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On Generalized Solutions of Some Differential Equations with Singular Coefficients

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Abstract: The article deals with the problem of existence of generalized solutions to simplest linear differential equation with a singular coefficient q. The generalized solution was introduced by choice one of the distribution Q corresponding q, and choice an approximation of Q by a family of common functions. The generalized solution is defined as the limit of the solutions of the approximating equations. It was demonstrated by model example that the existence of such generalized solutions depends essentially on the method of approximation.

Keywords: Distribution, equation with generalized coefficients, generalized solution

1 Introduction

The problems considered in the paper appear at the analysis of the solutions of differential equations with a singular coefficients. We will consider the equations of the form

$$u'(x) - q(x)u(x) = 0, \tag{1.1}$$

where function q has singularity at the point 0.

We remind that the distribution (generalized function) is a linear continuous functional on the Schwartz space $\mathscr{D}(\mathbb{R})$ [1,2]. Any locally integrable function u generates the distribution

$$< U, \varphi >= \int u(x)\varphi(x)dx, \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

The classical solution u of (1.1) can be not integrable and a set of distribution U corresponds to such function in the sense described in the section 2. But after the substitution of U in the equation the product qU appears which is not determined in the distributions theory.

More general, the substitution of a distribution U in any equation of the form

$$u' - Qu = 0, \tag{1.2}$$

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where Q is a distribution, leads to the product QU, which is not determined in the distributions theory.

Therefore, the first problem is to give sense to notion of generalized solution of (1.1) and (1.2), the second problem is to construct such solutions.

If q is a continuous or locally integrable function, (1.1) is the most simple differential equation and its solutions can be constructed by separation of variables. After the division on u and integration we have equality

$$\ln|u(x)| = \int_{x_0}^{x} q(t)dt + C$$

and the solution of the Cauchy problem $u(x_0) = C$ can be given by expression

$$u(x) = Cexp[\int_{x_0}^{x} q(t)dt] = C\exp[g(x) - g(x_0)], \tag{1.3}$$

where $g(x)$ is a antiderivative of q.

But for equation (1.2) this approach can not be applied, because all used transformations (multiplication, division, integration by segment, calculations of logarithm and exponent) are not determined for distributions. In particular, for the distribution Q there exists antiderivative G, but its exponent $exp\ G$ is not determined and final expression (1.3) give not a distribution.

Therefore, in order to construct generalized solutions of equations with generalized coefficients we need use other approach.

Analysis of equations with generalized coefficients is connected with calculations of a product of distribution by the following sense.

Let a family $w_\varepsilon(x)$ converges to the distribution W as $\varepsilon \to 0$ and a family $v_\varepsilon(x)$ converges to the distribution V. If there exists limit F of the products $f_\varepsilon = w_\varepsilon v\varepsilon$, then F is called the *product of W and V under given methods of approximations*. In general case the question about the existence of the product is a hard problem and it is solved only for some special W and V and special approximations.

As the main examples we consider equations

$$u'(x) + \frac{s}{x}u(x) = 0, s \in \mathbb{N}. \tag{1.4}$$

and

$$u'(x) + \frac{1}{x^2}u(x) = 0. \tag{1.5}$$

In contrast to case of continuous q, the set of classical solutions of such equations depends on two arbitrary constant. For equation (1.4) these solutions are functions

$$u(x) = \begin{cases} \frac{C_1}{x^s}, & x < 0; \\ \frac{C_2}{x^s}, & x > 0, \end{cases} \tag{1.6}$$

where C_1 and C_2 are arbitrary constants.

For equation (1.5) the classical solutions are

$$u(x) = \begin{cases} C_1 exp\frac{1}{x}, & x < 0; \\ C_2 exp\frac{1}{x}, & x > 0. \end{cases} \tag{1.7}$$

By the Cauchy condition $u(-1) = B$ only the constant C_1 is determined and the solution is determined uniquely only for $x < 0$.

In order to give a definition of the generalized solutions we use the following way. We will consider equation (1.1) under condition that there exist a set of distributions, which coincides with q on the set $\mathbb{R} \setminus 0$. This condition is fulfilled for (1.4) and (1.5).

The first step consists of a choice of distribution Q corresponding to the coefficient q and passing to the equation with generalized coefficient (1.2).

