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КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

FOURIER SERIES

LAPLACE TRANSFORMS

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**

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Настоящая методическая разработка предназначена для иностранных студентов технических специальностей. Данная разработка содержит необходимый материал по разделам «Ряды Фурье» и «Операционное исчисление (Преобразования Лапласа)». Изложение теоретического материала сопровождается рассмотрением большого количества примеров и задач, некоторые понятия и примеры проиллюстрированы.

Составители: Гладкий И.И., доцент
Дворниченко А.В., старший преподаватель
Каримова Т.И., к.ф.-м.н., доцент
Лебедь С.Ф., к.ф.-м.н., доцент
Шишко Т.В., преподаватель кафедры иностранных языков по техническим специальностям

Рецензент: Мирская Е.И., доцент кафедры алгебры, геометрии и математического моделирования учреждения образования «Брестский государственный университет им. А.С. Пушкина», к.ф.-м.н., доцент.

1 FOURIER SERIES

1.1 Trigonometric Polynomials and Series

Many phenomena in the applications of the natural and engineering sciences are periodic in nature. Examples are the vibrations of strings, springs and other objects, rotating parts in machines, the movement of the planets around the sun, the tides of the sea, the movement of a pendulum in a clock, the voltages and currents in electrical networks, electromagnetic signals emitted by transmitters in satellites, light signals transmitted through glass fibers, etc. Seemingly, all these systems operate in complicated ways; the phenomena that can be observed often behave in an erratic way. In many cases, however, they do show some kind of repetition. In order to analyse these systems, one can make use of elementary periodic functions or signals from mathematics, the sine and cosine functions. For many systems, the response or behaviour can be completely calculated or measured by exposing them to influences or inputs given by these elementary functions. When, moreover, these systems are linear, then one can also calculate the response to a linear combination of such influences, since this will result in the same linear combination of responses.

Hence, for the study of the aforementioned phenomena, two matters are of importance.

On the one hand, one should look at how systems behave under influences that can be described by elementary mathematical functions. Such an analysis will in general require specific knowledge of the system being studied. This may involve knowledge about how forces, resistances, and inertias influence each other in mechanical systems, how fluids move under the influence of external forces, or how voltages, currents and magnetic fields are mutually interrelated in electrical applications.

In this book we will not go into these analyses, but the results, mostly in terms of mathematical formulations, will often be chosen as a starting point for further considerations.

On the other hand, it is of importance to examine if and how an arbitrary periodic function can be described as a linear combination of elementary sine and cosine functions. This is the central theme of the theory of Fourier series: determine the conditions under which periodic functions can be represented as linear combinations of sine and cosine functions.

The central problem of the theory of Fourier series is how arbitrary periodic functions or signals might be written as a series of sine and cosine functions. The sine and cosine functions are also called sinusoidal functions. In this section we will first look at the functions that can be constructed if we start from the sine and cosine functions. Next we will examine how, given such a function, one can recover the sinusoidal functions from which it is build up. In the next section this will lead us to the definition of the Fourier coefficients and the Fourier series for arbitrary periodic functions.

The period of periodic functions will always be denoted by T. We would like to approximate arbitrary periodic functions with linear combinations of sine and cosine functions. These sine and cosine functions must then have period T as well. One can easily check that the functions $\sin\left(\frac{2\pi t}{T}\right)$, $\cos\left(\frac{2\pi t}{T}\right)$, $\sin\left(\frac{4\pi t}{T}\right)$, $\cos\left(\frac{4\pi t}{T}\right)$, $\sin\left(\frac{6\pi t}{T}\right)$, $\cos\left(\frac{6\pi t}{T}\right)$ and so on all have period T. The constant function also has period T. Jointly, these functions can be represented by $\sin\left(\frac{2\pi nt}{T}\right)$ and $\cos\left(\frac{2\pi nt}{T}\right)$, where $n \in N$. Instead of $\frac{2\pi}{T}$ one often writes ω_0, which means that the functions can be denoted by $\sin n\omega_0 t$ and $\cos n\omega_0 t$, where $n \in N$. All these functions are periodic with period T. In this context, the constant ω_0 is called the fundamental frequency: $\sin\omega_0 t$ and $\cos\omega_0 t$ will complete exactly one cycle on an interval of length T, while all functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$ with $n > 1$ will complete several cycles. The frequencies of these functions are thus all integer multiples of ω_0.

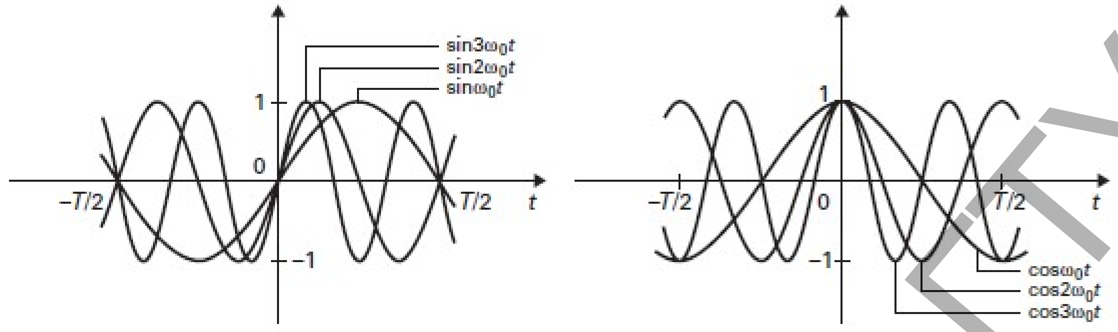


Fig.1

See Fig.1, where the functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$ are sketched for $n = 1, 2, 3$. Linear combinations, also called superpositions, of the functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$ are again periodic with period T. If in such a combination we include a finite number of terms, then the expression is called a trigonometric polynomial. Besides the sinusoidal terms, a constant term may also occur here. Hence, a trigonometric polynomial $f(t)$ with period T can be written as

$$f(t) = A + a_1\cos\omega_0 t + b_1\sin\omega_0 t + a_2\cos 2\omega_0 t + b_2\sin 2\omega_0 t + ... + a_n\cos n\omega_0 t +$$

$$+ b_n\sin n\omega_0 t,\ \ \omega_0 = \frac{2\pi}{T}.$$

In Fig.2a some examples of trigonometric polynomials are shown with $\omega_0 = 1$ and so $T = 2\pi$. The polynomials shown are

$f_1(t) = 2\sin t$,

$f_2(t) = 2(\sin t - \frac{1}{2}\sin 2t)$,

$f_3(t) = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t\right)$,

$f_4(t) = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t\right)$.

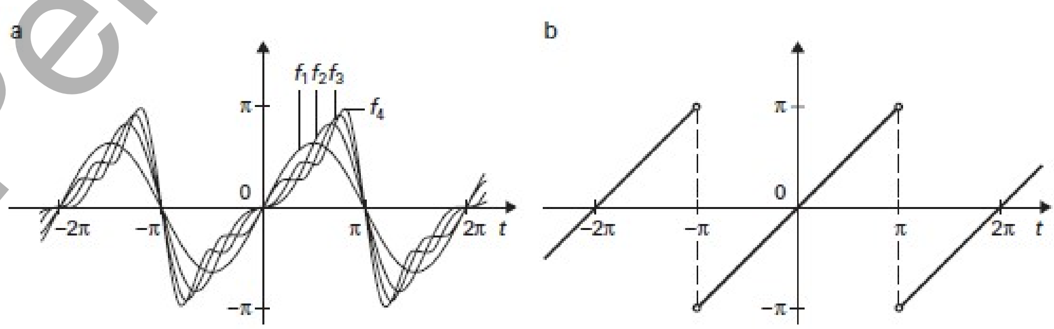


Fig.2

In Fig.2b the sawtooth function is drawn. It is defined as follows. On the interval $\left(-\frac{T}{2},\frac{T}{2}\right)=(-\pi,\pi)$ one has $f(t)=t$, while elsewhere the function is extended periodically, which means that it is defined by $f(t+kT)=f(t)$ for all $k\in Z$. The function $f(t)$ is then periodic with period T and is called the periodic extension of the function $f(t)=t$. The function values at the endpoints of the interval $\left(-\frac{T}{2},\frac{T}{2}\right)$ are not of importance for the time being and are thus not taken into account for the moment. Comparing the Fig.2a and 2b suggests that the sawtooth function, a periodic function not resembling a sinusoidal function at all, can in this case be approximated by a linear combination of sine functions only. The trigonometric polynomials f_1, f_2, f_3, f_4 above are partial sums of the infinite series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin nt$. It turns out that as more terms are being included in the partial sums, the approximations improve. When an infinite number of terms is included, one no longer speaks of trigonometric polynomials, but of trigonometric series. The most important aspect of such series is, of course, how well they can approximate an arbitrary periodic function. In the next chapter it will be shown that for a piecewise smooth periodic function it is indeed possible to find a trigonometric series whose sum converges at the points of continuity and is equal to the function.

