and a point on the parabola $(a;b)$. Find the square of the distance between the point $(a;b)$ and the focus. You may want to review the technique of minimizing a quadratic expression.)

31. By using the definition for a parabola, find the equation in standard form of the parabola with $V(0;0)$, $F(-c;0)$ and the directrix $x = c$.

32. Sketch a graph of $4(y-2) = (x|x|-1)$.

Individual Tasks 11

1. Find the vertex, the focus, and the directrix of the parabola given by each equation. Sketch the graph.

2. Identify the graph of the equation as a parabola. Sketch the graph.

3. Find an equation for the parabola that satisfies the given condition(s).

I.	**II.**
1. $x + y^2 - 3y + 4 = 0$	1. $x - y^2 - 4y + 9 = 0$
2. $y = -5 + \sqrt{-3x - 21}$	2. $x = -4 + 3\sqrt{y+5}$
3. With the focus $(3;-3)$ and the directrix $y = -5$.	3. With the vertex $(3;-5)$, the axis of symmetry parallel to the x-axis, and passing through the point $(4;3)$.
4. $x^2 + (y-1)^2 = 4$	4. $4x^2 - (y-1)^2 = 16$
5. $3x - 4y^2 + 8y + 2 = 0$	5. $3x + 2y^2 - 4y - 7 = 0$
6. $4x^2 - 9y^2 - 8x + 12y - 144 = 0$	6. $x^2 + 4y^2 - 6x + 8y - 3 = 0$
7. $x^2 + y^2 - 6x + 8y + 25 = 0$	7. $x^2 - y^2 - 6x - 8y - 7 = 0$
III.	**IV.**
1. $x^2 + 3x + 3y - 1 = 0$	1. $x^2 + 5x - 4y - 1 = 0$
2. $x = 3 + \sqrt{-y-2}$	2. $y = -1 + 2\sqrt{x-1}$
3. With the vertex $(-4;1)$, the axis of symmetry parallel to the y-axis, and passing through the point $(-2;2)$.	3. With the focus $(-2;4)$ and the directrix $x = 4$.
4. $y^2 = 16(x+3)$	4. $4x^2 + (y-1)^2 = 16$

5. $9x^2+4y^2+36x-8y+4=0$ 6. $11x^2-25y^2-44x-50y-256=0$ 7. $x^2+y^2-6x+8y+29=0$	5. $3x^2-4y^2+12x-24y-36=0$ 6. $4x^2+28x+32y+81=0$ 7. $x^2+y^2+4x-8y+20=0$

2.2.3 Ellipses

Definition An ***ellipse*** is a set of all points in the plane, and the sum of its distances from two fixed points (foci) is a positive constant (see Figure 2.6).

This definition can be used to draw an ellipse using a piece of string and two. Tack the ends of the string to the foci, and trace a curve with a pencil held tight against the string. The resulting curve is an ellipse. The positive constant is the length of the string.

Ellipse with the center at $(0;0)$

The graph of an ellipse is oval-shaped, with two axes of symmetry. The longer axis is called the ***major axis***. The foci of the ellipse are on the major axis. The shorter axis is called the ***minor axis***. It is customary to denote the length of the major axis as $2a$ and the length of the minor axis as $2b$. The length of the ***semiaxes*** is one-half the axes. Thus the length of the ***semi major axis*** is denoted by a and the length of the ***semi minor axis*** by b. The ***center*** of the ellipse is the midpoint of the major axis. The endpoints of the major axis are the ***vertices*** (plural of vertex) of the ellipse.

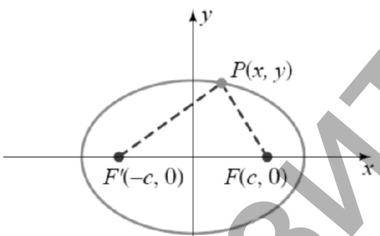


Figure 2.6

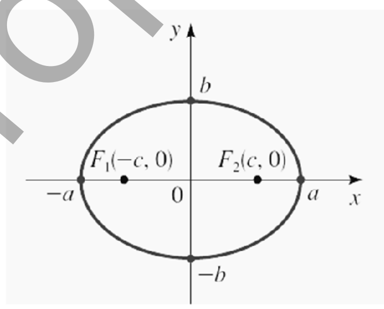


Figure 2.7

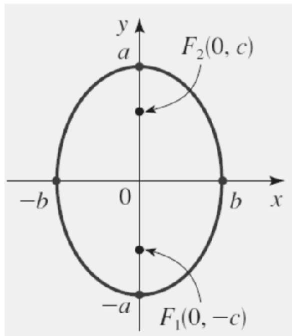


Figure 2.8

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

The Equation 1is called the equation of the ellipse in the ***standard form***.

Standard forms of the equation of an ellipse with the center at the origin

1. Major Axis on the x-axis. The standard form of the equation of an ellipse with the center at the origin and the major axis on the x-axis is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ , } a>b .$$

The coordinates of the vertices are $(a;0)$ and $(-a;0)$, and the coordinates of the foci are $(c;0)$ and $(-c;0)$, where $c^2=a^2-b^2$ (see Figure 2.7).

2. Major Axis on the y-axis. The standard form of the equation of an ellipse with the center at the origin and the major axis on the y-axis is given by

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \ a>b .$$

The coordinates of the vertices are $(0;a)$ and $(0;-a)$, and the coordinates of the foci are $(0;c)$ and $(0;-c)$, where $c^2=a^2-b^2$(see Figure 2.8).

Standard form of the equation of an ellipse with the center at $(x_0;y_0)$

1. Major Axis Parallel to the x-axis. The standard form of the equation of an ellipse with the center at $(x_0;y_0)$ and the major axis parallel to the *x*-axis is given by

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \text{ , } a>b .$$

2. Major Axis Parallel to the y-axis.The standard form of the equation of an ellipse with the center at $(x_0;y_0)$ and the major axis parallel to the *y*-axis is given by

$$\frac{(x-x_0)^2}{b^2}+\frac{(y-y_0)^2}{a^2}=1 \text{ , } a>b .$$

Eccentricity of an Ellipse

The graph of an ellipse can be very long and thin, or it can be much like a circle. The eccentricity of an ellipse is a measure of its « roundness».

Definition The ***eccentricity*** e of an ellipse is the ratio of c to a, where c is the distance from the center to the focus and a is the length of the semi major axis.

$$e=\frac{c}{a} \tag{3}$$

Because $c<a$, for an ellipse, $0<e<1$. If $c\approx 0$, then $e\approx 0$ and the graph will be almost like a circle.

If $c\approx a$, then $e\approx 1$ and the graph will be long and thin. In Figure 2.9 we show a number of ellipses to demonstrate the effect of varying the eccentricity e.

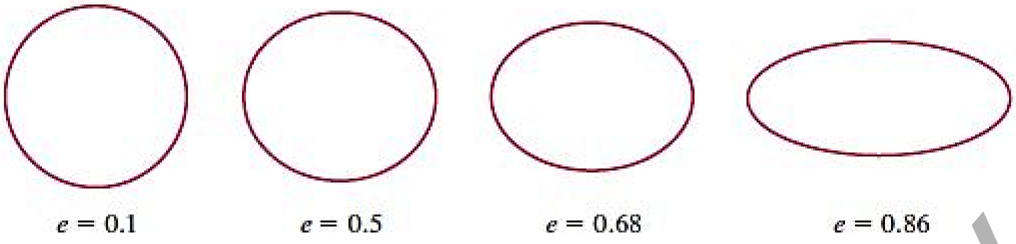


Figure 2.9

Recall that a parabola has a directrix that is a line perpendicular to the axis of symmetry. An ellipse has two ***directrixes***, both of which are perpendicular to the major axis and outside the ellipse. For an ellipse with the center at the origin and which major axis is the x-axis, the equations of the directrixes are $x = \pm a^2 / c$.

Exercise Set 12

In Exercises 1 to 16, find the vertices and foci of the ellipse given by each equation. Sketch the graph.

1. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

2. $\dfrac{x^2}{49} + \dfrac{y^2}{36} = 1$

3. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

4. $\dfrac{x^2}{64} + \dfrac{y^2}{25} = 1$

5. $3x^2 + 4y^2 = 12$

6. $5x^2 + 4y^2 = 20$

7. $25x^2 + 16y^2 = 400$

8. $25x^2 + 12y^2 = 300$

9. $64x^2 + 25y^2 = 400$

10. $9x^2 + 64y^2 = 144$

11. $5x^2 + 9y^2 - 20x + 54y + 56 = 0$

12. $9x^2 + 16y^2 + 36x - 16y - 104 = 0$

13. $16x^2 + 9y^2 - 64x - 80 = 0$

14. $16x^2 + 9y^2 + 36y - 108 = 0$

15. $25x^2 + 16y^2 + 50x - 32y - 359 = 0$

16. $16x^2 + 9y^2 - 64x - 54y + 1 = 0$

In Exercises 17 to 26, find the equation in the standard form of each ellipse, given the information provided.

17. The center $(0;0)$, the major axis of length 10, foci at $(4;0)$ and $(-4;0)$.

18. The center $(0;0)$, the minor axis of length 6, foci at $(0;4)$ and $(0;-4)$.

19. Vertices $(6;0)$, $(-6;0)$; the ellipse passes through $(0;4)$ and $(0;-4)$.

20. Vertices $(5;0)$, $(-5;0)$, the ellipse passes through $(0;7)$ and $(0;-7)$.

21. The major axis of length 12 on the x-axis, the center at $(0;0)$, and passing through $(2;-3)$.

22. The minor axis of length 8, the center at $(0;0)$, and passing through $(-2;2)$.

23. The center $(-2;4)$, vertices $(-6;4)$ and $(2;4)$, foci $(-5;4)$ and $(1;4)$.

24. The center $(0;3)$, the minor axis of length 4, foci $(0;0)$ and $(0;6)$.

25. The center $(2;4)$, the major axis parallel to the y-axis and of length 10, the ellipse passes through the point $(3;3)$.