The second step consists of a choice of some approximation for the distribution Q by a family of locally integrable (may be smooth) functions q_ε. After this we have a family of approximating equations

$$u'_\varepsilon - q_\varepsilon u_\varepsilon = 0. \tag{1.8}$$

Let u_ε are the solution of the Cauchy problem $u_\varepsilon(x_0) = C$ for (1.8).

Definition 1. *If there exists a limit U of the family u_ε in the distributions space, then the distribution U is called generalized solution of the Cauchy problem for* (1.2) *under the given approximation of the coefficient Q.*

One meets differential equations with generalized coefficients in many applications and vast literature is devoted to their analysis (see, for example, [3,4]), but the questions under consideration not discussed in previous publications.

Since the distribution Q coincides with q on the set $\mathbb{R} \setminus 0$, the antiderivative G of Q coincides on $\mathbb{R} \setminus 0$ with an antiderivative of q and is an ordinary function $g(x)$. Therefore, on set $\mathbb{R} \setminus 0$ a classical solution of (1.2) singled out

$$u(x) = Cexp[g(x) - g(x_0)], \tag{1.9}$$

which is said to be *the formal solution of the Cauchy problem.*

Here we can see that the choice of a distribution Q corresponding to coefficient q get us one classical solution determined on all line $\mathbb{R}$.

It is natural presuppose that the generalized solution U coincides on $\mathbb{R} \setminus 0$ with the formal solution u. But it can be that such distributions not exist and we obtain necessary condition: must be that the distributions, corresponding to the formal solution, exist.

For formal solutions of the form (1.7) such distributions not exist and for equation (1.5) the generalized solutions not exist.

At the case of equation (1.4) to any formal solution u of the form (1.6) corresponds a set of distributions U named regularizations of u (see Section 2) and the problem of generalized solutions is rich in content.

The main problem is *to clarify for which distributions Q and their approximations q_ε the generalized solution of* (1.2) *exists.*

The global answer it is not known even for the model equations (1.4) with $q(x) = \frac{s}{x}$.

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2 Regularization of the function with singularity

Let the function f be continuous at $x \neq 0$, the estimation

$$|f(x)| \leq const \frac{1}{|x|^s}$$

holds in a neighborhood of zero with some $s \in \mathbb{N}$, and $x^{s-1}f(x)$ is not integrable. Then integral

$$\int f(x)\varphi(x)dx,\ \varphi \in \mathscr{D}(\mathbb{R}), \tag{2.1}$$

exists only if

$$\varphi \in L_s = \{\varphi \in \mathscr{D}(\mathbb{R}) : \varphi(0) = \varphi'(0) = ... = \varphi^{(s-1)}(0) = 0\},$$

and (2.1) defines a linear continuous functional F_s on subspace L_s. We call a distribution F *corresponding* to f, if F is a continuation of F_s on all space $\mathscr{D}(\mathbb{R})$. Such continuation is said to be *regularization of divergent integral* (2.1).

The following way of regularization can be used [1, 2]. Let $\eta(x)$ be such bounded function that $\eta(x) = 1$, if $|x| < h$, and $\eta(x) = 0$, if $|x| > 2h$. Then formula

$$< F_\eta, \varphi >= \int f(x)\{\varphi(x) - [\sum_{k=0}^{s-1} \frac{1}{k!}\varphi^{(k)}(0)x^k]\eta(x)\}dx$$

give us some continuation of F_s. In particular, if

$$\eta(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases} \tag{2.2}$$

this continuation is

$$< F_0, \varphi >= \int_{|x|<1} f(x)\big[\varphi(x) - \sum_{k=0}^{s-1} \frac{1}{k!}\varphi^{(k)}(0)x^k\big]dx + \int_{|x|>1} f(x)\{\varphi(x)dx. \tag{2.3}$$

The set of all regularizations can be described by following proposition. Remind that Dirac δ-function is functional $\langle \delta, \varphi \rangle = \varphi(0)$ and its derivatives determined as

$$\left\langle \delta^{(k)}, \varphi \right\rangle = (-1)^k \varphi^{(k)}(0).$$

Proposition 1. *Let $\eta \in \mathscr{D}(\mathbb{R})$. The distributions corresponding to given function f have form*

$$F = F_\eta + \sum_{k=0}^{s-1} M_k \delta^{(k)},$$

where M_k are arbitrary constants.