At this point it suffices to observe that in this way a large class of periodic functions can be constructed, namely the trigonometric polynomials and series, all based upon the functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$. All functions $f(t)$ which can be obtained as linear combinations or superpositions of the constant function and the sinusoidal functions with period T can be represented as follows:

$$f(t)=A+\sum_{n=1}^{\infty}\left(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t\right) \text{ with } \omega_0=\frac{2\pi}{T} \qquad (1).$$

This, of course, only holds under the assumption that the right-hand side actually exists, that is, converges for all t. Let us now assume that a function from the previously described class is given, but that the values of the coefficients are unknown. We thus assume that the right-hand side of (1) exists for all t. It is then relatively easy to recover these coefficients. In doing so, we will use the trigonometric identities

$$\sin\alpha\cos\beta=\frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right),$$
$$\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right),$$
$$\cos\alpha\cos\beta=\frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right).$$

Using these formulas one can derive the following results for $n,m\in Z$ with $n\neq 0$.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos n\omega_0 t\,dt=0,\quad \int_{-\frac{T}{2}}^{\frac{T}{2}}\sin n\omega_0 t\,dt=0,\quad \int_{-\frac{T}{2}}^{\frac{T}{2}}\cos^2 n\omega_0 t\,dt=\frac{T}{2},\quad \int_{-\frac{T}{2}}^{\frac{T}{2}}\sin^2 n\omega_0 t\,dt=\frac{T}{2}.$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos m\omega_0 t \sin n\omega_0 t dt = 0, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin m\omega_0 t \sin n\omega_0 t dt = 0,$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos m\omega_0 t \cos n\omega_0 t dt = 0, n \neq m.$$

After this enumeration of results, we now return to (1) and try to determine the unknown coefficients A, a_n, b_n for a given $f(t)$. To this end we multiply the left-hand and right-hand side of (1) by $\cos m\omega_0 t$ and then integrate over the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$. It then follows that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos m\omega_0 t dt = a_m \frac{T}{2}, \quad a_m = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos m\omega_0 t dt \tag{2}$$

This means that for a given $f(t)$, it is possible to determine a_m using (2). In an analogous way an expression can be found for b_m. Multiplying (1) by $\sin m\omega_0 t$ and again integrating over the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$, one obtains an expression for b_m

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin m\omega_0 t dt = b_m \frac{T}{2}, \quad b_m = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin m\omega_0 t dt \tag{3}$$

A direct integration of (1) over $\left(-\frac{T}{2}, \frac{T}{2}\right)$ gives an expression for the constant A:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = AT, \quad A = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = \frac{a_0}{2} \tag{4}$$

All coefficients in (1) can thus be determined if $f(t)$ is a given trigonometric polynomial or series. The calculations are summarized in the following two expressions, from which the coefficients can be found for all functions in the class of trigonometric polynomials and series, in so far as these coefficients exist and interchanging the order of summation and integration, mentioned above, is allowed:

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega_0 t dt, \ n = 0,1,2,... \tag{5}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_0 t dt, \ \ n = 1,2,... \tag{6}$$

In these equations, the interval of integration is $\left(-\frac{T}{2},\frac{T}{2}\right)$. This interval is precisely of length one period. To determine the coefficients a_n, b_n, one can in general integrate over any other arbitrary interval of length T. Sometimes the interval $(0,T)$ is chosen.

1.2 Definition of Fourier Series

In the previous section we demonstrated how, starting from a collection of elementary periodic functions, one can construct new periodic functions by taking linear combinations. The coefficients in this combination could be recovered using formulas (5) and (6). These formulas can in principle be applied to any arbitrary periodic function with period T, provided that the integrals exist. This is an important step: the starting point is now an arbitrary periodic function. To it, we then apply formulas (5) and (6), which were originally only intended for trigonometric polynomials and series. The coefficients a_n, b_n thus defined are called the Fourier coefficients. The series in (1), which is determined by these coefficients, is called the Fourier series.

Definition 1 (Fourier coefficients) Let $f(x)$ be a periodic function with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, then the Fourier coefficients a_n, b_n of $f(x)$, if they exist, are defined by

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos nx\,dx \quad (n = 0, 1, \ 2,...) \tag{1}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin nx\,dx \quad (n = 1, \ 2, \ 3,...) \tag{2}$$

In fact, in definition1 a mapping or transformation is defined from functions to number sequences. This is also denoted as a transformation pair:

$$f(x) \leftrightarrow a_n, b_n.$$

One should pronounce this as: "to the function $f(x)$ belong the Fourier coefficients a_n, b_n". This mapping is the Fourier transform for periodic functions. The function $f(x)$ can be complex-valued. In that case, the coefficients a_n, b_n will also be complex. Using definition 1 one can now define the Fourier series associated with a function $f(x)$.

Definition 2 (Fourier series) When a_n, b_n are the Fourier coefficients of the periodic function $f(x)$ with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, then the Fourier series of $f(x)$ is defined by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega_0 x+b_n\sin n\omega_0 x\right) \tag{3}$$

Example 1 Determine the Fourier coefficients of the sawtooth function given by $f(x)=x$ on the interval $(-\pi,\pi)$ and extended periodically elsewhere, and sketch the graph.

Solution

In the present situation we have $T=2\pi$, so $\omega_0=\frac{2\pi}{T}=1$. The definition of Fourier coefficients can immediately be applied to the function $f(x)$. Using integration by parts it follows for $n\geq 1$ that Fourier series

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nxdx=\frac{1}{\pi n}\left[x\sin nx\right]_{x=-\pi}^{x=\pi}-\frac{1}{\pi n}\int_{-\pi}^{\pi}\sin nxdx=$$

$$=\frac{1}{n^2\pi}\left[\cos nx\right]_{x=-\pi}^{x=\pi}=0.$$

For $n=0$ we have

$$a_0=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}xdx=\frac{1}{\pi}\left[\frac{1}{2}x^2\right]_{x=-\pi}^{x=\pi}=0.$$

For the coefficients b_n we have that

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nxdx=-\frac{1}{\pi n}\left[x\cos nx\right]_{x=-\pi}^{x=\pi}+\frac{1}{\pi n}\int_{-\pi}^{\pi}\cos nxdx=$$

$$=-\frac{1}{\pi n}\left(\pi\cos\pi n-(-\pi)\cos(-\pi n)\right)-\frac{1}{n^2\pi}\left[\sin nx\right]_{x=-\pi}^{x=\pi}=-\frac{2\pi}{\pi n}\cos n\pi=(-1)^{n-1}\frac{2}{n}.$$

Here we used that $\cos\pi n=(-1)^n$ for $n\in N$. Hence, the Fourier coefficients an are all equal to zero, while the coefficients b_n are equal to $2\frac{(-1)^{n-1}}{n}$. The Fourier series of the sawtooth function is thus indeed equal to

$$\sum_{n=1}^{\infty}2\frac{(-1)^{n-1}}{n}\sin nx.$$

That the partial sums of the series are a good approximation of the sawtooth function can be seen in figure 3, where $\sum_{n=1}^{10}2\frac{(-1)^{n-1}}{n}\sin nx$ is sketched.

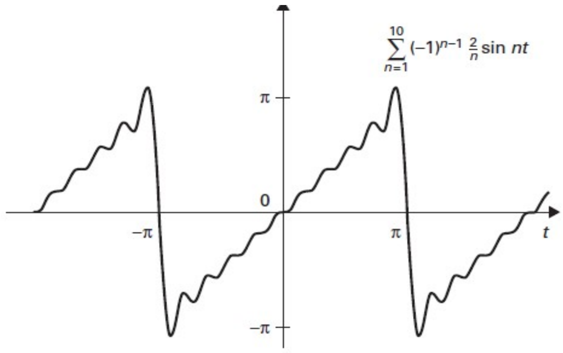


Fig.3

1.3 Fourier Cosine and Fourier Sine Series

In section 1.2 we showed that the ordinary Fourier series of an even periodic function contains only cosine terms and that the Fourier series of an odd periodic function contains only sine terms. For the standard functions we have seen that the periodic block function and the periodic triangle function , which are even, do indeed contain cosine terms only and that the sawtooth function, which is odd, contains sine terms only. Sometimes it is desirable to obtain for an arbitrary function on the interval $(0,T)$ a Fourier series containing only sine terms or containing only cosine terms. Such series are called Fourier sine series and Fourier cosine series. In order to find a Fourier cosine series for a function defined on the interval $(0,T)$, we extend the function to an even function on the interval $(-T,T)$ by defining $f(-x)=f(x)$ for $-T<x<0$ and subsequently extending the function periodically with period $2T$ .The function thus created is now an even function and its ordinary Fourier series will contain only cosine terms, while the function is equal to the original function on the interval $(0,T)$.

In a similar way one can construct a Fourier sine series for a function by extending the function defined on the interval $(0,T)$ to an odd function on the interval $(-T,T)$ and subsequently extending it periodically with period $2T$. Such an odd function will have an ordinary Fourier series containing only sine terms. Determining a Fourier sine series or a Fourier cosine series in the way described above is sometimes called a forced series development.

Example 2 Determine the Fourier coefficients of the sawtooth function given by $f(x)=x^2$ on the interval $(-1,1)$.

Solution

Let the function $f(x)$ be given by $f(x)=x^2$ on the interval (0, 1). We wish to obtain a Fourier sine series for this function. We then first extend it to an odd function on the interval (−1, 1) and subsequently extend it periodically with period 2. The function and its odd and periodic extension are drawn in figure 4.