26. The center $(-4;1)$, the minor axis parallel to the y-axis of length 8 and the ellipse passes through the point $(0;4)$.

In Exercises 27 to 34, use the eccentricity of the ellipse to find the equation in the standard form of each of the following ellipse.

27. Eccentricity 2/5, the major axis on the x-axis of length 10, and the center at $(0;0)$.

28. Eccentricity 3/4, foci at $(-9;0)$.

29. Eccentricity 2/5, foci $(-1;3)$ and $(3;3)$.

30. Eccentricity 1/4, foci $(-2;4)$ and $(-2;-2)$.

31. Eccentricity 2/3, the major axis of length 24 on the y-axis, the center at $(0;0)$.

32. Eccentricity 3/5, the major axis of length 15 on the x-axis, the center at $(0;0)$.

33. Explain why the graph of the equation $4x^2+9y^2-8x+36y+76=0$ is or is not an ellipse.

34. Explain why the graph of the equation $4x^2+9y-16x-2=0$ is or is not an ellipse. Sketch the graph of this equation.

In Exercises 35 and 37 find the latus rectum of the given ellipse. The line segment with endpoints on the ellipse that is perpendicular to the major axis and passes through the focus is the ***latus rectum*** of the ellipse.

35. Find the length of the latus rectum of the ellipse given by $\dfrac{(x-1)^2}{9}+\dfrac{(y+1)^2}{16}=1$.

36. Find the length of the latus rectum of the ellipse given by $9x^2+16y^2-36x+96y+36=0$.

37. Show that for any ellipse, the length of the latus rectum is $2b^2/a$.

38. Let $P(x;y)$ be a point on the ellipse $\dfrac{x^2}{12}+\dfrac{y^2}{8}=1$. Show that the distance from the point P to the focus $(2;0)$ divided by the distance from the point P to the directrix $x=6$ equals the eccentricity.

39. Let $P(x;y)$ be a point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. Show that the distance from the point P to the focus $(3;0)$ divided by the distance from the point to the directrix $x=25/3$ equals the eccentricity.

Individual Tasks 12

1. Find the vertex, the focus, and the directrix of the ellipse given by each equation. Sketch the graph.

2. Identify the graph of equation as an ellipse. Sketch the graph.

3. Find an equation for the ellipse that satisfies the given condition(s).

I. 1. $8x^2+25y^2-48x+50y+47=0$ 2. $y=1-\frac{4}{3}\sqrt{-6x-x^2}$ 3. Foci at $(0;-4)$ and $(0;4)$, eccentricity 2/3.	**II.** 1. $x^2+9y^2+6x-36y+36=0$ 2. $x=-2+\sqrt{-5-6y-y^2}$ 3. Vertices at $(-7;-1)$ and $(5;-1)$, foci at $(-5;-1)$ and $(3;-1)$.
III. 1. $4x^2+y^2-24x-8y+48=0$ 2. $y=-2+\sqrt{6+4x-2x^2}$ 3. Vertices at $(5;6)$ and $(5;-4)$, foci at $(5;4)$ and $(5;-2)$.	**IV.** 1. $4x^2+9y^2+24x+18y+44=0$ 2. $x=3+\sqrt{2-3y^2+6y}$ 3. Foci at $(0;-3)$ and $(0;3)$, eccentricity $1/4$.

2.2.4 Hyperbolas

Definition A ***hyperbola*** is a set of all points in the plane, the difference of its distances from two fixed points (foci) is a positive constant (see Figure 2.10).

Hyperbolas with the center at $(0;0)$

The ***transverse axis*** is the line segment joining the intercepts through the foci of a hyperbola. The midpoint of the transverse axis is called the ***center*** of the hyperbola. The ***conjugate axis*** passes through the center of the hyperbola and is perpendicular to the transverse axis. The hyperbola consists of two parts called its ***branches***.

The length of the transverse axis is customarily denoted by $2a$, and the distance between the two foci is denoted by $2c$. The length of the conjugate axis is denoted by $2b$.

The ***vertices*** of a hyperbola are the points where the hyperbola intersects the transverse axis.

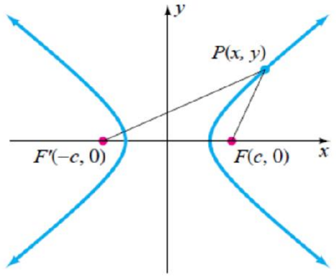


Figure 2.10

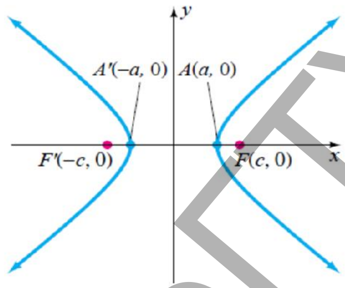


Figure 2.11

Standard forms of the equation of a hyperbola with the center at the origin

1. Transverse axis on the x-axis. The standard form of the equation of a hyperbola with the center at the origin and the transverse axis on the x-axis is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (1)$$

The coordinates of the vertices are $(a;0)$ and $(-a;0)$, the coordinates of the foci are $(c;0)$ and $(-c;0)$, where $c^2=a^2+b^2$ (see Figure 2.11).

2. Transverse axis on the y-axis. The standard form of the equation of a hyperbola with the center at the origin and the transverse axis on the y-axis is given by

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \qquad (2)$$

The coordinates of the vertices are $(0;a)$ and $(0;-a)$, the coordinates of the foci are $(0;c)$ and $(0;-c)$, where $c^2=a^2+b^2$.

Remark By looking at the equations, note that it is possible to determine the transverse axis by finding which term in the equation is positive. If the x^2 term is positive, then the transverse axis is on the x-axis. When the y^2 term is positive, the transverse axis is on the y-axis.

The asymptotes of the hyperbola are a useful guide to sketching the graph of the hyperbola. Each hyperbola has two asymptotes that pass through the center of the hyperbola.

Asymptotes of a Hyperbola with the Center at the Origin

The ***asymptotes*** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by the equations $y=\pm\frac{b}{a}x$.
The asymptotes of the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are given by the equations $y=\pm\frac{a}{b}x$.

Remark One method for remembering the equations of the asymptotes is to write the equation of a hyperbola in the standard form but replace 1 by 0 and then solve for y.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$y^2=\frac{b^2}{a^2}x^2$ or $y=\pm\frac{b}{a}x$
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=0$	$y^2=\frac{a^2}{b^2}x^2$ or $y=\pm\frac{a}{b}x$

To sketch the graph, we draw a rectangle with its center at the origin that has dimensions equal to the lengths of the transverse and conjugate axes. The asymptotes are extensions of the diagonals of the rectangle (see Figure 2.12).

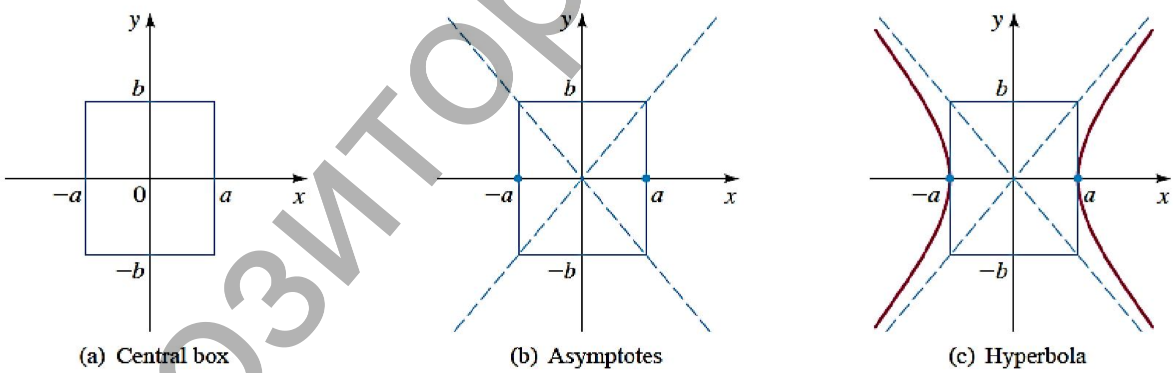


Figure 2.12

Standard form of hyperbolas with the center at $(x_0;y_0)$

1. Transverse Axis Parallel to the x-axis. The standard form of the equation of a hyperbola with the center $(x_0;y_0)$ and the transverse axis parallel to x-axis is given by

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1.$$

2. Transverse Axis Parallel to the y-axis. The standard form of the equation of a hyperbola with the center $(x_0;y_0)$ and The center transverse axis parallel to the y-axis is given by

$$\frac{(y-y_0)^2}{a^2}-\frac{(x-x_0)^2}{b^2}=1.$$

Example 1 Find the vertices, foci, and asymptotes of the hyperbola given by the equation $4x^2-9y^2-16x+54y-29=0$.

Solution Write the equation of the hyperbola in the standard form by completing the square.

$$4x^2-9y^2-16x+54y-29=0$$

$4x^2-16x-9y^2+54y=29$	Rearrange terms
$4(x^2-4x)-9\left(y^2-6y\right)=29$	Factor
$4\left(x^2-4x+4\right)-9\left(y^2-6y+9\right)=29+16-81$	Complete the square
$4(x-2)^2-9(y-3)^2=-36$	Factor
$\frac{(y-3)^2}{4}-\frac{(x-2)^2}{9}=1$	Divide by -36

The coordinates of the center are (2; 3). Because the term containing $(y-3)^2$ is positive, the transverse axis is parallel to the y-axis. We know $a^2=4$; thus $a=2$. The vertices are (2; 5) and (2; 1). $c^2=a^2+b^2=4+9\Rightarrow c=\sqrt{13}$. The foci are $\left(2;3+\sqrt{13}\right)$ and $\left(2;3-\sqrt{13}\right)$. We know $b^2=9$, thus $b=3$. The equations of the asymptotes are $y=\frac{2}{3}x+\frac{5}{3}$ and $y=-\frac{2}{3}x+\frac{13}{3}$.

