АНАЛИТИЧЕСКИЕ И ЧИСЛЕННЫЕ МЕТОДЫ ИССЛЕДОВАНИЙ В МАТЕМАТИКЕ И ИХ ПРИЛОЖЕНИЯ

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FOURIER SERIES BEHAVIOR IN THE NEIGHBORHOOD OF A DISCONTINUITY

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The fundamental theorem of Fourier series was formulated for piecewise smooth functions. According to this theorem, the series converges pointwise to the function. Possible points of discontinuity were excluded here. At these points, the series converges to the average value of the left- and right-hand limits of the function. We already noted that in a small neighborhood of a discontinuity, the series will approximate the function much slower. This had already been observed by Wilbraham in 1848, but his results fell into oblivion. In 1898 the physicist Michaelson published an article in the magazine Nature, in which he doubted the fact that 'a real discontinuity (of a function f) can replace a sum of continuous curves' (i.e., the terms in the partial sums $S_n(t)$). This is because Michaelson had constructed a machine which calculated the nth partial sum of the Fourier series of a function up to $n = 80$. In a small neighborhood of a discontinuity, the partial sums $S_n(t)$ did not behave as he had expected: the sums continued to deviate and the largest deviation, the so-called *overshoot* of $S_n(t)$ relative to $f(t)$, did not decrease with increasing n.

In a letter to Nature from 1899, Gibbs explained this phenomenon and showed that $S_n(t)$ will always have an overshoot of about 9% of the magnitude of the jump at the discontinuity.

The sine integral is the function $Si(x)$ defined by $Si(x) = \int\limits_0^x \frac{\sin t}{t} dt$.

Since $\left|\frac{\sin t}{t}\right| \le 1$ for all $t \ne 0$, the integrand is bounded and the integral well defined. The sine integral cannot be determined analytically, but there are tables containing function values. In particular one has

$$Si(\pi) = \int\limits_0^{\pi} \frac{\sin t}{t} dt = 1{,}852..$$

The definition of $Si(x)$ can also be used for negative values of x, from which it follows that $Si(x)$ is an odd function. $Si(x)$ converges to π/2 for $x \to \infty$. Although $Si(x)$ cannot be calculated analytically, one is able to determine its limit.

Theorem For the sine integral one has $\lim\limits_{x\to\infty} Si(x) = \int\limits_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$.

Gibbs' phenomenon is a rather technical treatment, which does not result in any specific new insight into Fourier series. We treat Gibbs' phenomenon using the periodic block function. Since it will result in much simpler formulas, we will not start from the periodic block function, but instead from the periodic function $f(t)$ defined on the interval $(-T/2;T/2)$ by

$$f(t)=\begin{cases}1/2, & if \quad 0<t<T/2,\\ -1/2, & if \quad -T/2<t<0.\end{cases}$$

This odd function has a Fourier sine series and the partial sums $S_n(t)$ of the Fourier series are

$$S_n(t)=\sum_{k=1}^{n}\frac{2}{(2k-1)\pi}\sin\left((2k-1)\omega_0 t\right).$$

Gibbs' phenomenon is clearly visible: immediately next to a discontinuity of the function $f(t)$, the partial sum overshoots the values of $f(t)$. We will now calculate the overshoot, that is, the magnitude of the maximum difference between the function and the partial sums immediately next to the discontinuity [1]. By determining the derivative of the partial sum, we can find out where the maximum difference occurs, and subsequently calculate its value. Differentiating $S_n(t)$ gives

$$S_n'(t)=\sum_{k=1}^{n}\frac{2}{\pi}\omega_0\cos\left((2k-1)\omega_0 t\right)=\sum_{k=1}^{n}\frac{4}{T}\cos\left((2k-1)\omega_0 t\right).$$

In order to determine the zeros of the derivative, we rewrite the last sum.

$$\cos\left((2k-1)\omega_0 t\right)=\frac{\sin 2k\omega_0 t-\sin\left((2k-2)\omega_0 t\right)}{2\sin\omega_0 t}.$$

By substituting this into getting expression for the derivative $S_n'(t)$ it follows that $S_n'(t)=\dfrac{2}{T}\dfrac{\sin 2n\omega_0 t}{\sin\omega_0 t}$.

The derivative is 0 if $2n\omega_0 t=k\pi$ for $k\in Z$ and $k\neq 0$. We thus have extrema for $S_n'(t)$ at $t=\dfrac{k\pi}{2n\omega_0}$. The value at the extremum can be found by substituting getting values t in $S_n(t)$. This gives

$$S_n\left(\frac{\pi}{2n\omega_0}\right)=\sum_{k=1}^{n}\frac{2}{(2k-1)\pi}\sin\left((2k-1)\frac{\pi}{2n}\right).$$

The last sum can be rewritten as an expression which is a Riemann sum with step size π/n for the function $\dfrac{\sin x}{x}$:

$$S_n\left(\frac{\pi}{2n\omega_0}\right)=\frac{1}{\pi}\sum_{k=1}^{n}\frac{\pi}{n}\cdot\frac{\sin\left(\left(2k-1\right)\frac{\pi}{2n}\right)}{\left(2k-1\right)\frac{\pi}{2n}}.$$

Taking the limit $n\to\infty$, the sums in the right-hand side converge to the integral $\int_0^{\pi}\frac{\sin x}{x}dx$. Note that for large values of n the contribution of the nth term gets smaller and smaller and will even tend to zero (since the series converges to the integral) [2]. The value of this integral was given in previous section and hence

$$\lim_{n\to\infty}S_n\left(\frac{\pi}{2n\omega_0}\right)=\lim_{n\to\infty}\frac{1}{\pi}\sum_{k=1}^{n}\frac{\pi}{n}\cdot\frac{\sin\left(\left(2k-1\right)\frac{\pi}{2n}\right)}{\left(2k-1\right)\frac{\pi}{2n}}=\frac{1}{\pi}\int_0^{\pi}\frac{\sin x}{x}dx=\frac{1}{\pi}\cdot 1{,}852=0{,}589\cdot$$

This establishes the value at the first maximum next to the jump. Since the jump has magnitude 1, the overshoot of the function value 0.5 is approximately 9% of the jump. Since the additional contribution for large values of n gets increasingly smaller, this overshoot will remain almost constant with increasing n. Furthermore we see that the value of t where the extrema is attained is getting closer and closer to the point of discontinuity.

The phenomenon occurs in a similar way for other piecewise smooth functions having points of discontinuity [3]. There is always an overshoot of the partial sums immediately to the left and to the right of the points of discontinuity, with a value approximately equal to 9% of the magnitude of the jump. As more terms are being included in the partial sums, the extrema are getting closer and closer to the point of discontinuity.

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ПРОИЗВОДНАЯ π-ДЛИНЫ π-РАЗРЕШИМОЙ ГРУППЫ, У КОТОРОЙ СИЛОВСКИЕ p-ПОДГРУППЫ ЛИБО БИЦИКЛИЧЕСКИЕ, ЛИБО ИМЕЮТ ПОРЯДОК НЕ ВЫШЕ p^5

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Рассматриваются только конечные группы. Все обозначения и используемые определения соответствуют [1].