Proof. Let us consider $s-$dimentional subspace

$$D_s = \{[\sum_{k=0}^{s-1} C_k x^k]\,\eta(x)\} \subset \mathscr{D}(\mathbb{R})$$

The mapping

$$P : \varphi(x) \to [\sum_{k=0}^{s-1} \frac{1}{k!}\varphi^{(k)}(0)x^k]\eta(x)$$

is a projection of $\mathscr{D}(\mathbb{R})$ on D_s and gives the decomposition

$$\mathscr{D}(\mathbb{R}) = D_s \oplus L_s.$$

The functional F_s is determined on L_s and in order to construct its continuation F we need define F on D_s. Since functions $e_k(x) = x^k\eta(x), k = 0,\dots,s-1$, form a basis in D_s, it is sufficient assign the values $< F, e_k >,\ k = 0,\dots,s-1$. Therefore

$$< F,\varphi >= \sum_{k=0}^{s-1} \frac{1}{k!} < F, e_k > \varphi^{(k)}(0) + < F_\eta,\varphi >=$$

$$=< F_\eta + \sum_{0}^{s-1} M_k\delta^{(k)},\varphi >,$$

where $M_k = \frac{(-1)^k}{k!} < F, e_k >$. □

Example 1 Let $f(x) = \frac{1}{x}$. Regularization (2.3) of $\frac{1}{x}$, denoted by $P(\frac{1}{x})$, is

$$< P(\frac{1}{x}),\varphi >= \int_{|x|<1} \frac{1}{x}[\varphi(x) - \varphi(0)]dx + \int_{|x|>1} \frac{1}{x}\varphi(x)dx. \tag{2.4}$$

This distribution can be given by principal part of integral

$$< P(\frac{1}{x}),\varphi >= \lim_{h\to 0}\int_{|x|>h} \frac{1}{x}\varphi(x)dx.$$

The distributions corresponding to $\frac{1}{x}$ are

$$Q_M = P\left(\frac{1}{x}\right) + M\delta,$$

where M are arbitrary constants.

Antiderivative for Q_M is locally integrable function $G_M(x) = \ln|x| + M\Theta(x)$, where Θ is Heaviside function. The formal solutions of equation $u' + sQ_M u = 0$ with condition $u(-1) = C$ is

$$u_M(x) = \begin{cases} \frac{C}{|x|^s}, & x < 0; \\ \frac{Ce^{-sM}}{|x|^s}, & x > 0. \end{cases} \tag{2.5}$$

3 Convergence of approximations

Let function $f_\varepsilon(x)$ be locally integrable and the family $f_\varepsilon(x)$ tends uniformly to a function $f(x)$ on set $\{x : |x| > h\}$ for all $h > 0$. Not arise from this that the family $f_\varepsilon(x)$ converges to any distribution.

Proposition 2. *Let the estimation*

$$|f_\varepsilon(x)| \le \frac{const}{|x|^\nu}, \quad where \quad \nu \ < s+1\,,$$

holds in the neighborhood of 0.

The family f_ε converges in distributions space if and only if there exist limits

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\frac{(-1)^j}{j!}f_\varepsilon(x)x^j dx := K_j,\ \ j = 0,1,\ldots,s-1\,. \tag{3.1}$$

Under these conditions the regularization F_0 of $f(x)$ given by (2.3) *is determined and*

$$\lim_{\varepsilon\to 0} f_\varepsilon := F = F_0 + \sum_{j=0}^{s-1} K_j \delta^{(j)}\,.$$

Proof. Use decomposition

$$\varphi(x) = [\sum_{j=0}^{s-1}\frac{1}{j!}\varphi^{(j)}(0)x^j]\eta(x) + \psi(x),$$

where η is given (2.2). Here $\psi^{(j)}(0) = 0$ for $j = 0,1,\ldots,s-1$ and $|\psi(x)| \le const|x|^s$, the estimation

$$|f_\varepsilon(x)\psi(x)| \le \frac{const}{|x|^{\nu-s}}$$

holds, the Lebesque theorem can be applied and

$$\int f_\varepsilon(x)\psi(x)dx \to \int f(x)\psi(x)dx =< F_0,\psi > \ .$$

Therefore f_ε converges if and only if there exist limits of integrals

$$\int f_\varepsilon(x)x^k\eta(x)dx.$$

It is equivalent to existence of limits in (3.1). □

4 Analytical representation

The existence and the form of generalized solution depend on the used approximation of the generalized coefficient Q. One of the most natural approximation method is named *analytical representation* [5].