Eccentricity of a hyperbola

The graph of a hyperbola can be very wide or very narrow. The ***eccentricity*** of a hyperbola is a measure of its "wideness".

Definition The ***eccentricity*** e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is the length of the semi-transverse axis $e=c/a$.

For a hyperbola, $c>a$ and therefore $e>1$. As the eccentricity of the hyperbola increases, the graph becomes wider and wider.

A hyperbola has two ***directrixes*** that are perpendicular to the transverse axis and outside the hyperbola. For a hyperbola with the center at the origin and the transverse axis on the x-axis, the equations of the directrixes are $x=\pm a^2/c$.

Exercise Set 13

In Exercises 1 to 20, find the center, vertices, foci, and asymptotes for the hyperbola given by each equation. Sketch the graph.

1. $\dfrac{x^2}{16}-\dfrac{y^2}{25}=1$

2. $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$

3. $\dfrac{y^2}{4}-\dfrac{x^2}{25}=1$

4. $\dfrac{y^2}{25}-\dfrac{x^2}{36}=1$

5. $\dfrac{(x-3)^2}{16}-\dfrac{(y+4)^2}{9}=1$

6. $\dfrac{(x+3)^2}{35}-\dfrac{y^2}{4}=1$

7. $\dfrac{(y+2)^2}{4}-\dfrac{(x-1)^2}{16}=1$

8. $\dfrac{(y-2)^2}{36}-\dfrac{(x+1)^2}{49}=1$

9. $x^2-y^2=9$

10. $4x^2-y^2=16$

11. $16y^2-9x^2=144$

12. $9y^2-25x^2=225$

13. $9y^2-36x^2=4$

14. $16x^2-25y^2=9$

15. $x^2-y^2-6x+8y-3=0$

16. $4x^2-25y^2+16x+50y-109=0$

17. $4x^2-y^2+32x+6y+39=0$

18. $x^2-16y^2+8x-64y+16=0$

19. $4x^2-9y^2+8x-18y-6=0$

20. $2x^2-9y^2-8x+36y-46=0$

In Exercises 21 to 32, find the equation in the standard form of the hyperbola satisfying the stated conditions.

21. Vertices at $(3;0)$ and $(-3;0)$, foci at $(4;0)$and $(-4;0)$.

22. Vertices at $(0;2)$ and $(0;-2)$ foci at $(0;3)$and $(0;-3)$.

23. Foci at $(0;5)$ and $(0;-5)$ asymptotes $y=2x$.

24. Foci at$(4;0)$ and $(-4;0)$, asymptotes $y=\pm x$.

25. Vertices at$(0;3)$ and $(0;-3)$, passing through $(2;4)$.

26. Vertices at$(5;0)$ and $(-5;0)$, passing through $(-1;3)$.

27. Vertices at$(6;3)$ and $(2;3)$, foci at$(7;3)$ and $(1;3)$.

28. Vertices at$(-1;5)$ and $(-1;-1)$, foci at$(-1;7)$ and $(-1;-3)$.

29. Foci at$(1;-2)$ and $(7;-2)$, slope of an asymptote $5/4$.

30. Foci at$(-3;-6)$ and $(-3;-2)$, slope of an asymptote 1.

31. Passing through $(9;4)$, slope of an asymptote $1/2$, center at $(7;2)$, transverse axis parallel to the y-axis.

32. Passing through $(6;1)$, slope of an asymptote 2, center at $(3;3)$, transverse axis parallel to the x-axis.

In Exercises 33 to 36, use the eccentricity to find the equation in the standard form of a hyperbola.

33. Vertices at $(1;6)$ and $(1;8)$, eccentricity 2.

34. Vertices at $(2;3)$ and $(-2;3)$, eccentricity $5/2$.

35. The center at $(4;1)$, conjugate axis length 4, eccentricity $4/3$.

36. The center at $(-3;-3)$, conjugate axis length 6, eccentricity 2.

In Exercises 37 to 56 identify the graph of each equation as a parabola, ellipse, or hyperbola. Sketch the graph.

37. $x = -4 + 3\sqrt{y+5}$

38. $y = 7 - \frac{3}{2}\sqrt{x^2 - 6x + 13}$

39. $x = 5 - \frac{3}{4}\sqrt{y^2 + 4y - 12}$

40. $y = -5 + \sqrt{-3x - 21}$

41. $y = 1 - \frac{4}{3}\sqrt{-6x - x^2}$

42. $x = -2 + \sqrt{-5 - 6y - y^2}$

43. $y = -7 + \frac{2}{5}\sqrt{-x^2 + 6x + 16}$

44. $x = -5 + \frac{2}{3}\sqrt{8 + 2y - y^2}$

45. $y = 3 - 4\sqrt{x-1}$

46. $x = 2 - \sqrt{6 - 2y}$

47. $x = -5\sqrt{-y}$

48. $y = \frac{2}{5}\sqrt{x^2 + 25}$

49. $4x^2 + 9y^2 - 16x - 36y + 16 = 0$.

50. $2x^2 + 3y - 8x + 2 = 0$.

51. $5x - 4y^2 + 24y - 11 = 0$.

52. $9x^2 - 25y^2 - 18x + 50y = 0$.

53. $x^2 + 2y - 8x = 0$.

54. $9x^2 + 16y^2 + 36x - 64y - 44 = 0$.

55. $25x^2 + 9y^2 - 50x - 72y - 56 = 0$.

56. $(x-3)^2 + (y-4)^2 = (x+1)^2$.

57. Let $P(x;y)$ be a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Show that the distance from the point P to the focus $(5;0)$ divided by the distance from the point P to the directrix $x = 9/5$ equals the eccentricity.

58. Let $P(x;y)$ be a point on the hyperbola $\frac{x^2}{7}-\frac{y^2}{9}=1$. Show that the distance from the point P to the focus $(4;0)$ divided by the distance from the point to the directrix $x=7/4$ equals the eccentricity.

Individual Tasks 13

1. Find the vertex, the focus, and the directrix of the hyperbola given by each equation. Sketch the graph.

2. Identify the graph of equation as a hyperbola. Sketch the graph.

3. Find an equation for the hyperbola that satisfies the given condition(s).

4-7. Determine the type of the curve. Sketch the graph.

I. 1. $9x^2-4y^2+36x-8y+68=0$ 2. $y=-1+\frac{2}{3}\sqrt{x^2-4x-5}$ 3. Eccentricity 2, foci at $(4;0)$ and $(-4;0)$.	**II.** 1. $2x^2-9y^2+12x-18y+18=0$ 2. $x=9-2\sqrt{y^2+4y+8}$ 3. Asymptotes $y=\pm\frac{2}{3}x$, vertices at $(6;0)$ and $(-6;0)$.
III. 1. $9x^2-16y^2-36x-64y+116=0$ 2. $y=2+\frac{2}{3}\sqrt{x^2-6x+10}$ 3. Asymptotes $y=\pm\frac{1}{2}x$, vertices at $(0;4)$ and $(0;-4)$.	**IV.** 1. $16x^2-9y^2-32x-54y+79=0$ 2. $x=-3-2\sqrt{y^2+6y+10}$ 3. Eccentricity $4/3$, foci at $(0;6)$ and $(0;-6)$.

2.2.5 Parametric Equations and Polar Coordinates

So far we have described plane curves by giving y as a function of x $(y=f(x))$ or x as a function of y or by giving a relation between x and y that define y implicitly as a function of x $(f(x,y)=0)$. In this chapter we discuss two new methods for describing curves.

Some curves, such as the ***cycloid***, are best handled when both x and y are given in terms of a third variable t called a parameter $(x=f(t), y=g(t))$. Other curves, such as the ***cardioid***, have their most convenient description when we use a new coordinate system, called the ***polar coordinate system***.

Curves defined by parametric equations

Imagine that a particle moves along the curve C shown in Figure 2.13. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

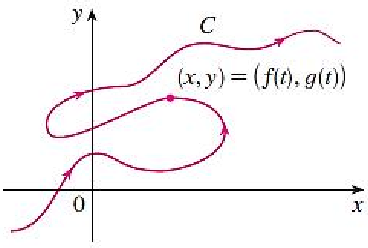


Figure 2.13

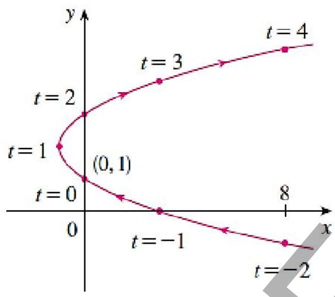


Figure 2.14

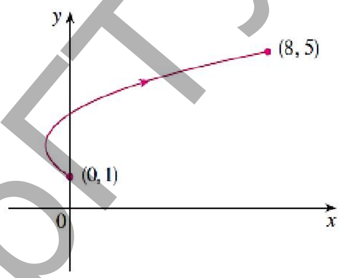


Figure 2.15

Suppose that x and y are both given as functions of a third variable t (called a ***parameter***) by the equations

$$x = f(t),\ \ y = g(t) \qquad (1)$$

(called ***parametric equations***). Each value of t determines a point $(x; y)$, which we can plot in a coordinate plane. As t varies, the point $(x; y) = \left(f(t); g(t)\right)$ varies and traces out a curve C , which we call a ***parametric curve***. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x; y) = \left(f(t); g(t)\right)$ as the position of a particle at time t.

Example 1 Sketch and identify the curve defined by the parametric equations

$x = t^2 - 2t,\ y = t + 1$

Solution Each values of t gives a point on the curve, as shown in the table. For instance, if $t = 0$, then $x = 0, y = 1$, and so the corresponding point is $(0;1)$. In Figure 2.14 we plot the points $(x; y)$ determined by several values of the parameter and we join them to produce a curve.

t	-2	-1	0	1	2	3	4
x	8	3	0	-1	0	3	8
y	-1	0	1	2	3	4	5

A particle which position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter t as follows. We obtain $t = y - 1$ from the second equation and substitute it into the first equation. This gives

$$x = t^2 - 2t = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$$

and so the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number. But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t, y = t + 1, 0 \le t \le 4$$

shown in Figure 2.15 is the part of the parabola in Example 1 that starts at the point $(0;1)$ and ends at the point $(8;5)$. The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve with parametric equations

$$x = f(t),\ y = g(t),\ a \le t \le b$$

has the initial point $(f(a); g(a))$ and the terminal point $(f(b); g(b))$.