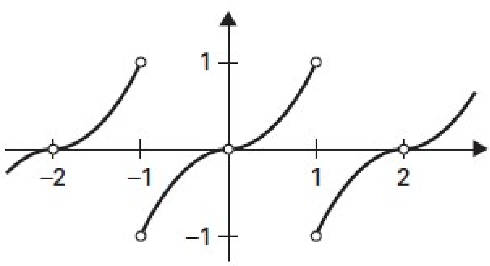


Fig.4

The ordinary Fourier coefficients of the function thus created can be calculated using (1) and (2). Since the function is odd, all coefficients a_n will equal 0. For b_n we have

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin n\,x\,dx = \int_{-1}^{0}(-x^2)\sin nxdx + \int_{0}^{1} x^2\sin nxdx = 2\int_{0}^{1} x^2\sin nxdx.$$

Applying integration by parts twice, it follows that

$$b_n = \frac{-2}{\pi n}\left(\left[x^2\cos\pi nx\right]_0^1 - \frac{2}{\pi n}\left[x\sin\pi nx\right]_0^1 - \frac{2}{\pi^2 n^2}\left[\cos\pi nx\right]_0^1\right) =$$

$$= \frac{2}{\pi n}\left(\frac{2((-1)^n - 1)}{\pi^2 n^2} - (-1)^n\right).$$

The Fourier sine series of $f(x) = x^2$ on the interval (0, 1) is thus equal to

$$\sum_{n=0}^{\infty}\frac{2}{\pi n}\left(\frac{2((-1)^n - 1)}{\pi^2 n^2} - (-1)^n\right)\sin\pi nx.$$

Example 3 Determine the Fourier coefficients of the function given by $f(x) = \sin x$ on the interval $(0,\pi)$.

Solution

In this final example we will show that one can even obtain a Fourier cosine series for the sine function on the interval $(0,\pi)$. To this end we first extend $f(x) = \sin x$ to an even function on the interval $(-\pi,\pi)$ and then extend it periodically with period $T = 2\pi$; see figure 5. The ordinary Fourier coefficients of the function thus created can be calculated using (1) and (2). Since the function is even, all coefficients will be equal to 0.

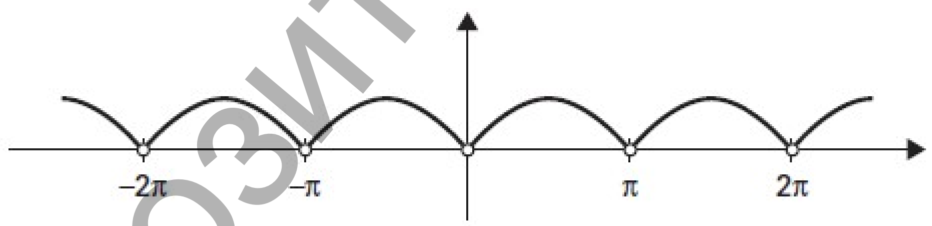


Fig.5

For a_n one has

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\sin x)\cos nxdx + \int_{0}^{\pi}\sin x\cos nxdx\right) = \frac{2}{\pi}\int_{0}^{\pi}\sin x\cos nxdx.$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi}\left(\sin(1+n)x + \sin(1+n)x\right)dx = \frac{1}{\pi}\left[\frac{-1}{1+n}\cos(1+n)x + \frac{-1}{1-n}\cos(1-n)x\right]_0^{\pi} =$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{1+n} + \frac{1-(-1)^{n-1}}{1-n}\right) = \frac{2(1-(-1)^{n-1})}{\pi(1-n^2)}.$$

If $n = 0$, then

$$a_0 = \frac{2(1-(-1)^{0-1})}{\pi(1-0^2)} = \frac{4}{\pi}.$$

The Fourier cosine series of the function $f(x) = \sin x$ on the interval $(0, \pi)$ is thus equal to

$$\frac{2}{\pi} + \sum_{n=0}^{\infty} \frac{2(1-(-1)^{n-1})}{\pi(1-n^2)} \cos nx.$$

Exercise Set 1

In Exercises 1 to 4 determine the Fourier coefficients of the given functions on the given intervals:

1. $f(x) = \begin{cases} 1 + \dfrac{x}{\pi}, & if \ -\pi \le x < 0, \\ 1 - \dfrac{x}{\pi}, & if \ \ 0 \le x < \pi. \end{cases}$

2. $f(x) = \begin{cases} 0, & if \ -\pi < x \le 0, \\ \dfrac{\pi x}{4}, & if \ \ 0 < x < \pi. \end{cases}$

3. $f(x) = x^2, \ \ x \in [-\pi; \pi].$

4. $f(x) = \begin{cases} ax, & if \ -\pi < x < 0, \\ bx, & if \ \ 0 \le x < \pi. \end{cases}$

In Exercises 5 to 8 determine the Fourier sine series of the given functions on the given intervals:

5. $y = 1 - \dfrac{x}{2}, x \in [0;2].$

6. $f(x) = 1 - x, \ x \in (0;2).$

7. $f(x) = x(1-x), \ x \in (0;1).$

8. $f(x) = \dfrac{1}{2}, \ x \in (0;2).$

In Exercises 9 to 10 determine the Fourier cosine series of the given functions on the given intervals:

9. $f(x) = 1 - x, \ x \in (0;2), \ l = 2.$

10. $f(x) = x(1-x), \ x \in (0;1), \ l = 1.$

11. $f(x) = \begin{cases} x, & if \ \ 0 < x \le 2, \\ 2, & if \ \ 2 < x < 4. \end{cases}$

12. $y = \cos x, x \in [0; \pi]$

2 LAPLACE TRANSFORMS

In physical reality we usually study signals that have been switched on at a certain moment in time. One then chooses this switch-on time as the origin of the time-scale. Hence, in such a situation we are dealing with functions on R which are zero for $t<0$, the so-called causal functions. The Fourier transform of such a function $f(t)$ is then given by

$$F(p)=\int_{0}^{+\infty} f(t)\mathrm{e}^{-pt}\,dt,$$

where $p\in\mathbb{C}$. A disadvantage of this integral is the fact that, even for very simple functions, it often does not exist. For the unit step function $\eta(t)$ for example, the integral does not exist and in order to determine the spectrum of $\eta(t)$ we had to resort to distribution theory.

The function $F(p)$ is called the Laplace transform of the causal function $f(t)$ and the mapping assigning the function $F(p)$ to $f(t)$ is called the Laplace transform. When studying phenomena where one has to deal with switched-on signals, the Laplace transform is often given preference over the Fourier transform. In fact, the Laplace transform has a better way 'to deal with the switch-on time $t=0$'. Another advantage of the Laplace transform is the fact that we do not need distributions very often, since the Laplace transform of 'most' functions exists as an ordinary integral. For most applications it therefore suffices to use only a very limited part of the distribution theory. Although the fundamental theorem of the Laplace transform can easily be derived from the one for the Fourier integral. In order to recover a function $f(t)$ from its Laplace transform $F(p)$ we will instead use a table (see chapter 3.3, Table 1), the properties of the Laplace transform and partial fraction expansions.

2.1 Definition and Existence of the Laplace Transform

Definition(causal function) A continuous-time signal $f(t)$, or a discrete-time signal $f[n]$ respectively, is called causal if

1) $f(t)=0$, where $t<0$;

2) for each increasing t condition $|f(t)|\le M\mathrm{e}^{\alpha t}$, $M,\alpha=const$, will be determined.

Definition (Laplace transform) Let $f(t)$ be a causal function, so $f(t)=0, t<0$. The Laplace transform $F(p)$ of $f(t)$ is the complex function defined for $p\in\mathbb{C}$ by

$$F(p)=\int_{0}^{+\infty} f(t)\mathrm{e}^{-pt}\,dt \tag{1}$$

provided the integral exists.

We will see in a moment that for many functions $f(t)$ the Laplace transform $F(p)$ exists (on a certain subset of $\mathbb{C}$). The mapping assigning the Laplace transform $F(p)$ to a function $f(t)$ in the time domain will also be called the Laplace transform. Furthermore, we will say that $F(p)$ is defined in the p-domain; one sometimes calls this p-domain the 'complex frequency domain' (although a physical interpretation can hardly be given for arbitrary $p\in\mathbb{C}$). Besides the notation $F(p)$ we will also use $L(f(t))$, so $F(p)=L(f(t))$. Often the notation $L(f(t))(p)$, although not very elegant, will be useful in the case of a concrete function.

Theorem 1 Let $f(t)$ be a causal function and consider the integral in (1). If the integral is absolutely convergent for a certain value $p = s_0 \in R$, then the integral is absolutely convergent for all $p \in \mathbb{C}$ with $\operatorname{Re} p = s > s_0$.

Note 1 If $F(p) = L(f(t))$, then $\lim\limits_{p \to \infty} F(p) = 0$.

Example 1 Determine for the following functions the Laplace transform:

a) the shifted unit step function $(1 \cdot \eta(t))$ in figure 6 is defined by $\eta(t) = \begin{cases} 1, & if \quad t \geq 0, \\ 0, & if \quad t < 0; \end{cases}$

b) $f(t) = \mathrm{e}^{at}$;

c) $f(t) = t$;

d) $f(t) = t^2$.