Let $Q^+(z)$ be an analytical function on upper half-plane and $Q^-(z)$ be an analytical function on lower half-plane. Is sed to be that these functions give *analytical representation of distribution* Q, if

$$Q = \lim_{\varepsilon\to 0}\,[Q^+(x+i\varepsilon) - Q^-(x-i\varepsilon)].$$

Here the family of smooth functions $q_\varepsilon(x) = Q^+(x+i\varepsilon) - Q^-(x-i\varepsilon)$ is an approximation of Q.

It is well known that the analytical representation of $P\left(\frac{1}{x}\right)$ is

$$\frac{1}{2}[\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}] = \frac{x}{x^2+\varepsilon^2}$$

and the analytical representation of δ-function is the family

$$\frac{i}{2\pi}\left[\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right] = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \tag{4.1}$$

Therefore analytical representation of the distribution $sQ_M = s\left[P\left(\frac{1}{x}\right) + M\delta\right]$ can be written in the form

$$q_\varepsilon(x) = \frac{\lambda s}{x+i\varepsilon} + \frac{(1-\lambda)s}{x-i\varepsilon}, \tag{4.2}$$

where $\lambda = \frac{1}{2} + \frac{Mi}{2\pi}$.

Such analytical representation of $s\left[P\left(\frac{1}{x}\right) + M\delta\right]$ was used in [6, 7] for the construction of generalized solutions of (1.4).

In particular, the family $\frac{1}{x\pm i\varepsilon}$ converges to distribution

$$\frac{1}{x\pm i0} := P\left(\frac{1}{x}\right) \pm i\pi\delta\,.$$

Since

$$\frac{1}{(x\pm i\varepsilon)^s} = \frac{(-1)^{s-1}}{(s-1)!}[\frac{1}{x\pm i\varepsilon}]^{(s-1)},$$

the family $\frac{1}{(x\pm i\varepsilon)^s}$ is the analytical representation of distribution

$$\frac{1}{(x\pm i0)^s} := \frac{(-1)^{s-1}}{(s-1)!}[P\left(\frac{1}{x}\right) \pm i\pi\delta]^{(s-1)}.$$

Theorem 1. [7] *Let u_ε be the solutions of the Cauchy problem $u_\varepsilon(-1) = (-1)^s$ for equations* (1.8)*, where $-q_\varepsilon$ is approximation of the distribution $sQ_M = s\left[P\left(\frac{1}{x}\right) + M\delta\right]$ given by* (4.2).

The family u_ε converges in the distributions space only in two following cases:
I) if $\lambda s \in \mathbb{Z}$ and $\lambda s \leq 0$, then

$$u_\varepsilon \to \frac{1}{(x - i0)^s};$$

II) if $\lambda s \in \mathbb{Z}$ and $\lambda s \geq s$, then

$$u_\varepsilon \to \frac{1}{(x + i0)^s}.$$

Let us note that the condition on λ in the Theorem 1 is equivalent to

$$M = \frac{\pi}{i}[\frac{2m}{s} - 1],$$

where $m \in \mathbb{Z}$ and $m \leq 0$ or $m \geq s$.

5 Main Theorem

By the Theorem 1, under the analytical representations the generalized solutions of equation

$$u' + sQ_M u = 0$$

exist only for special values of M. In particular, the distribution $P(\frac{1}{x})$ is the most natural among corresponding to $\frac{1}{x}$, but the equation with such coefficient have not generalized solutions under the analytical representations.

The question arises: *Can it be a generalized solution of the equation*

$$u' + P(\frac{1}{x})u = 0 \tag{5.1}$$

under some kind of approximation of $P(\frac{1}{x})$?

The answer is the main result of this paper.