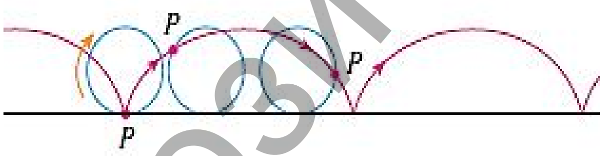


Figure 2.16

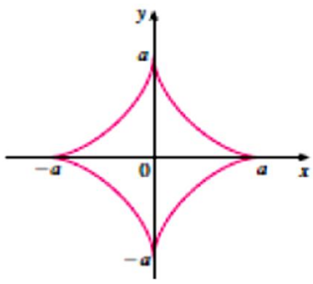


Figure 2.17

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a ***cycloid*** (see Figure 2.16). If the circle has a radius r and rolls along the x-axis and if one position of P is the origin, parametric equations for the cycloid are

$$x = r(t - \sin t), y = r(1 - \cos t) \tag{2}$$

The curve defined by the parametric equations

$$x = a\cos^3 t, y = a\sin^3 t \qquad (3)$$

is called a ***hypocycloid of four cusps***, or an ***asteroid*** (See Figure 2.17).

Polar coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the ***polar coordinate system***, which is more convenient for many purposes.

We choose a point in the plane that is called the ***pole*** (or origin) and is labeled as O. Then we draw a ray (half-line) starting at O called the ***polar axis***. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

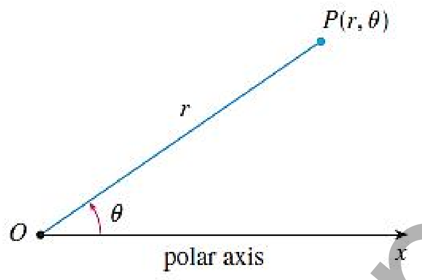


Figure 2.18

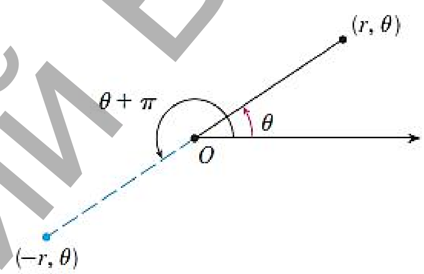


Figure 2.19

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 2.18. Then the point P is represented by the ordered pair $(r;\theta)$ and numbers r, θ are called ***polar coordinates*** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0;\theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates $(r;\theta)$ to the case in which r is negative by agreeing that, as in Figure 2.19, the points $(-r;\theta)$ and $(r;\theta)$ lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point $(r;\theta)$ lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that point $(-r;\theta)$ represents the same point as $(r;\theta + \pi)$.

The connection between polar and Cartesian coordinates can be seen from Figure 2.20, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates $(x;y)$ and polar coordinates $(r;\theta)$, then, from the figure, we have

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases} \qquad (4)$$

Although Equations 4 were deduced from Figure 2.20, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ.

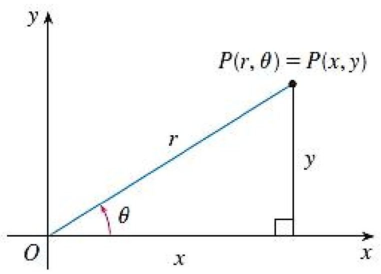


Figure 2.20

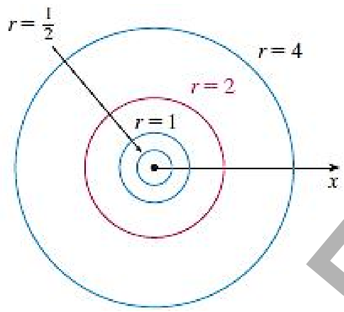


Figure 2.21

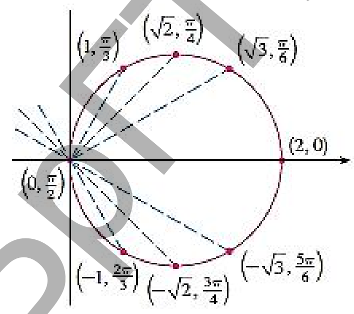


Figure 2.22

Equations 4 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

$$\begin{cases} r^2 = x^2 + y^2, \\ tg\theta = y/x. \end{cases} \qquad (5)$$

The ***graph of a polar equation*** $r = f(\theta)$ or more generally $F(r,\theta) = 0$, consists of all points P that have at least one polar representation whose coordinates $(r;\theta)$ satisfy the equation.

Example 2 (a) Sketch the curve with the polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

Solution

(a) In Figure 2.22 we find the values of r for some convenient values of θ and plot the corresponding points $(r;\theta)$. Then we join these points to sketch the curve, which appears to be a circle. We have used only the values of θ between 0 and π, since if we let θ increase beyond π, we obtain the same points again.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 2\cos\theta$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	-$\sqrt{2}$	-$\sqrt{3}$	-2

(b) To convert the given equation to a Cartesian equation we use Equations 4 and Equations 5. From $x = r\cos\theta$ we have $\cos\theta = x/r$, so the equation $r = 2\cos\theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \Rightarrow x^2 - 2x + y^2 = 0$$

Completing the square, we obtain

$$(x-1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1;0)$ and radius 1.

Note The curve $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$ is called ***cardioid***, because it's shaped like a heart (See Figure 2.23 $r = 1 + \sin\theta$).

The curve $r = a\sin n\theta$, $n \in N$ is called the ***n-leaved rose*** (See Figure 2.24 $r = a\cos 2\theta$).

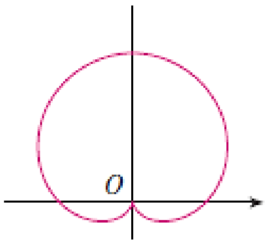


Figure 2.23

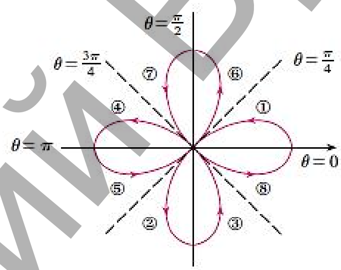


Figure 2.24

Exercise Set 14

In Exercises 1 to 4 plot the point whose polar coordinates are given. Find the Cartesian coordinates of the point.

1. $M_1\left(2;\frac{\pi}{6}\right)$ ***2.*** $M_2\left(1;\frac{3\pi}{4}\right)$ ***3.*** $M_3\left(3;\frac{5\pi}{4}\right)$ ***4.*** $M_4\left(2;\frac{5\pi}{6}\right)$

In Exercises 5 to 12 sketch the curve with the given polar equation. Identify the curve by finding a Cartesian equation for the curve.

5. $r = 5$ ***6.*** $\varphi = \frac{\pi}{3}$ ***7.*** $r = a\varphi$ ***8.*** $r = a\cos 3\varphi$

9. $r\cos\varphi = 2$ ***10.*** $r^2 = a^2\cos 2\varphi$ ***11.*** $r = \frac{4}{1-\cos\varphi}$ ***12.*** $r = a(1-\cos\varphi)$

In Exercises 13 to 15 sketch the curve

13. $\left(x^2+y^2\right)^3=4x^2y^2$ **14.** $\left(x^2+y^2\right)^2=y^2$ **15.** $3x^2-y^2=\left(x^2+y^2\right)^{\frac{3}{2}}$

Individual Tasks 14

1. Plot the point whose polar coordinates are given. Find the Cartesian coordinates of the point.

2-3. Sketch the curve with the given polar equation. Identify the curve by finding a Cartesian equation for the curve.

I. 1. $M_1\left(3;\dfrac{\pi}{2}\right)$ 2. $r=\dfrac{3}{2-\cos\varphi}$ 3. $r=5\sin 3\varphi$	**II.** 1. $M_1\left(2;\dfrac{3\pi}{2}\right)$ 2. $r=\dfrac{8}{1-3\cos\varphi}$ 3. $r=10\sin\varphi$
III. 1. $M_2\left(3;\dfrac{7\pi}{4}\right)$ 2. $r=\dfrac{2}{1-\cos\varphi}$ 3. $r=2(1-\cos\varphi)$	**IV.** 1. $M_2\left(4;\dfrac{\pi}{3}\right)$ 2. $r=\dfrac{9}{4-5\cos\varphi}$ 3. $r=2\cos 4\varphi$

2.3 Planes, Lines in Space, Cylinders and Quadric Surfaces

2.3.1 Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the “direction” of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0\left(x_0;y_0;z_0\right)$ in the plane and a vector $\vec{n}=\left(A;B;C\right)$ that is orthogonal to the plane. This orthogonal vector $\vec{n}$ is called a ***normal vector***.

Equations of a plane

1. Vector equation of the plane

Let $P(x;y;z)$ be an arbitrary point in the plane, and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P. Then the vector $\vec{r_0}-\vec{r}$ is represented by $\overrightarrow{P_0P}$ (see Figure 3.1). The normal vector $\vec{n}$ is orthogonal to every vector in the given plane and so we have

$$\vec{n}\cdot\left(\vec{r_0}-\vec{r}\right)=0 \qquad (1)$$

Equation 1 is called the ***vector equation*** of the plane.