Solution

a)

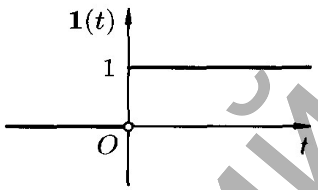


Fig.6

$$F(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,dt = \int_0^{+\infty} 1 \cdot \mathrm{e}^{-pt}\,dt = \lim_{b\to\infty}\int_0^{b} \mathrm{e}^{-pt}\,dt = \lim_{b\to\infty} -\frac{1}{p}\Big[\mathrm{e}^{-pt}\Big]_0^b = \frac{1}{p}.$$

$$L(1 \cdot \eta(t)) = \frac{1}{p}.$$

b) $$F(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,dt = \int_0^{+\infty} e^{at} \cdot \mathrm{e}^{-pt}\,dt = \lim_{b\to\infty}\int_0^{b} \mathrm{e}^{-(p-a)t}\,dt =$$

$$= -\lim_{b\to\infty}\frac{1}{p-a}\Big[\mathrm{e}^{-(p-a)t}\Big]_0^b = \lim_{b\to\infty}\left(\frac{1}{p-a} - \frac{\mathrm{e}^{-(p-a)b}}{p-a}\right) = \frac{1}{p-a}.$$

$$L\left(\mathrm{e}^{at} \cdot \eta(t)\right) = \frac{1}{p-a}.$$

c) For $p = 0$ the Laplace transform does not exist, while for $p > 0$ it follows from integration by parts that

$$F(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,dt = \int_0^{+\infty} t \cdot \mathrm{e}^{-pt}\,dt = \lim_{b\to\infty}\int_0^{b} t\,\mathrm{e}^{-pt}\,dt =$$

$$= \lim_{b\to\infty}\left(-\left[\frac{t}{p}\mathrm{e}^{-pt}\right]_0^b - \frac{1}{p^2}\Big[\mathrm{e}^{-pt}\Big]_0^b\right) = \frac{1}{p^2}.$$

$$L(t \cdot \eta(t)) = \frac{1}{p^2}.$$

d) For $p = 0$ the Laplace transform does not exist, while for $p > 0$ it follows from integration by parts that

$$F(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,dt = \int_0^{+\infty} t^2 \cdot \mathrm{e}^{-pt}\,dt = \lim_{b\to\infty}\int_0^{b} t^2\,\mathrm{e}^{-pt}\,dt =$$

$$= \lim_{b\to\infty}\left(-\left[\frac{t^2}{p}\mathrm{e}^{-pt}\right]_0^b - \frac{2}{p^2}\left[t\,\mathrm{e}^{-pt}\right]_0^b - \frac{2}{p^3}\left[\mathrm{e}^{-pt}\right]_0^b\right) = \frac{2}{p^3}.$$

$$L\left(t^2 \cdot \eta(t)\right) = \frac{2}{p^3}.$$

2.2 Linearity, Shifting and Scaling

1. Linearity

As for the Fourier transform, the linearity of the Laplace transform follows immediately from the linearity of integration (see section 2.4). For $A, B \in \mathbb{C}$ one thus has

$$L\left(Af(t) + B\varphi(t)\right) = AL\left(f(t)\right) + BL\left(\varphi(t)\right) = AF(p) + B\Phi(p) \qquad (1)$$

in the half-plane where $F(p) = L(f(t))$ and $\Phi(p) = L(\varphi(t))$ both exist.

2. Shift in the time domain

The unit step function is often used to represent the switching on of a signal $f(t)$ at time $t = 0$ (see figure 7a). When several signals are switched on at different moments in time, then it is convenient to use the shifted unit step function $\eta(t-b)$. In fact, when the signal f is switched on at time $t = b, b \ge 0$, then this can simply be represented by the function $f(t-b)\cdot\eta(t-b)$ (See figure 7.b). Using the functions $\eta(t-b)$ it is also quite easy to represent combinations of shifted (switched on) signals.

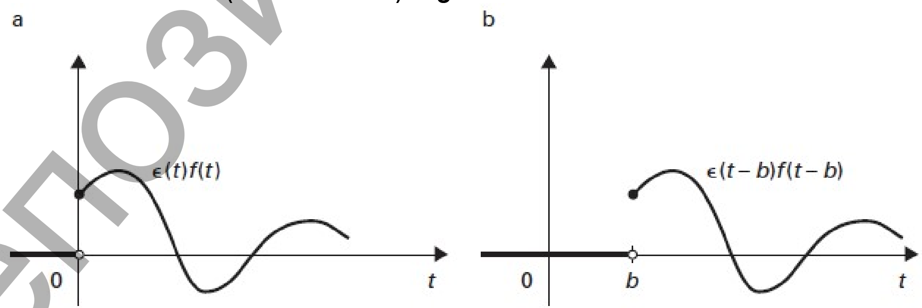


Fig.7

Figure 8, for example, shows the graph of the causal function

$$f(t) = 3 - 2(t-1)\eta(t-1) + 2(t-3)\eta(t-3).$$

In fact, $f(t) = 3$ for $0 \le t < 1$, $f(t) = 3 - 2(t-1) = 5 - 2t$ for $1 \le t < 3$ and $f(t) = 3 - 2(t-1) + 2(t-3) = -1$ for $t \ge 3$.

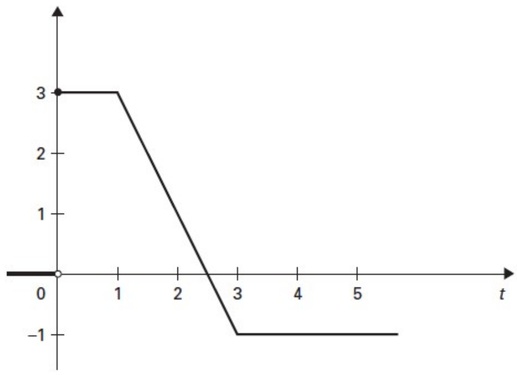


Fig.8

Let $f(t)$ be a function with Laplace transform $F(p)$ for $\operatorname{Re} p = s > s_0$ and let $b \geq 0$. Then one has for $\operatorname{Re} p = s > s_0$ that

$$L(f(t-b)\eta(t-b)) = \mathrm{e}^{-pb} F(p) \tag{2}$$

3. Shift in the p-domain

Let $f(t)$ be a function with Laplace transform $F(p)$ for $\operatorname{Re} p = s > s_0$ and let $b \in \mathbb{C}$. Then one has for $\operatorname{Re} p = s > s_0 + \operatorname{Re} b$ that

$$L(\mathrm{e}^{bt} f(t)) = F(p-b) \tag{3}$$

4. Scaling

Let $f(t)$ be a function with Laplace transform $F(p)$ for $\operatorname{Re} p = s > s_0$ and let $b \geq 0$. Then one has for $\operatorname{Re} p = s > bs_0$ that

$$L(f(bt)) = \frac{1}{b} F\left(\frac{p}{b}\right) \tag{4}$$

Example 2 Determine the Laplace transform $F(p)$ of the following functions:

a) $\sin(wt)$, $\cos(wt)$; **b)** $\operatorname{ch}(wt)$, $\operatorname{sh}(wt)$; **c)** $f(t) = (5\cos t - 3\sin t)\,\eta(t)$;

d) $f(t) = \mathrm{e}^{4t}\cos t$; **e)** $f(t) = \begin{cases} t, & if\ \ 0 \leq t \leq 2, \\ 4-t, & if\ 2 < t < 4, \\ 0, & if\ \ t < 0, t \geq 4; \end{cases}$ **f)** $f(t) = (2t^2 - 6t) \cdot \eta(t-3)$.

Solution

a) $L(\sin(wt)) = L\left(\dfrac{\mathrm{e}^{iwt} - \mathrm{e}^{-iwt}}{2i}\right) = \dfrac{1}{2i}\left(\dfrac{1}{p-iw} - \dfrac{1}{p+iw}\right) = \dfrac{w}{p^2+w^2}$,

$L(\cos(wt)) = L\left(\dfrac{\mathrm{e}^{iwt} + \mathrm{e}^{-iwt}}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{p-iw} + \dfrac{1}{p+iw}\right) = \dfrac{p}{p^2+w^2}$.

b) $L(\operatorname{ch}(wt)) = L\left(\dfrac{\mathrm{e}^{wt} + \mathrm{e}^{-wt}}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{p-w} - \dfrac{1}{p+w}\right) = \dfrac{p}{p^2-w^2}$,

$L(\operatorname{sh}(wt)) = L\left(\dfrac{e^{wt} - e^{-wt}}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{p-w} + \dfrac{1}{p+w}\right) = \dfrac{w}{p^2-w^2}$.

c) $L(f(t)) = L(5\cos t - 3\sin t)\ \eta(t) = 5\dfrac{p}{p^2+1^2} - 3\dfrac{1}{p^2+1^2} = \dfrac{5p-3}{p^2+1}$.

d) $L(f(t)) = L(\mathrm{e}^{4t}\cos t\eta(t)) = \dfrac{p-4}{(p-4)^2+1^2} = \dfrac{p-4}{p^2-8p+17}$.

e) Figure 9 shows the graph of the causal function $f(t) = \begin{cases} t, & if\ \ 0 \le t \le 2, \\ 4-t, & if\ 2 < t < 4, \\ 0, & if\ \ t < 0, t \ge 4. \end{cases}$

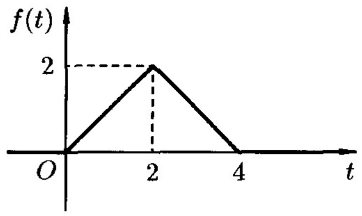


Fig.9

Rewrite the casual function as analytic expression with help unit step function $\eta(t-b)$ and $\eta(t)$

$f(t) = t\cdot\eta(t) - t\cdot\eta(t-2) + (4-t)\cdot\eta(t-2) - (4-t)\cdot\eta(t-4)$,

$f(t) = t\cdot\eta(t) - (t-2+2)\cdot\eta(t-2) - (t-2-2)\cdot\eta(t-2) + (t-4)\cdot\eta(t-4)$,

$f(t) = t\cdot\eta(t) - 2(t-2)\cdot\eta(t-2) + (t-4)\cdot\eta(t-4)$,

$L(f(t)) = L(t\cdot\eta(t) - 2(t-2)\cdot\eta(t-2) + (t-4)\cdot\eta(t-4)) = \dfrac{1}{p^2} - 2\dfrac{1}{p^2}\mathrm{e}^{-2p} + \dfrac{1}{p^2}\mathrm{e}^{-4p}$.

f) $L(f(t)) = L((2t^2-6t)\cdot\eta(t-3)) =$

$$= \begin{bmatrix} t-3 = a, t = a-3 \\ 2t^2 - 6t = 2(a-3)^2 - 6(a-3) = 2a^2 - 18a + 36 \end{bmatrix} =$$

$$= L(2a^2 - 18a + 36)\eta(a) = \left(\frac{2}{p^3} - \frac{18}{p^2} + \frac{36}{p}\right)e^{-3p}.$$

Example 3 Determine a function $f(t)$ whose Laplace transform $F(p)$ is given by

a) $F(p) = \dfrac{p}{p^2-25}$; **b)** $F(p) = \dfrac{p-2}{p^2-4p-5}$; **c)** $F(p) = \dfrac{3p-2}{p^2-2p-8}$.

Solution

a) Using the properties of scaling we have

$$f(t) = L^{-1}(F(p)) = L^{-1}\left(\frac{p}{p^2-25}\right) = L^{-1}\left(\frac{p}{p^2-5^2}\right) = \mathrm{ch}\,5t.$$

b) Using the properties of shift in the p-domain, scaling we have

$$L^{-1}(F(p)) = L^{-1}\left(\frac{p-2}{p^2-4p-5}\right) =$$

$$=\left[\frac{p-2}{p^2-4p-5}=\frac{p-2}{(p-2)^2-9}=\frac{p-2}{(p-2)^2-3^2}\right]=\mathrm{e}^{2t}\cdot \operatorname{ch}3t.$$

c) Using the properties of linearity, shift in the p-domain, scaling we have

$$L^{-1}\big(F(p)\big)=L^{-1}\left(\frac{3p-2}{p^2-2p-8}\right)=$$

$$=\left[\frac{3p-2}{p^2-2p-8}=\frac{3(p-1)+1}{(p-1)^2-9}=\frac{3(p-1)}{(p-1)^2-3^2}+\frac{1}{3}\cdot\frac{3}{(p-1)^2-3^2}\right]=$$

$$=3L^{-1}\left(\frac{(p-1)}{(p-1)^2-3^2}\right)+\frac{1}{3}L^{-1}\left(\frac{3}{(p-1)^2-3^2}\right)=\mathrm{e}^{t}\left(3\operatorname{ch}3t+\frac{1}{3}\operatorname{sh}3t\right).$$

Other way:

$$L^{-1}\big(F(p)\big)=L^{-1}\left(\frac{3p-2}{p^2-2p-8}\right)=$$

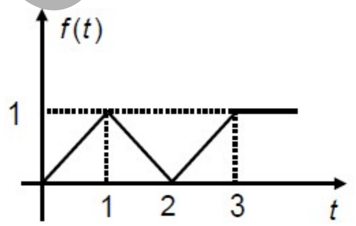
$$=\left[\begin{array}{l}\dfrac{3p-2}{p^2-2p-8}=\dfrac{3p-2}{(p+2)(p-4)}=\dfrac{A}{p+2}+\dfrac{B}{p-4}=\\ =\dfrac{(A+B)p+2B-4A}{(p+2)(p-4)}\Rightarrow A+B=3,2B-4A=-2\Rightarrow A=\dfrac{4}{3},B=\dfrac{5}{3}\end{array}\right]=$$

$$=\frac{4}{3}L^{-1}\left(\frac{1}{p+2}\right)+\frac{5}{3}L^{-1}\left(\frac{1}{p-4}\right)=\frac{4}{3}\mathrm{e}^{-2t}+\frac{5}{3}\mathrm{e}^{4t}.$$

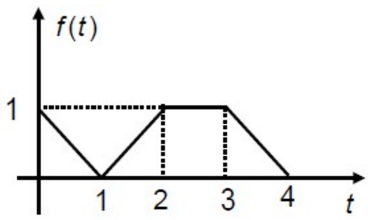
Exercise Set 2

In Exercises 1 to 17 determine the Laplace transform $F(p)$ of the following functions:

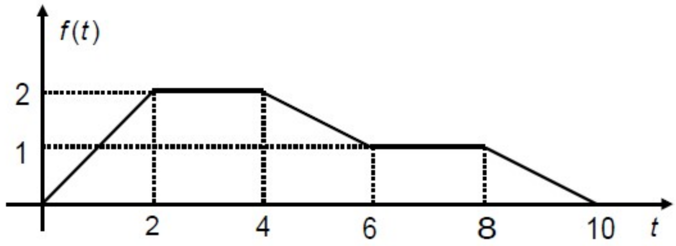
1. $f(t)=(3t^4-2t^3+t^2-7)\,\eta(t)$;
2. $f(t)=(5\cos t-3\sin t)\,\eta(t)$;
3. $f(t)=(\mathrm{e}^{-7t}+4\operatorname{sh}7t-2\operatorname{ch}7t)\eta(t)$;
4. $f(t)=\cos^2 8t\cdot\eta(t)$;
5. $f(t)=((t-1)^2+4(t-1)+6))\cdot\eta(t-1)$;
6. $f(t)=(t^2-4t)\cdot\eta(t-2)$;
7. $f(t)=(t^3-6t^2+4t-8)\cdot\eta(t-1)$;
8. $f(t)=\sin 2(t-3)\cdot\eta(t-3)$;
9. $f(t)=(3\mathrm{e}^{it}-4\sin t+7\cos t)\cdot\eta(t)$;
10. $f(t)=(4\mathrm{e}^{3t}+2\operatorname{sh}3t-6\operatorname{sh}3t)\cdot\eta(t)$;
11. $f(t)=\sin 6t\cdot\cos 4t\cdot\eta(t)$;
12. $f(t)=\sin 8t\cdot\sin 2t\cdot\eta(t)$;
13. $f(t)=(t^2-t+2)\cdot\eta(t-1)$;
14. $f(t)=(t^2+2t+5)\cdot\eta(t-3)$;
15.



16.



17.



In Exercises 18 to 27 determine a function $f(t)$ whose Laplace transform $F(p)$ is given by:

18. $F(p)=\dfrac{3}{(p-2)^2+9}$;

19. $F(p)=\dfrac{5}{(p-4)^2-9}$;

20. $F(p)=\dfrac{p}{p^2+36}$;

21. $F(p)=\dfrac{5p}{p^2-49}$;

22. $F(p)=\dfrac{p+1}{p^2+2p}$;

23. $F(p)=\dfrac{p-10}{p^2+4p}$;

24. $F(p)=\dfrac{p+2}{p^2+4p-5}$;

25. $F(p)=\dfrac{p-4}{p^2-8p+17}$;

26. $F(p)=\dfrac{5p-3}{p^2-4p-12}$;

27. $F(p)=\dfrac{7p-3}{p^2-6p+10}$.

2.3 Differentiation in the Time Domain and in the p-Domain. Integration in the Time Domain. Convolution

Theorem 1 Let $f(t)$ be a causal function which, in addition, is differentiable on $\mathbb{R}$. In a half-plane where $L(f(t))$ and $L\left(f'(t)\right)$ both exist one has

$$L\left(f'(t)\right)=pF(p)-f(0) \tag{1}$$

By repeatedly applying theorem 1, one can obtain the Laplace transform of the higher derivatives of a function. Of course, the conditions of theorem 1 should then be satisfied throughout. Suppose, for example, that a causal function $f(t)$ is continuously differentiable on $\mathbb{R}$ (so $f'(t)$ exists and is continuous on $\mathbb{R}$) and that $f'(t)$ is differentiable on $\mathbb{R}$. By applying theorem 1 twice in a half-plane where all Laplace transforms exist, it then follows that

$$L\left(f''(t)\right)=p^2F(p)-f(0)p-f'(0)$$

$$L\left(f'''(t)\right)=p^2F(p)-f(0)p^2-f'(0)p-f''(0)$$

$$\ldots$$

$$L\left(f^{(n)}(t)\right)=p^nF(p)-f(0)p^{n-1}-f'(0)p^{n-2}-\ldots-f^{(n-1)}(0) \tag{2}$$

In a half-plane where all Laplace transforms exist and $f(0)=f'(0)=\ldots=f^{(n)}(0)=0$, we then have the following differentiation rule in the time domain:

$$L\left(f^{(n)}(t)\right)=p^nF(p) \tag{3}$$

Theorem 2 Let $f(t)$ be a function with Laplace transform $F(p)$ and let s_0 be the abscissa of absolute convergence. Then $F(p)$ is an analytic function of p for $\mathrm{Re}\, p=s>s_0$ and

$$L((-t)f(t)=F'(p),\quad L((-t)^2 f(t)=F''(p),\quad \ldots \quad L((-t)^n f(t)=F^{(n)}(p) \tag{4}$$

Theorem 3 Let $f(t)$ be a causal function which is continuous on $\mathbb{R}$ and has Laplace transform $F(p)$. Then one has in a half-plane contained in the region $\operatorname{Re} p = s > 0$

$$L\left(\int_0^t f(\tau)d\tau\right)(p) = \frac{F(p)}{p} \tag{5}$$

Theorem 4 Let $f(t)$ be a function with Laplace transform $F(p)$ and $\int_p^\infty F(\rho)d\rho$ is absolutely convergent for a certain value $\operatorname{Re} p = s > s_0$, then

$$L^{-1}\left(\int_p^\infty F(\rho)d\rho\right) = \frac{f(t)}{t} \tag{6}$$

Example 4 Determine for the following functions the Laplace transform:

a) $f(t)=t^3$, **b)** $f(t)=t^n$, **c)** $f(t)=t^n\,\mathrm{e}^{at}$,

d) $f(t)=t\sin wt$, **e)** $f(t)=\mathrm{e}^{at}\,t\sin wt$.

Solution

a) Let $L(1\cdot\eta(t))=\dfrac{1}{p}$ and $L(t\cdot\eta(t))=\dfrac{1}{p^2}$.