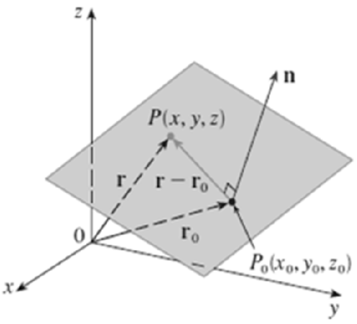


Figure 3.1

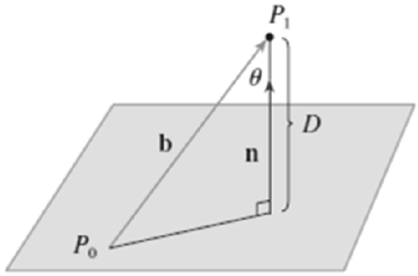


Figure 3.2

2. Equation of the Plane Passing through the Point $P_0(x_0;y_0;z_0)$ ***and Perpendicular to*** $\vec{n}=(A;B;C)$.

To obtain a scalar equation for the plane, we write $\vec{n}=(A;B;C)$, $\vec{r}=(x;y;z)$, and $\vec{r}_0=(x_0;y_0;z_0)$. Then the vector equation (1) becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \qquad (2)$$

3. General Linear Equation

By collecting terms in Equation 2, we can rewrite the equation of a plane as

$$Ax+By+Cz+D=0 \qquad (3)$$

Equation 3 is called a ***linear equation*** of the plane.

Theorem 1 Let A,B,C, and D be constants such that not all A,B and C are 0. Then the equation $Ax+By+Cz+D=0$ describes a plane. Moreover, the vector $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}=(A;B;C)$ is perpendicular to this plane.

4. Three-Point Equation of a Plane

Let points $M_1(x_1;y_1;z_1)$, $M_2(x_2;y_2;z_2)$ and $M_3(x_3;y_3;z_3)$ are any three distinct fixed points on the plane. The equation of the plane passing through these points can be described by the following formula

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \qquad (4)$$

Equation 4 is called the ***three-point equation of a plane***.

5. Equation of a plane in the «intercept form»

Let Δ be any plane, which makes an intercept «a» on the $x-$axis, an intercept «b» on the $y-$axis and an intercept «c» on the $z-$ axis ($a\neq 0, b\neq 0, c\neq 0$).

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (5)$$

Equation 5 is called the ***«intercept form» of the equation of a plane***.

The distance from the point to the plane

Theorem 2 The distance from the point $P_1(x_1;y_1;z_1)$ to the plane Δ: $Ax+By+Cz+D=0$ is the number calculated by the formula (see Figure 3.2):

$$d(P_1;\Delta)=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}} \qquad (6)$$

Relations between the planes

Definition The angle between two planes is the angle between their normal vectors.

The following results can be obtained, depending on the planes are given by the general linear equation.

	$\Delta_1: A_1x+B_1y+C_1z+D=0$ $\Delta_2: A_2x+B_2y+C_2z+D=0$
The angle between Δ_1 to Δ_2	$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$
Perpendicular planes ($\Delta_1 \perp \Delta_2$)	$A_1A_2+B_1B_2+C_1C_2=0$
Parallel planes ($\Delta_1 \parallel \Delta_2$)	$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}\neq\frac{D_1}{D_2}$

Exercise Set 15

In Exercises 1 to 4 find the general linear equation for planes.

1. The plane through $(1;2;-1)$ perpendicular to $\vec{n}=\vec{i}+\vec{j}$.

2. The plane through $(1;5;-1)$ perpendicular to $\vec{n}=\vec{i}+2\vec{j}-\vec{k}$.

3. The plane through $(1;0;1)$ parallel to $x+2y+z=0$.

4. The plane through $(1;2;-1)$ parallel to $x+y+z=1$.

5. Explain why a plane cannot:

a) contains $(1;2;3)$ and $(2;3;4)$ be perpendicular to $\vec{n}=\vec{i}+\vec{j}$;

b) be perpendicular to $\vec{n}=\vec{i}+\vec{j}$ and parallel to $\vec{m}=\vec{i}+\vec{k}$;

c) contains $(1;0;0)$, $(0;1;0)$, $(0;0;1)$ and $(1;1;1)$;

d) passes through the origin and have the equation $ax+by+cz=1$.

6. Find the equation of a plane if:

a) it is parallel to the plane OXZ and passes through the point $M_0(7;-3;5)$;

b) it passes through the $z-axis$ and through the point $M_0(-3;1;-2)$;

c) it is parallel to the $x-axis$ and passes through the points $M_1(4;0;-2)$; $M_2(5;1;7)$;

d) it passes through the point $M_0(2;1;-1)$ and has the normal vector $\vec{n}=(1;-2;3)$;

e) it is parallel to the vectors $\vec{a}=(3;1;-1)$; $\vec{b}=(1;-2;1)$ and passes through the point $M_0(3;4;-5)$;

f) it passes through the points $M_1(1;1;1)$; $M_2(2;3;4)$ and is perpendicular to the plane $2x-7y+5z+9=0$.

7. Find the angle between $x+2y+2z=0$ and:

a) $x+2z=0$; b) $x+2z=5$; c) $x=0$.

8. How far is the plane $x+y-z=1$ from $(0;0;0)$ and also from $(1;1;-1)$? Find the nearest points.

9. Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

a) $3x+2y-z+2=0$; $6x+4y-2z+1=0$

b) $x-3z+2=0$; $2x-6z-7=0$

c) $3x+y-5z-12=0$; $2x+6z-3=0$

d) $2x-3y+z+8=0$; $4x-6y-3z-7=0$

10. Find the distance between the planes $2x-2y+z=1$ and $2x-2y+z=3$.

11. Find the distance between the planes $x+y+5z=7$ and $3x+2y+z=1$.

12. A plane passes through the points $(1;1;2)$, $(1;3;4)$ and $(2;1;-1)$. A second plane passes through $(2;1;-1)$, $(1;0;2)$ and $(3;4;1)$.

a) Find a normal line to each plane;

b) Find the cosine of the angle between the two planes;

c) Find the angle between the planes.

13. Find the distance from the point $M(2;2;-1)$ to the plane that passes through $(1;4;3)$ and has the normal vector $2\vec{i}-7\vec{j}+2\vec{k}$.

14. Find the distance from the point $M(0;0;0)$ to the plane that passes through $(4;1;0)$ and is perpendicular to the vector $\vec{i}+\vec{j}+\vec{k}$.

15. Find the coordinates of a point Q symmetrical to the point $P(-3;1;-9)$ with respect to the plane $4x-3y-z-7=0$.

In Exercises 16 to 18 find the equation of the bisector plane of the angle from Δ_1 to Δ_2 .

16. $3x-y+7z-4=0$; $5x+3y-5z+2=0$;
17. $2x-y+5z+3=0$; $2x-10y+4z-2=0$;
18. $5x-2y+5z-3=0$; $2x+y-7z+2=0$.

Individual Tasks 15

1. Find the general linear equation for planes.
2. Find the angle between $x-2y+3z-5=0$ and the given plane.
3. Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.
4. Find the coordinates of a point Q symmetrical to the point $P_1(x_1;y_1;z_1)$ with respect to the plane Δ: $Ax+By+Cz+D=0$.
5. Find the equation of the bisector plane of the angle from Δ_1 to Δ_2.

I. 1. The plane through $(-1;2;1)$ parallel to $3x+2y-z=0$. 2. $2x-10y+4z-2=0$ 3. $3x+7y+z+4=0$, $9x+21y+3z+12=0$ 4. $P(-2;1;0)$, $2x-2y+z=1$ 5. $x-3z+2=0$, $2x+6z-3=0$	**II.** 1. Find the equation of the plane $A_1A_2A_3$, where $A_1(0;4;5)$; $A_2(3;-2;-1)$; $A_3(4;5;6)$. 2. $4x-3y-z-7=0$ 3. $5x+2y-3z-5=0$, $10x+4y-6z+5=0$ 4. $P(-1;0;5)$, $3x+2y+z=1$ 5. $2x-2y+z=3$, $3x-y+7z-4=0$
III.	**IV.** 1. The plane through $(1;0;-1)$ parallel to $-2x+3y+z-1=0$.

1. Find the equation of the plane $A_1A_2A_3$, where $A_1(2;1;7)$; $A_2(3;3;6)$; $A_3(1;2;5)$. 2. $2x-y+4z-2=0$ 3. $3x+6y+2z-15=0$, $3x+6y+2z+13=0$ 4. $P(-2;1;0)$, $3x+6y+2z+13=0$ 5. $2x-7y+5z+9=0$, $-62x-2y+22z+42=0$.	2. $x-5y+2z-1=0$ 3. $2x-7y+5z+9=0$, $-62x-2y+22z+42=0$ 4. $P(-2;1;0)$, $2x-7y+5z+9=0$ 5. $x-2y+2z-3=0$, $3x-4y+5=0$.

2.3.2 Lines in Space

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Equations of a line

1. Vector equation

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0;y_0;z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\vec{s}$ be a vector parallel to L. Let $P(x;y;z)$ be an arbitrary point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\vec{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 3.3, then the Triangle Law for vector addition gives $\vec{r}=\vec{r_0}+\vec{a}$. But, since $\vec{a}$ and $\vec{v}$ are parallel vectors, there is a scalar t such that $\vec{a}=t\vec{v}$. Thus

$$\vec{r}=\vec{r_0}+t\vec{v} \qquad (1)$$

which is a ***vector equation*** of L.

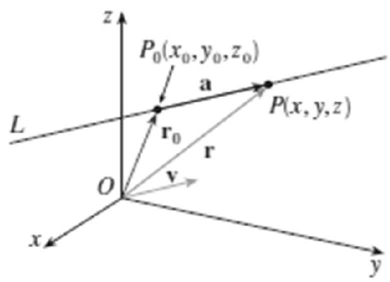


Figure 3.3

2. Parametric equation

If the vector $\vec{v}$ that gives the direction of the line L is written in the component form as $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1;a_2;a_3)$, then we have the three scalar equations:

$$\begin{cases} x = x_0 + a_1 t \\ y = y_0 + a_2 t \\ z = z_0 + a_3 t \end{cases} \qquad (2)$$

These equations are called ***parametric equations*** of the line L through the point $P_0(x_0;y_0;z_0)$ and parallel to the vector $\vec{v} = (a_1;a_2;a_3)$.