Using (4)

$$L(-t^2\cdot\eta(t))=\left(\frac{1}{p^2}\right)'_p=-\frac{2}{p^3},$$

$$L(t^2\cdot\eta(t))=\left(\frac{1}{p^2}\right)'_p=\frac{2!}{p^3}.$$

b) The method from the example above can be used to determine $L(t^n\cdot\eta(t))$ for every $n\ge 2, n\in N$. In fact, the function $t^n\cdot\eta(t)$ satisfies the conditions of theorem 2 for $n\ge 2, n\in N$ and so it follows from (4) that

$$L(t^n\cdot\eta(t))=\frac{n!}{p^{n+1}}.$$

c) Using the property of shift in the p-domain, we have

$$L(t^n e^{at}\eta(t))=\frac{n!}{(p-a)^{n+1}}.$$

d) If $L\big(\sin(wt)\big)=\dfrac{w}{p^2+w^2}$:

$$L(-t\sin wt\cdot\eta(t))=\left(\frac{w}{p^2+w^2}\right)'_p=-\frac{2wp}{\left(p^2+w^2\right)^2}.$$

e) Using (d) and the property of shift in the p-domain, we have

$$L(e^{at}t\sin wt\cdot\eta(t))=-\frac{2w(p-a)}{\left((p-a)^2+w^2\right)^2}.$$

Example 5 Determine for the following functions the Laplace transform:

a) $f(t)=1-\cos t$, **b)** $f(t)=\dfrac{\sin t}{t}$.

Solution

a) The causal function $f(t)=\sin t$ is continuous on $\mathbb{R}$ and since $\int_0^t \sin\tau d\tau=1-\cos t$, it then follows from theorem 3 that

$$L(1-\cos t)(p)=\frac{L(\sin t)(p)}{p}=\frac{1}{p(p^2+1)}.$$

This result is easy to verify since we know, that $L\left(\cos(wt)\right)=\dfrac{p}{p^2+w^2}$ and $L(1\cdot\eta(t))=\dfrac{1}{p}$. Hence,

$$L(1-\cos t)(p)=\frac{1}{p}-\frac{p}{p^2+1}=\frac{1}{p(p^2+1)}.$$

b) If $L\left(\sin(wt)\right)=\dfrac{w}{p^2+w^2}$:

$$L\left(\frac{\sin t}{t}\cdot\eta(t)\right)=\int_p^{\infty}\frac{1}{\rho^2+1}d\rho=\frac{\pi}{2}-\operatorname{arctg} p=\operatorname{arcctg} p.$$

Example 6 Determine a function $f(t)$ whose Laplace transform $F(p)$ is given by

$$F(p)=\frac{4}{p(p^2+4)}.$$

Solution Let $f(t)$ be a function with Laplace transform $F(p)=\dfrac{4}{p(p^2+4)}$. If we ignore the factor $\dfrac{1}{p}$, then we are looking for a function $g(t)$ having Laplace transform $G(p)=\dfrac{4}{p^2+4}$. This is easy: $g(t)=2\sin 2t$. Integrating $g(t)$ we find $f(t)$:

$$g(t)=\int_o^t 2\sin 2\tau d\tau=\left[-\cos 2\tau\right]_0^t=1-\cos 2t.$$

Example 7 Determine for the following expression the Laplace transform:

$$x'''(t)-2x''(t)-3x'(t)+2x(t)+2 \quad \text{if} \quad x(0)=3, x'(0)=0, x''(0)=-2.$$

Solution Let $L(x(t))=X(p)=X$. Using the property of linearity and (1)-(3) we have

$$L\big(x'(t)\eta(t)\big)(p) = pX - 3,$$
$$L\big(x''(t)\eta(t)\big)(p) = p^2X - 3p - 0,$$
$$L\big(x'''(t)\eta(t)\big)(p) = p^3X - 3p^2 - 0p + 2,$$
$$L(2\cdot\eta(t)) = \frac{2}{p}.$$

Then

$$L\Big(\big(x'''(t) - 2x''(t) - 3x'(t) + 2x(t) + 2\big)\eta(t)\Big)(p) =$$
$$= p^3X - 3p^2 + 2 - 2(p^2X - 3p) - 3(pX - 3) + 2X + \frac{2}{p}.$$

When, moreover, $f(t)$ and $\varphi(t)$ are both causal, then one has for $t > 0$ that

$$\int_0^t f(u)\varphi(t-u)\,du = f * \varphi \tag{7}$$

since the integrand is zero for both $u < 0$ and $t - u < 0$.

Theorem 5 (Convolution theorem) Let $f(t)$ and $\varphi(t)$ be piecewise smooth and causal functions. Let the Laplace transforms for $L(f(t)) = F(p)$ and $L(\varphi(t)) = \Phi(p)$ exist as absolutely convergent integrals in a half-plane $\operatorname{Re} p = s > s_0$. Then $L(f(t) * \varphi(t))$ exists for $\operatorname{Re} p = s > s_0$ and

$$L(f(t) * \varphi(t)) = F(p)\cdot\Phi(p) \tag{8}$$

Example 8 Determine a function $f(t)$ whose Laplace transform $F(p)$ is given by

$$F(p) = \frac{1}{\left(p^2 + w^2\right)^2}.$$

Solution Let

$$F(p) = \frac{1}{(p^2 + w^2)^2} = \frac{1}{(p^2 + w^2)}\cdot\frac{1}{(p^2 + w^2)},$$
$$L\left(\frac{1}{w}\sin(wt)\right) = \frac{1}{p^2 + w^2}.$$

Then $L^{-1}\left(\dfrac{1}{(p^2 + w^2)^2}\right) = \displaystyle\int_0^t \frac{1}{w}\cdot\sin wu\cdot\frac{1}{w}\cdot\sin w(t-u)\,du =$

$$= \frac{1}{2w^2}\cdot\int_0^t \big(\cos w(2u - t) - \cos wt\big)du = \frac{1}{2w^2}\cdot\left(\frac{1}{2w}\big[\sin w(2u - t)\big]_0^t - \cos wt\,[u]_0^t\right) =$$
$$= \frac{1}{2w^2}\left(\frac{1}{w}\sin wt - t\cos wt\right) = \frac{1}{2w^3}\big(\sin wt - wt\cdot\cos wt\big).$$
$$L^{-1}\left(\frac{1}{(p^2 + w^2)^2}\right) = \frac{1}{2w^3}\big(\sin wt - wt\cdot\cos wt\big).$$

Using the properties of linearity, shift in the p-domain, scaling, differentiation in the time domain, differentiation in the p-domain, integration in the time domain, convolution summary table (1) establishes a correspondence between some of the causal functions and their Laplace transform

Table 1

№	$f(t)$	$F(p)$
1.	$1\cdot\eta(t)$	$\frac{1}{p}$
2.	e^{at}	$\frac{1}{p-a}$
3.	t	$\frac{1}{p^2}$
4.	$\sin wt$	$\frac{w}{p^2+w^2}$
5.	$\cos wt$	$\frac{p}{p^2+w^2}$
6.	$\mathrm{sh}\, wt$	$\frac{w}{p^2-w^2}$
7.	$\mathrm{ch}\, wt$	$\frac{p}{p^2-w^2}$
8.	$e^{at}\cdot\sin wt$	$\frac{w}{\left((p-a)^2+w^2\right)}$
9.	$e^{at}\cdot\cos wt$	$\frac{p-a}{(p-a)^2+w^2}$
10.	$e^{at}\cdot\mathrm{sh}\, wt$	$\frac{w}{(p-a)^2-w^2}$
11.	$e^{at}\cdot\mathrm{ch}\, wt$	$\frac{p-a}{(p-a)^2-w^2}$
12.	t^n	$\frac{n!}{p^{n+1}}$
13.	$e^{at}\cdot t^n$	$\frac{n!}{(p-a)^{n+1}}$
14.	$t\cdot\sin wt$	$\frac{2wp}{\left(p^2+w^2\right)^2}$
15.	$t\cdot\cos wt$	$\frac{p^2-w^2}{\left(p^2+w^2\right)^2}$

№	$f(t)$	$F(p)$
16.	$t\cdot\operatorname{sh} wt$	$\dfrac{2wp}{\left(p^2-w^2\right)^2}$
17.	$t\cdot\operatorname{ch} wt$	$\dfrac{p^2+w^2}{\left(p^2-w^2\right)^2}$
18.	$\mathrm{e}^{at}\cdot t\cdot\cos wt$	$\dfrac{(p-a)^2-w^2}{\left((p-a)^2+w^2\right)^2}$
19.	$\mathrm{e}^{at}\cdot t\cdot\sin wt$	$\dfrac{2w(p-a)}{\left((p-a)^2+w^2\right)^2}$

Exercise Set3

In Exercises 1 to 10 determine the Laplace transform $F(p)$ of the following functions:

1. $f(t)=\operatorname{ch} t\cdot\sin t$.
2. $f(t)=\operatorname{ch} 2t\cdot\operatorname{sh} t$.
3. $f(t)=-t\cos 3t$.
4. $f(t)=\mathrm{e}^{4t}\cos t$.
5. $f(t)=t\cdot\mathrm{e}^{-3t}$.
6. $f(t)=t^2\mathrm{e}^{5t}$.
7. $f(t)=\mathrm{e}^{2(t-1)}\cos 3(t-1)\cdot\eta(t-1)$.
8. $f(t)=t\cdot\sin t+\cos t$.
9. $f(t)=\mathrm{e}^{-t}\sin^2 t$.
10. $f(t)=\dfrac{1}{2}\left(\operatorname{ch} t\cdot\sin t+\operatorname{sh} t\cdot\cos t\right)$.