3. Symmetric Equation

Another way of describing a line L is to eliminate the parameter from Equation 2. If none of a_1, a_2, a_3 is 0, we can solve each of these equations for t, equate the results, and obtain

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \qquad (3)$$

These equations are called ***symmetric equations*** of L. Notice that the numbers a_1, a_2, a_3 that appear in the denominators of Equations 3 are ***direction numbers*** of L, that is, components of a vector parallel to L. If one of a_1, a_2, a_3 is 0, we can eliminate t.

4. Two - point equation of a line

In general, direction numbers of the line L through the points $P_0(x_0;y_0;z_0)$and $P_1(x_1;y_1;z_1)$are $x_1 - x_0$, $y_1 - y_0$, $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} \qquad (4)$$

This equation is called ***two - point equation*** of a line.

5. General linear equation

A line L can be defined as the intersection line of two planes $\Delta_1:\ A_1x + B_1y + C_1z + D = 0$ and $\Delta_2:\ A_2x + B_2y + C_2z + D = 0$.

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad (5)$$

This equation is called ***the general linear equation***. Using the cross product, directional numbers of L can be determined as scalar components of vector $\vec{n_1} \times \vec{n_2}$, where $\vec{n_1} = (A_1; B_1; C_1)$ and $\vec{n_2} = (A_2; B_2; C_2)$ are normal vectors of corresponding planes.

Relations between the lines

Definition The angle between two lines is the angle between their directional vectors.

Definition The lines L_1 and L_2 are called the ***skew lines*** if they do not intersect and are not parallel (and therefore do not lie in the same plane).

The following results can be obtained, depending on the lines are given by the symmetric equation of the line.

	$L_1:\ \dfrac{x-x_1}{m_1} = \dfrac{y-y_1}{n_1} = \dfrac{z-z_1}{p_1}$ $L_2:\ \dfrac{x-x_2}{m_2} = \dfrac{y-y_2}{n_2} = \dfrac{z-z_2}{p_2}$
The angle from L_1 to L_2	$\cos\varphi = \cos(\vec{s_1}, \vec{s_2}) = \dfrac{m_1m_2 + n_1n_2 + p_1p_2}{\sqrt{m_1^{\ 2} + n_1^{\ 2} + p_1^{\ 2}} \cdot \sqrt{m_2^{\ 2} + n_2^{\ 2} + p_1^{\ 2}}}$
Perpendicular lines ($L_1 \perp L_2$)	$m_1m_2 + n_1n_2 + p_1p_2 = 0$
Parallel lines ($L_1 \parallel L_2$)	$\dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = \dfrac{p_1}{p_2}$
Skew lines ($L_1 \dot{-} L_2$)	$\overrightarrow{M_1M_2} \cdot \left(\vec{s_1} \times \vec{s_2}\right) = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{vmatrix} \neq 0$

Distance between skew lines	$d=\dfrac{\left\lvert\overrightarrow{M_1M_2}\cdot(\vec{s}_1\times\vec{s}_2)\right\rvert}{\lvert\vec{s}_1\times\vec{s}_2\rvert}$
Distance from the point M_0 to the line L_1	$d=\dfrac{\lvert\overrightarrow{M_0M_1}\times\vec{s}_1\rvert}{\lvert\vec{s}_1\rvert}$

Relations between the line and the plane

Definition The angle between a line L and a plane Δ is the angle between the line and its projection onto the plane.

The following results can be obtained, depending on the line is given by the symmetric equation of the line and the plane is given by the general linear equation.

	$L:\dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}$ Δ: $Ax+By+Cz+D=0$
Angle between L and Δ	$\sin\varphi=\dfrac{\lvert Am+Bn+Cp\rvert}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{m^2+n^2+p^2}}$
Line L perpendicular to the plane Δ $(L\perp\Delta)$	$\dfrac{A}{m}=\dfrac{B}{n}=\dfrac{C}{p}$
Line L parallel to the plane Δ $(L\parallel\Delta)$	$\begin{cases}Am+Bn+Cp=0,\\ Ax_0+By_0+Cz_0+D\neq 0\end{cases}$
Line L intersects the plane Δ	$Am+Bn+Cp\neq 0$
Line L belongs to the plane Δ	$\begin{cases}Am+Bn+Cp=0,\\ Ax_0+By_0+Cz_0+D=0\end{cases}$

Exercise Set 16

1. The line is given by the general equation $\begin{cases} x-y+2z+4=0; \\ 3x+y-5z-8=0. \end{cases}$ Find its symmetric equation.

2. Give the symmetric equation for the line through the point $(2;0;-3)$ and parallel to the vector $\vec{s}=(2;-3;5)$.

3. Give sthe ymmetric equation for the line through the point $(2;0;-3)$ and parallel to the to the line $\begin{cases} 2x-y+3z-11=0; \\ 5x+4y-z+8=0. \end{cases}$

4. Give the symmetric equation for the line through the points $(1;0;3)$ and $(2;1;-1)$.

5. Give the symmetric equation for the line through the points $(7;-1;5)$ and $(4;3;2)$.

6. a) How far is the point $(1;1;1)$ from the line through the points $(2;1;3)$ and $(1;4;5)$?

b) Find the point on the line nearest to $(1;1;1)$.

7. Let L be the line in which the planes $x+y+3z=5$ and $2x-y+z=2$ intersect.

a) Find the vector parallel to L;

b) Find a point on L;

c) Find parametric equations for L.

8. Where does the line of intersection of the planes $x+2y+z=4$ and $2x-y+z=1$ meet the plane $3x+2y+z=6$?

9. The planes $x+2y+3z=6$ and $2x-3y+4z=8$ intersect in a line L:

a) Find the vector parallel to L;

b) Find a point on L;

c) Find parametric equations of L;

d) Find symmetric equations of L.

In Exercises 10 to 16 find the point at which the line intersects the given plane.

10. $L: \dfrac{x-1}{2}=\dfrac{y+3}{-1}=\dfrac{z+2}{5}$ $\quad \Delta: 4x+3y-z+3=0$

11. $L: \begin{cases} 2x-y+3z+4=0, \\ x-2y+z+3=0. \end{cases}$ $\quad \Delta: 4x-5y-z+8=0$

12. $L:\begin{cases} 2x-3y-3z-9=0, \\ x-2y+z+3=0. \end{cases}$ $\Delta: x-2y+z-1=0$

13. $L:\dfrac{x+1}{2}=\dfrac{y-3}{4}=\dfrac{z}{3}$ $\Delta: 3x-3y+2z-5=0$

14. $L:\dfrac{x-13}{8}=\dfrac{y-1}{2}=\dfrac{z-4}{3}$ $\Delta: x+2y-4z+1=0$

15. $L:\dfrac{x-7}{5}=\dfrac{y-4}{1}=\dfrac{z-5}{4}$ $\Delta: 3x-y+2z-5=0$

16. $L:\dfrac{x-1}{2}=\dfrac{y+2}{1}=\dfrac{z-2}{1}$ $\Delta: 3x-y+2z+5=0$

In Exercises 17 to 21 determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection. If they skew, find the distance between them.

17. $L_1:\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z}{2}$; $L_2:\dfrac{x-7}{3}=\dfrac{y-1}{4}=\dfrac{z-3}{2}$

18. $L_1:\dfrac{x-2}{1}=\dfrac{y+2}{-3}=\dfrac{z+1}{-2}$; $L_2: \dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z-1}{1}$

19. $L_1:\dfrac{x+2}{-1}=\dfrac{y-3}{2}=\dfrac{z-4}{3}$; $L_2:\dfrac{x}{3}=\dfrac{y+4}{2}=\dfrac{z-3}{5}$

20. $L_1:\begin{cases} 2x-3z+2=0, \\ 2y-z-6=0. \end{cases}$; $L_2:\begin{cases} x-12z+49=0, \\ 4y-37z+148=0. \end{cases}$

21. $L_1:\begin{cases} x=3z-1, \\ y=-5z+7. \end{cases}$; $L_2:\begin{cases} y=2x-5, \\ z=7x+2. \end{cases}$

22. Find the coordinates of a point Q symmetrical to the point $P(6;-5;5)$ with respect to the line $L:\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z}{2}$.

23. Find the coordinates of a point Q symmetrical to the point $P(2;-1;3)$ with respect to the line $\begin{cases} x=3t; \\ y=5t-7;. \\ z=2t+2. \end{cases}$

24. Find the angle between the line $\begin{cases} x-2y+3=0 \\ 3y+z-1=0 \end{cases}$ and the plane $2x+3y-z+1=0$

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25. Find the equation of a plane passing through the line $\frac{x-2}{5}=\frac{y-3}{1}=\frac{z+1}{2}$ and perpendicular to the plane $x+4y-3z+7=0$.

26. Find the distance between parallel lines $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z}{2}$ and $\frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$.

27. Are the lines $\frac{x+2}{-1}=\frac{y-3}{2}=\frac{z-4}{3}$ and $\frac{x}{3}=\frac{y+4}{2}=\frac{z-3}{5}$ intersect?

28. Find the equation of projection of a line $\frac{x-2}{6}=\frac{y+1}{-5}=\frac{z-5}{4}$ to the plane $x-3y+2z-7=0$.

Let A_1, A_2, A_3, A_4 be the points of space. In Exercises 29 and 30 find: a) the equation of the line A_1A_2; b) the equation of the line A_4M perpendicular to the plane $A_1A_2A_3$; c) the equation of the line A_3N parallel to A_1A_2; d) the equation of the plane passing through A_4 perpendicular to the line A_1A_2; e) the angle between the line A_1A_4 and the plane $A_1A_2A_3$.

29. $A_1(0;\,4;\,5)$, $A_2(3;\,-2;\,-1)$, $A_3(4;\,5;\,6)$, $A_4(3;\,3;\,2)$;

30. $A_1(2;\,-1;\,7)$, $A_2(6;\,3;\,1)$, $A_3(3;\,2;\,8)$, $A_4(2;\,-3;\,7)$.

Individual Tasks 16

1. Give symmetric and parametric equations for the line passing through the given points.

2. Find the point at which the line intersects the given plane.

3. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection. If they skew, find the distance between them.

4. Find the coordinates of a point Q symmetrical to the point $P_1(x_1;y_1;z_1)$ with respect to the plane Δ: $Ax+By+Cz+D=0$.

5. Let A_1, A_2, A_3, A_4 be the points of space. Find: a) the equation of the line A_1A_2; b) the equation of the line A_4M perpendicular to the plane $A_1A_2A_3$; c) the equation of the line A_3N parallel to A_1A_2; d) the equation of the plane passing through A_4 perpendicular to the line A_1A_2; e) the angle between the line A_1A_4 and the plane $A_1A_2A_3$.

I.	**II.**
1. $P(2;0;-1)$, $Q(4;1;0)$	1. $P(3;0;1)$, $Q(-1;2;5)$

2. $L: \dfrac{x-2}{-2} = \dfrac{y-3}{3} = \dfrac{z-1}{2}$, $\Delta: 4x + 2y + z + 24 = 0$ 3. $L_1: \dfrac{x-1}{6} = \dfrac{y+2}{2} = \dfrac{z}{-1}$, $L_2: \begin{cases} x - 2y + 2z - 8 = 0, \\ x + 6z - 6 = 0. \end{cases}$ 4. $P(-2;1;0)$, $L: \dfrac{x+1}{2} = \dfrac{y-3}{4} = \dfrac{z}{3}$ 5. $A_1(2;\ 1;\ 7)$, $A_2(3;\ 3;\ 6)$, $A_3(2;\ -3;\ 9)$, $A_4(1;\ 2;\ 5)$	2. $L: \dfrac{x-2}{-1} = \dfrac{y+1}{4} = \dfrac{z+5}{2}$, $\Delta: 4x + y - z = 0$ 3. $L_1: \dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-1}{6}$, $L_2: \begin{cases} 2x + y - 4z + 2 = 0, \\ 4x - y - 5z + 4 = 0. \end{cases}$ 4. $P(-1;0;5)$, $L: \dfrac{x-7}{5} = \dfrac{y-4}{1} = \dfrac{z-5}{4}$ 5. $A_1(2;\ 1;\ 6)$, $A_2(1;\ 4;\ 9)$, $A_3(2;\ -5;\ 8)$, $A_4(5;\ 4;\ 2)$
III. 1. $P(3;2;-1)$, $Q(-4;1;0)$ 2. $L: \dfrac{x-2}{-4} = \dfrac{y-3}{6} = \dfrac{z-1}{4}$, $\Delta: 8x + 4y + 2z + 24 = 0$ 3. $L_1: \dfrac{x-1}{3} = \dfrac{y+2}{1} = \dfrac{z-6}{-1}$, $L_2: \begin{cases} x - 2y + 2z - 8 = 0, \\ x + 6z - 6 = 0. \end{cases}$ 4. $P(-2;1;0)$, $L: \dfrac{x+1}{-2} = \dfrac{y-3}{-4} = \dfrac{z}{-3}$ 5. $A_1(0;\ 4;\ 5)$, $A_2(3;\ -2;\ -1)$, $A_3(4;\ 5;\ 6)$, $A_4(3;\ 3;\ 2)$	**IV.** 1. $P(-3;0;3)$, $Q(-1;-2;5)$ 2. $L: \dfrac{x-2}{2} = \dfrac{y+1}{-8} = \dfrac{z+5}{-4}$, $\Delta: -4x - y + z = 0$ 3. $L_1: \dfrac{x+1}{2} = \dfrac{y-3}{-1} = \dfrac{z-1}{6}$, $L_2: \begin{cases} 2x + y - 4z + 2 = 0, \\ 4x - y - 5z + 4 = 0. \end{cases}$ 4. $P(-1;0;5)$, $L: \dfrac{x-7}{-15} = \dfrac{y-4}{-3} = \dfrac{z-5}{-12}$ 5. $A_1(2;\ -1;\ 7)$, $A_2(6;\ 3;\ 1)$, $A_3(3;\ 2;\ 8)$, $A_4(2;\ -3;\ 7)$

2.3.3 Cylinders and Quadric Surfaces

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with the planes parallel to the coordinate planes. These curves are called ***traces*** (or ***cross-sections***) of the surface.

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Example 1 Sketch the graph of the surface $z = x^2$.

Solution Notice that the equation of the graph $z = x^2$ doesn't involve y. This means that any vertical plane with the equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with the equation $z = x^2$. So these vertical traces are parabolas. Figure 3.4 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis. The graph is a surface, called a ***parabolic cylinder***, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the y-axis.

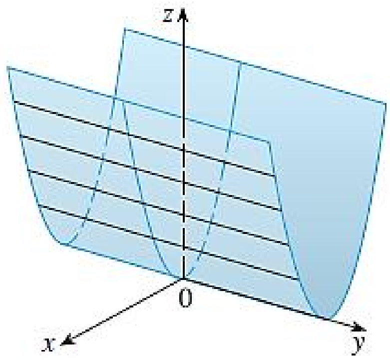

Figure 3.4

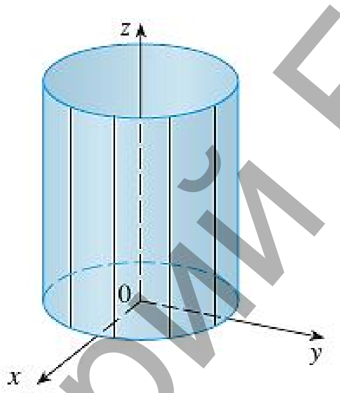

Figure 3.5

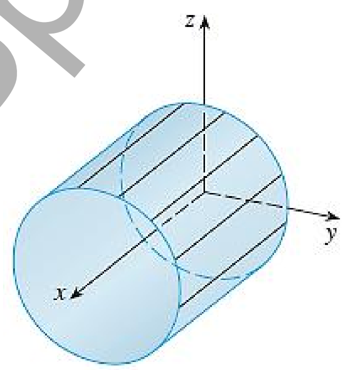

Figure 3.6

We noticed that the variable y is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables x, y or z is missing from the equation of a surface, then the surface is a cylinder.

Example 2 Identify and sketch the surfaces (a) $x^2 + y^2 = 1$; (b) $y^2 + z^2 = 1$.

Solution (a) Since z is missing and the equations $x^2 + y^2 = 1$, $z = k$ represent a circle with radius 1 in the plane $z = k$, the surface $x^2 + y^2 = 1$ is a circular cylinder whose axis is the z-axis (see Figure 3.5). Here the rulings are vertical lines.

(b) In this case x is missing and the surface is a circular cylinder whose axis is the x-axis (see Figure 3.6). It is obtained by taking the circle $y^2 + z^2 = 1, x = k$ in the yz-plane and moving it parallel to the x-axis.

Note When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1, z = 0$.

A ***quadric surface*** is the graph of a second-degree equation in three variables x, y and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \tag{1}$$

where $A, B, C, ..., \mathrm{J}$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \text{ or } Ax^2 + By^2 + Cz^2 + Iz = 0.$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal. Table 2 (see Appendix) shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Example 3 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Solution By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2.$$

Comparing this equation with Table 2, we see that it represents an ***elliptic paraboloid***. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3;1;0)$. The traces in the plane $y = k,\ k > 0$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1.$$

The trace in the xy-plane is the parabola with the equation $y = (x - 3)^2 + 1,\ z = 0$.

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are reflected to the receiver at the focus and are therefore amplified. Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes.

Exercise Set 17

In Exercises 1 to 4 use traces to sketch and identify the surface.

1. $x = y^2 + 4z^2$

2. $9x^2 - y^2 + z^2 = 0$

3. $25x^2 + 4y^2 + z^2 = 100$

4. $4x^2 - 16y^2 + z^2 = 16$

In Exercises 5 to 12 reduce the equation to one of the standard forms, classify the surface, and sketch it.

5. $x = 2y^2 + 3z^2$

6. $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$

7. $4x - y^2 + 4z^2 = 0$

8. $4y^2 + z^2 - x - 16y - 4z + 20 = 0$

9. $x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$

10. $5x^2 + y^2 + 10x - 6y - 10z + 14 = 0$

11. $x^2 + y^2 - z^2 - 2x + 2y + 4z + 2 = 0$

12. $x^2 + y^2 + z^2 = 2z$

In Exercises 13 to 18 sketch the region bounded by the given surfaces

13. $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 = 1$ for $1 \le z \le 2$.

14. $z = x^2 + y^2$, $z = 2 - x^2 - y^2$

15. $x + y + z = 4$, $3x + y = 4$, $\frac{3}{2}x + y = 4$, $y = 0$, $z = 0$

16. $x^2 + z^2 = 9$, $y = 0$, $z = 0$, $y = x$

17. $z = x^2 + y^2$, $y = x^2$, $y = 1$, $z = 0$

18. $4z = x^2 + y^2$, $x^2 + y^2 + z^2 = 12$

19. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the y-axis.