In Exercises 11 to 20 determine a function $f(t)$ whose Laplace transform $F(p)$ is given by:

11. $F(p)=\dfrac{1}{p^{10}}$.
12. $F(p)=\dfrac{1}{(p-3)^5}$.
13. $F(p)=\dfrac{3}{(p-2)^2+9}$.
14. $F(p)=\dfrac{5}{(p-1)^2-9}$.
15. $F(p)=\dfrac{p}{p^2-25}$.
16. $F(p)=\dfrac{4p}{p^2+25}$.
17. $F(p)=\dfrac{\mathrm{e}^{-2p}}{(p+1)^3}$.
18. $F(p)=\dfrac{\mathrm{e}^{-4p}}{(p+7)^5}$.
19. $F(p)=\dfrac{4}{p^2(p^2+16)}$.
20. $F(p)=\dfrac{4}{p(p^2+9)}$.

2.4 The Inverse Laplace Transform. Applications of the Laplace Transform

Theorem 1 (Fundamental theorem of the Laplace transform) Let $f(t)$ be a piecewise smooth (and causal) function of exponential order $\alpha\in\mathbb{R}$. Let $F(p)$ be the Laplace transform of $f(t)$. Then one has for $t\geq 0$ and $p=s+i\sigma$ with $\sigma>\alpha$ that

$$f(t) = L^{-1}F(p) = \frac{1}{2\pi i}\int\limits_{\alpha - i\infty}^{\alpha + i\infty} F(p)e^{pt}\, dp \qquad (1)$$

As for the Fourier transform, theorem 1 tells us precisely how we can recover the function $f(t)$ from $F(p)$. Obtaining $f(t)$ from $F(p)$ is called the inverse problem and therefore theorem 1 is also known as the inversion theorem and (1) as the inversion formula. We will call the function $f(t)$ the inverse Laplace transform of $F(p)$. Still, (1) will not be used for this purpose. In fact, calculating the integral in (1) requires a thorough knowledge of the integration of complex functions over lines in $\mathbb{C}$, an extensive subject which is outside the scope of this book. Hence, the fundamental theorem of the Laplace transform will not be used in the remainder of this book, except in the form of the frequent (implicit) application of the fact that the Laplace transform is one-to-one. Moreover, in practice it is often a lot easier to determine the inverse Laplace transform of a function $F(p)$ by using tables, applying the properties of the Laplace transform, and using partial fraction expansions.

We will now describe in a number of steps how the inverse Laplace transform of such a rational function $F(p)$ can be determined.

Step 1 If the degree of the numerator is greater than or equal to the degree of the denominator, then we perform a division. The function $F(p)$ is then the sum of a polynomial and a rational function for which the degree of the numerator is smaller than the degree of the denominator. The polynomial gives rise to distributions in the inverse Laplace transform since

$$p^n = \left(Lg^{(n)}(t)\right)(p).$$

Example 9 We want to determine the function/distribution $f(t)$ having Laplace transform

$$F(p) = \frac{p^3 - p^2 + p}{p^2 + 1}.$$

Solution Since the degree of the numerator is greater than the degree of the denominator, we first divide: $F(p) = p - 1 + \frac{1}{p^2+1}$. Now $L(g) = 1, L(g') = p$ and $L(\sin t) = \frac{1}{p^2+1}$, so $f(t) = g'(t) - g(t) + \sin t$.

Step 2 From step 1 it follows that henceforth we may assume that $F(p)$ is a rational function for which the degree of the numerator is smaller than the degree of the denominator.

From the results of section 2.2 it then follows that $F(p)$ can be written as a sum of fractions of the form

$$\frac{A}{(p+a)^k} \quad \text{and} \quad \frac{Bp + C}{\left(p^2 + qp + c\right)^j}$$

with $k, j \in \mathbb{N}$ and where all constants are real and $p^2 + qp + c$ cannot be factorized into factors with real coefficients. This latter fact means that the discriminant of $p^2 + qp + c$ is negative. We will now determine the inverse Laplace transform for each of these fractions separately.

Step 3 From table 1 we immediately obtain the inverse Laplace transform of $\frac{A}{(p+a)^k}$

$$L\left(t^{k-1}e^{at}\eta(t)\right)=\frac{(k-1)!}{(p-a)^k}.$$

Step 4 In order to determine the inverse Laplace transform of $\dfrac{Bp+C}{\left(p^2+qp+c\right)^j}$, we complete the square in the denominator: $p^2+qp+c=\left(p+\dfrac{q}{2}\right)^2+\left(c-\dfrac{q^2}{4}\right)$. For convenience we write the positive constant $\left(c-\dfrac{q^2}{4}\right)$ simply as c^2 (for some new constant c), which means that we want to determine the inverse Laplace transform of the function

$$\frac{Bp+C}{\left(\left(p+\dfrac{q}{2}\right)^2+c^2\right)^j} \tag{2}$$

For $j=1$ we obtain the inverse Laplace transform of this function from (2) by taking a suitable linear combination of the following results from table 1:

$$L\left(e^{at}\sin wt\cdot\eta(t)\right)=\frac{w}{\left((p-a)^2+w^2\right)},$$

$$L\left(e^{at}\cos wt\cdot\eta(t)\right)=\frac{p-a}{\left((p-a)^2+w^2\right)} \tag{3}$$

Example 10 Determine a function $f(t)$ whose Laplace transform $F(p)$ is given by

a) $F(p)=\dfrac{1}{p(p^2+1)}$; **b)** $F(p)=\dfrac{1}{(p^2+16)(p^2+4)}$;

c) $F(p)=\dfrac{1}{p^3(p-1)}$; **d)** $F(p)=\dfrac{e^{-4p}(p+1)}{p^2(p+5)}+\dfrac{p+2}{p^2+2p+5}$.

Solution

a) A partial fraction expansion leads to $F(p)=\dfrac{1}{p}-\dfrac{p}{p^2+1}=\dfrac{1}{p(p^2+1)}$. From a very simple case of (3) it then follows that $f(t)=1-\cos t$.

b) Since $F(p)$ is a function of p^2, we put $y=p^2$ and apply partial fraction expansion to the function $\dfrac{1}{(y+16)(y+4)}$, resulting in $\dfrac{1}{12(y+4)}-\dfrac{1}{12(y+16)}$.

Hence, $F(p)=\dfrac{1}{12(p^2+4)}-\dfrac{1}{12(p^2+16)}$ and then it again follows from a simple case of (3) that $f(t)=\dfrac{\sin 2t}{24}-\dfrac{\sin 4t}{48}$.

c) A partial fraction expansion leads to

$$F(p)=\frac{A}{p}+\frac{B}{p^2}+\frac{C}{p^3}+\frac{N}{p-1}=\frac{1}{p^3(p-1)}=-\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}+\frac{1}{p-1}.$$

From a very simple case of table (1) it then follows that

$$f(t)=-1-t-\frac{t^2}{2}+\mathrm{e}^t.$$

d) Since the discriminant of p^2+2p+5 is negative, we complete the square: $(p+1)^2+4$. To the first term we apply partial fraction expansion:

$$F(p)=\mathrm{e}^{-4p}\left(\frac{1}{5p^2}+\frac{4}{25p}-\frac{4}{25(p+5)}\right)+\frac{p+1}{(p+1)^2+4}+\frac{1}{(p+1)^2+4}.$$

Using table (1) and the shift property in the time domain we obtain

$$f(t)=\frac{1}{25}\left(5(t-4)+4-4\mathrm{e}^{-5(t-4)}\right)\eta(t-4)+\frac{\mathrm{e}^{-t}}{2}\left(2\cos 2t+\sin 2t\right).$$

Using an elementary example we will illustrate how the Laplace transform can be used to obtain solutions to linear differential equations with constant coefficients. Moreover, we will show the difference between the method using the Laplace transform and the ‘classical’ solution method using the homogeneous and particular solutions. This classical solution method has already been explained in differential equation section and can also be found in many introductions to this subject.

We will now use the Laplace transform to solve the initial value problem for example 11

Example 11 Consider for the unknown function $y=y(t)$ the initial value problem

$$y''-y=2t,\quad y(0)=y'(0)=0.$$

Solution Apply the Laplace transform to both sides of the differential equation $y''-y=2t$. Assume that $L(y(t))=Y(p)$ exists in a certain half-plane and that moreover the differentiation rule in the time domain (table 1) can be applied. Then $L(y''-y)(p)=p^2Y-Y$ and since $L(t)=\frac{1}{p^2}$, the initial value problem transforms into

$$Y\left(p^2-1\right)=\frac{2}{p^2}.$$

Note that instead of a differential equation for $y=y(t)$ we now have an algebraic equation for $Y(p)$. Solving for $Y(p)$ and applying a partial fraction expansion we obtain that

$$Y(p)=\frac{2}{p^2(p^2-1)}=\frac{2}{p^2-1}-\frac{2}{p^2}.$$

The inverse Laplace transform $y=y(t)$ of $Y(p)$ follows from table 1:

$$y=y(t)=2\,\mathrm{sh}\,t-2t=\mathrm{e}^t-\mathrm{e}^{-t}-2t.$$

We first summarize the most important steps of the solution method in example 11.

Step 1 The Laplace transform is applied to the differential equation for $y=y(t)$. Here we assume that the Laplace transform $Y(p)$ of the unknown function $y=y(t)$ exists and that the differentiation rule in the time domain may be applied (either in the ordinary sense or in the sense of distributions). From the differential equation for $y = y(t)$ we obtain an algebraic equation for $Y(p)$ which is much easier to solve.

Step 2 The algebraic equation in the p-domain is solved for $Y(p)$.

Step 3 The solution we have found in the p-domain is then transformed back into the t-domain. For this we use tables, the properties of the Laplace transform and partial fraction expansions. For the solution $y = y(t)$ found in this way, one can verify whether it satisfies the differential equation and the initial conditions.

Example 12 Consider for the unknown function $y = y(t)$ the initial value problem

$$y'' + 4y = \begin{cases} \frac{1}{2}t, & if\, 0 \le t < 2, \\ 3 - t, & if\, 2 \le t < 3, \\ 0, & if\; t < 0, t \ge 3, \end{cases}$$

$$y(0) = y'(0) = 0.$$

Solution

Figure 10 shows the graph of the causal function

$$y'' + 4y = \begin{cases} \frac{1}{2}t, & if\;\, 0 \le t < 2, \\ 3 - t, & if\;\, 2 \le t < 3, \\ 0, & if\; t < 0, t \ge 3. \end{cases}$$

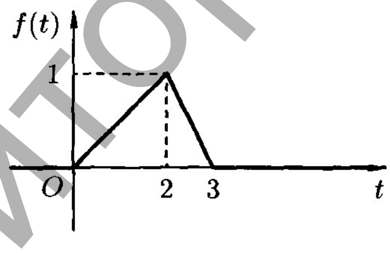


Fig.10

Rewrite the causal function as an analytic expression with the help of unit step function $\eta(t-b)$ and $\eta(t)$

$$f(t) = \frac{1}{2}t \cdot \eta(t) - \frac{1}{2}t \cdot \eta(t-2) + (3-t) \cdot \eta(t-2) - (3-t) \cdot \eta(t-3),$$

$$f(t) = \frac{1}{2}t \cdot \eta(t) - \frac{3}{2}(t-2) \cdot \eta(t-2) + (t-3) \cdot \eta(t-3),$$

$$y'' + 4y = \frac{1}{2}t \cdot \eta(t) - \frac{3}{2}(t-2) \cdot \eta(t-2) + (t-3) \cdot \eta(t-3).$$

The initial value problem transforms into

$$p^2 Y + 4Y = \frac{1}{2} \cdot \frac{1}{p^2} - \frac{3}{2} \cdot \frac{1}{p^2} \mathrm{e}^{-2p} + \frac{1}{p^2} \mathrm{e}^{-3p}.$$

Solving for $Y(p)$ and applying a partial fraction expansion we obtain that

$$Y(p)=\frac{1}{2}\cdot\frac{1}{p^2(p^2+4)}-\frac{3}{2}\cdot\frac{1}{p^2(p^2+4)}\mathrm{e}^{-2p}+\frac{1}{p^2(p^2+4)}\mathrm{e}^{-3p},$$

$$\frac{1}{p^2(p^2+4)}=\frac{1}{4}\left(\frac{1}{p^2}-\frac{1}{p^2+4}\right)=\frac{1}{4}\left(\frac{1}{p^2}-\frac{1}{2}\cdot\frac{2}{p^2+2^2}\right).$$

Using the properties of linearity, shift in the p-domain, scaling we have

$$y(t)=\frac{1}{8}\cdot(t-\sin 2t)-\frac{3}{8}\cdot\left(t-2-\frac{1}{2}\sin 2(t-2)\right)\eta(t-2)+$$
$$+\frac{1}{4}\cdot\left(t-3-\frac{1}{2}\sin 2(t-3)\right)\eta(t-3).$$

Using the Laplace transform one can also solve systems of several coupled ordinary linear differential equations with constant coefficients. We confine ourselves here to systems of two such coupled differential equations, since these can still be solved relatively easy without using techniques from matrix theory. Systems of more than two coupled differential equations will not be considered in this book. We merely note that they can be solved entirely analogously, although matrix theory becomes indispensable.

In general, a system of two coupled ordinary linear differential equations with constant coefficients and of first order has the following form:

$$\begin{cases} a_{11}x'+a_{12}y'+b_{11}x+b_{12}y=u_1(t) \\ a_{21}x'+a_{22}y'+b_{21}x+b_{22}y=u_2(t) \end{cases}$$

with initial conditions certain given values for $x(0)$ and $y(0)$. Similarly, one can describe the general system of second order with initial conditions $x(0)$, $x'(0)$, $y(0)$ and $y'(0)$ (for higher order and/or more differential equations the vector and matrix notation is much more convenient). The solution method based on the Laplace transform again consists of Laplace transforming all the functions that occur, and then solving the resulting system of linear equations with polynomials in p as coefficients.

Example 13 Consider the system

$$\begin{cases} 7x'+y'+2x=0 \\ x'+3y'+y=0 \end{cases}$$

with initial conditions $x(0)=1$ and $y(0)=0$.

Solution

Let $X(p)$ and $Y(p)$ be the Laplace transforms of $x=x(t)$ and $y=y(t)$. One then has

$$L\left(7x'+y'+2x\right)(p)=7\left(pX-x(0)\right)+\left(pY-y(0)\right)+2X$$

and by substituting the initial conditions one obtains that $7(pX-1)+pY+2X=0$, or $(7p+2)X+pY=7$. Transforming the second differential equation of the system in a similar way, we see that the Laplace transform turns the system into the algebraic system

$$\begin{cases} (7p+2)X+pY=7, \\ pX+(3p+1)Y=1. \end{cases}$$

Solving this system of two linear equations in the unknowns $X=X(p)$ and $Y=Y(p)$, we find that

$$X(p) = \frac{7(3p+1) - p}{(3p+1)(7p+2) - p^2} = \frac{20p+7}{(4p+1)(5p+2)},$$

$$Y(p) = \frac{7p - (7p+2)}{p^2 - (7p+2)(3p+1)} = \frac{2}{(4p+1)(5p+2)},$$

($X = X(p)$ is found by multiplying the first and the second equation by, respectively, $3p+1$ and p and subtracting; $Y(p)$ follows similarly). Using partial fraction expansions we obtain that

$$X(p) = \frac{8}{3(4p+1)} + \frac{5}{3(5p+2)} = \frac{2}{3\left(p+\frac{1}{4}\right)} + \frac{1}{3\left(p+\frac{2}{5}\right)},$$

$$Y(p) = \frac{8}{3(4p+1)} - \frac{10}{3(5p+2)} = \frac{2}{3\left(p+\frac{1}{4}\right)} - \frac{2}{3\left(p+\frac{2}{5}\right)}$$

and by inverse transforming this we see that the solution to the system is given by

$$\begin{cases} x(t) = \dfrac{2}{3}\mathrm{e}^{-\frac{t}{4}} + \dfrac{1}{3}\mathrm{e}^{-\frac{2t}{5}} \\ y(t) = \dfrac{2}{3}\mathrm{e}^{-\frac{t}{4}} - \dfrac{2}{3}\mathrm{e}^{-\frac{2t}{5}} \end{cases}$$

Exercise Set 4

In Exercises 1 to 8 determine a function $f(t)$ whose Laplace transform $F(p)$ is given by

1. $F(p) = \dfrac{p+2}{(p+1)(p-2)(p^2+1)}$.
2. $F(p) = \dfrac{5p+8}{p^4 - 13p^2 + 36}$.
3. $F(p) = \dfrac{4p}{(p^2+16)(p^2+25)}$.
4. $F(p) = \dfrac{p^2}{(p^2+4)(p^2+9)}$.
5. $F(p) = \dfrac{4p+5}{(p-2)(p^2+4p+5)}$.
6. $F(p) = \dfrac{p}{(p+1)(p^2+4p+5)}$.
7. $F(p) = \dfrac{p}{p^4+5p^2+4}$.
8. $F(p) = \dfrac{p+3}{p^3+2p^2+3p}$.

In Exercises 9 to 14 Consider for the unknown function $y = y(t)$ the initial value problem

9. $y'' + y' - 2y = -2(t+1),\ y(0) = 1,\ y'(0) = 1$;
10. $y'' + y' - 2y = \mathrm{e}^{-t},\ \ y(0) = -1,\ \ y'(0) = 0$;
11. $y'' + 4y = \sin 2t,\ \ y(0) = 0,\ \ y'(0) = 1$;
12. $y'' + 2y' + 10y = 2\mathrm{e}^{-t}\cos 3t,\ \ y(0) = 5,\ \ y'(0) = 1$;
13. $y'' - 9y = \sin t - \cos t,\ \ y(0) = -3,\ \ y'(0) = 2$
14. $y'' + y' - 6y = t + 3,\ y(0) = 1, y'(0) = 0$.

In Exercises 15 to 19 consider the system with initial conditions

15. $\begin{cases} x' = 2x + 2y + 2, \\ y' = 4y + 1, \end{cases}$ $\quad x(0) = 2,\ y(0) = 1.$

16. $\begin{cases} x' = 2x + 5y, \\ y' = x - 2y + 2, \end{cases}$ $\quad x(0) = 1,\ y(0) = 1.$

17. $\begin{cases} x'' - y' = e^t, \\ x' + y'' - y = 0, \end{cases}$ $\quad x(0) = 1,\ y(0) = -1,\ x'(0) = y'(0) = 0.$

18. $\begin{cases} x' = -2x + 5y + 1, \\ y' = x + 2y + 1, \end{cases}$ $\begin{cases} x(0) = 0, \\ y(0) = 2. \end{cases}$

19. $\begin{cases} x' = x + 4y, \\ y' = 2x - y + 9, \end{cases}$ $\begin{cases} x(0) = 1, \\ y(0) = 0. \end{cases}$

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Contents

УЧЕБНОЕ ИЗДАНИЕ

Составители:
Гладкий Иван Иванович
Дворниченко Александр Валерьевич
Каримова Татьяна Ивановна
Лебедь Светлана Федоровна
Шишко Татьяна Витальевна

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