20. Find an equation for the surface obtained by rotating the line $x = 3y$ about the x-axis.

Individual Tasks 17

1. Reduce the equation to one of the standard forms, classify the surface, and sketch it.

2. Sketch the region bounded by the given surfaces.

3. Find an equation for the surface obtained by rotating the given curve about the indicated axis.

I.	II.
1. $x^2+y^2+z^2-6x+4y-4z=0$ 2. $y=\sqrt{x}$, $y=2\sqrt{x}$, $x+z=6$, $z=0$ 3. $y=x^3$, $x\geq 0$, OX	1. $x^2+3z^2-8x+18z+34=0$ 2. $z=x^2-y^2$, $y=0$, $z=0$, $x=1$ 3. $x=y^2$, $y\geq 0$, OY
III.	**IV.**
1. $36x^2+16y^2-9z^2+18z=9$ 2. $x=2$, $y=0{,}5x$, $y=0$, $z=0$, $z=1$ 3. $y=x^2$, $x\geq 0$, OX	1. $4x^2-y^2-16z^2+16=0$ 2. $z^2+y^2=2x$, $y^2+z^2=4$ 3. $2x^2+3y^3=10$, OX

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APPENDIX

Table 1

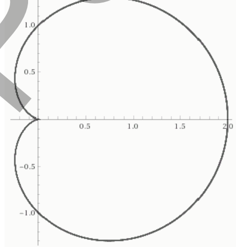
Cartesian coordinates	Polar coordinate system	Parametric equations
Line passing through the origin $y = kx$	$tg\varphi = k$	$\begin{cases} x = t \\ y = t \end{cases}$
Circle: $x^2 + y^2 = a^2$ $(x-a)^2 + y^2 = a^2$ $x^2 + (y-a)^2 = a^2$	$r = a$ $r = 2a\cos\varphi$ $r = 2a\sin\varphi$	$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $\begin{cases} x = a\cos t + a \\ y = a\sin t \end{cases}$ $\begin{cases} x = a\cos t \\ y = a\sin t + a \end{cases}$
Parabola $y^2 = 2px$	$r = \dfrac{a}{1-\cos\varphi}$, a some number	$\begin{cases} y = t \\ x = \dfrac{t^2}{2p} \end{cases}$
Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$r = \dfrac{a}{1-e\cos\varphi}$, where $e<1$ eccentricity, a some number	$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$
Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$r = \dfrac{a}{1-e\cos\varphi}$, where $e>1$ eccentricity, a some number	$\begin{cases} x = a\,\mathrm{c}ht \\ y = b\,\mathrm{s}ht \end{cases}$

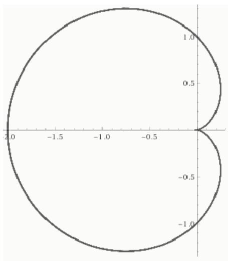
Cardioids

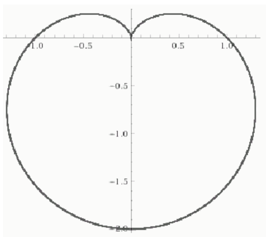
$$r = a(1-\cos\varphi) \Leftrightarrow \left(x^2 + y^2 + 2ax\right)^2 - 4a^2\left(x^2 + y^2\right) = 0 \Leftrightarrow$$

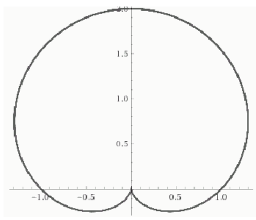
$$\Leftrightarrow \begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$$

$r = a(1+\cos\varphi)$ $r = a(1-\cos\varphi)$ $r = a(1-\sin\varphi)$ $r = a(1+\sin\varphi)$



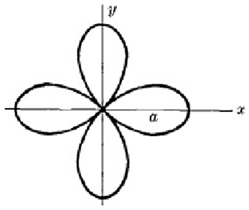




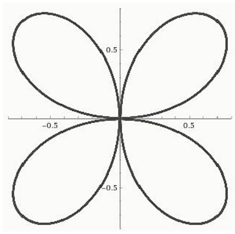


Roses

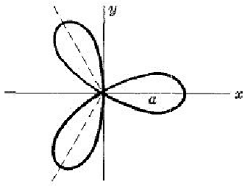
$r = a\cos 2\varphi$



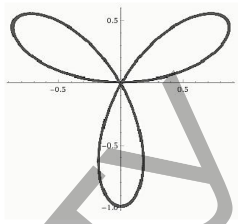
$r = a\sin 2\varphi$



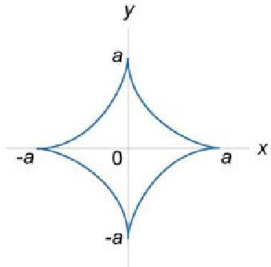
$r = a\cos 3\varphi$



$r = a\sin 3\varphi$

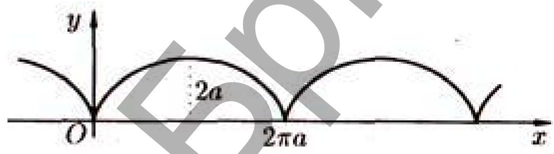


Asteroid



$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \Leftrightarrow \begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$$

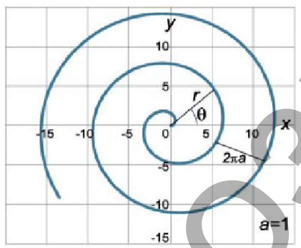
Cycloid



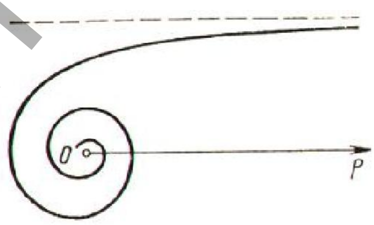
$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

Spirals

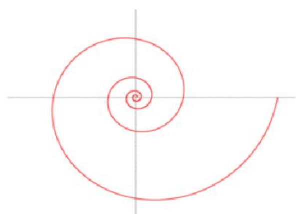
Spiral of Archimedes
$r = a\varphi$



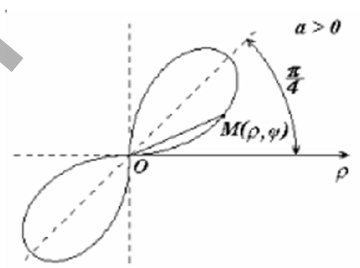
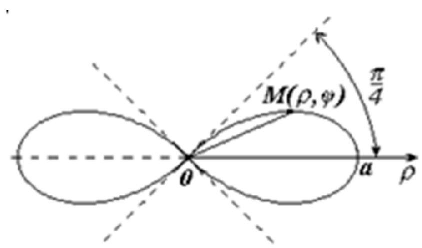
Hyperbolic spiral $r = \dfrac{a}{\varphi}$



Logarithmic spiral $r = a^{\varphi}$

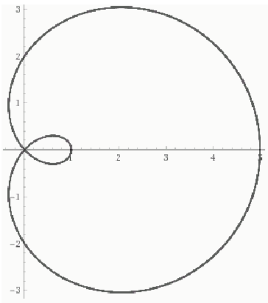


Lemniscates

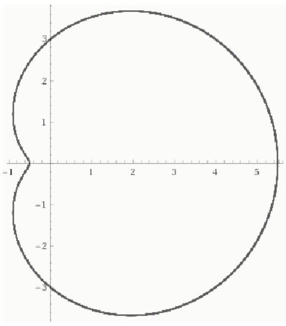
$r^2 = a^2 \sin 2\varphi$	$r^2 = a^2 \cos 2\varphi \Leftrightarrow \left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right)$
	

Limacons

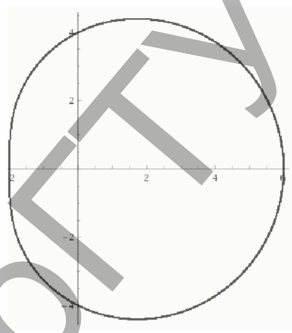
Cardioid is a special kind of limacons family of related curves. Limacon equations:
$r = b + a\cos\theta$ (horizontal) or $r = b + a\sin\theta$ (vertical)



$b < a$
looped

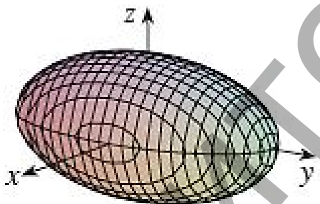
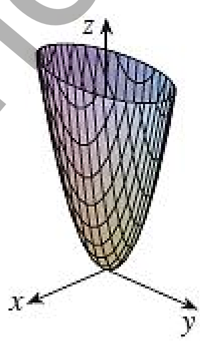


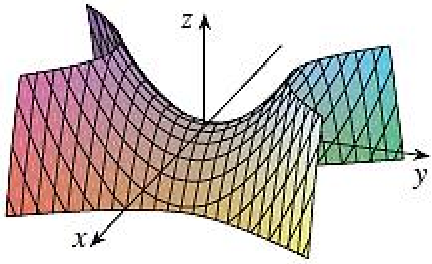
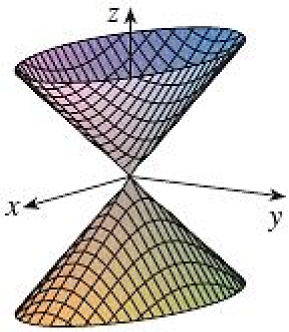
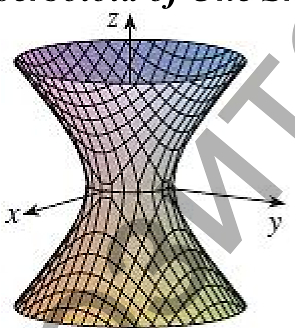
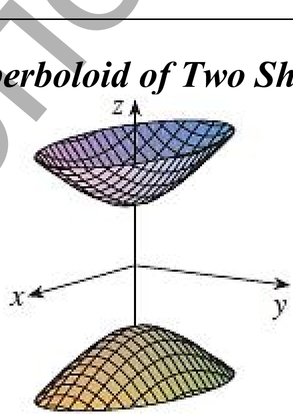
$a < b < 2a$
dimpled



$2a \le b$
convex

Table 2

Surface	**Equation**
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

Hyperbolic Paraboloid 	$$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where is $c<0$ illustrated.
Cone 	$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Hyperboloid of One Sheet 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ Horizontal traces are ellipse. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperboloid of Two Sheets 	$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

УЧЕБНОЕ ИЗДАНИЕ

Составители:
Дворниченко Александр Валерьевич
Крагель Екатерина Александровна
Лебедь Светлана Федоровна
Бань Оксана Васильевна
Гладкий Иван Иванович

Elements of Algebra and Analytic Geometry

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**

Ответственный за выпуск: Лебедь С.Ф.
Редактор: Боровикова Е.А.
Компьютерная вёрстка: Дворниченко А.В.

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