The formal solution of (5.1) with the condition $u(-1) = 1$ is function $u(x) = \frac{1}{|x|}$. The corresponding distributions are $U = U_0 + K\delta$, where

$$< U_0, \varphi >= \int_{|x|<1} \frac{1}{|x|}[\varphi(x) - \varphi(0)]dx + \int_{|x|>1} \frac{1}{|x|}\varphi(x)dx =$$

$$= \lim_{\varepsilon\to 0}\big[\int_{|x|<\varepsilon} \frac{1}{|x|}\varphi(x)dx - 2\ln\frac{1}{\varepsilon}\varphi(0)\big]. \tag{5.2}$$

Remark that here the principal part of the integral in (5.2) not exists, the integrals

$$\int_{\varepsilon<|x|} \frac{1}{|x|}\varphi(x)dx$$

grow as $2\ln\frac{1}{\varepsilon}\varphi(0)$ and the regularization (5.2) is obtained by the subtract of these values.

We need to construct such approximation q_ε of $P(\frac{1}{x})$ that the solutions of (1.8) make up an approximation of a distribution of the form $U_0 + K\delta$.

According (5.2) the family

$$v_\varepsilon(x) = \begin{cases} \frac{1}{|x|}, & |x| > \varepsilon; \\ -\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}, & |x| < \varepsilon \end{cases} \tag{5.3}$$

is an approximation of the distribution (5.2).

But the functions $v_\varepsilon(x)$ are discontinuous and can not be the solutions of a differential equations.

An approximations of the distribution (5.2) by continuous functions $f_\varepsilon(x)$ can be constructed. But, if the function $f_\varepsilon(x)$ are real-valued, there exist points x_0 such that $f_\varepsilon(x_0) = 0$ and these functions can not be the solutions of a differential equations.

Therefore we must construct the required approximation of $P(\frac{1}{x})$ by using complex-valued functions.

Theorem 2. *For any $K \in \mathbb{R}$ there exists approximation $q_\varepsilon(x)$ of the distribution $P(\frac{1}{x})$ such that the corresponding solutions u_ε converge to the distribution $U_0 + K\delta$.*

Proof. The representation of $P(\frac{1}{x})$ in the form (2.4) means that family

$$p_\varepsilon(x) = \begin{cases} \frac{1}{x}, & |x| > \varepsilon; \\ 0, & |x| < \varepsilon \end{cases}$$

is an approximation of $P(\frac{1}{x})$. Antiderivatives of $p_\varepsilon(x)$ are continuous functions

$$g_\varepsilon(x) = \begin{cases} \ln|x|, & |x| > \varepsilon; \\ ln\frac{1}{\varepsilon}, & |x| < \varepsilon. \end{cases}$$

and the solutions of the Cauchy problem $u_\varepsilon(-1) = 1$ are

$$u_\varepsilon(x) = \begin{cases} \frac{1}{|x|}, & |x| > \varepsilon; \\ \frac{1}{\varepsilon}, & |x| < \varepsilon. \end{cases}.$$

Here integrals

$$\int u_\varepsilon(x)dx = \int_{|x|>\varepsilon} \frac{1}{|x|}dx + \frac{1}{\varepsilon}\int_{|x|<\varepsilon} \varphi(x)dx$$

grow as $2\ln\frac{1}{\varepsilon}\varphi(0)$ and the limit not exist.

Consider some other approximations of $P(\frac{1}{x})$. Let

$$q_\varepsilon(x) = \begin{cases} \ln|x|, & |x| > \varepsilon; \\ \frac{1}{\varepsilon}\left[\frac{C(\varepsilon)}{\varepsilon} + i\pi\right], & -\varepsilon < x < 0; \\ -\frac{1}{\varepsilon}\left[\frac{C(\varepsilon)}{\varepsilon} + i\pi\right], & 0 < x < \varepsilon. \end{cases} \tag{5.4}$$

where $C(\varepsilon)$ is an indetermined function. If $C(\varepsilon)\to 0$, expression (5.4) give some approximation of $P(\frac{1}{x})$. The antiderivatives are functions

$$v_\varepsilon(x)=g_\varepsilon(x)+\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]\,\psi(\frac{x}{\varepsilon}),$$

where

$$\psi(x)=\begin{cases}x+1, & -1<x<0,\\ 1-x, & 0<x<1,\\ 0, & |x|>1,\end{cases}.$$

The corresponding solutions are

$$w_\varepsilon(x)=e^{v_\varepsilon(x)}=\begin{cases}\frac{1}{|x|}, & |x|>\varepsilon;\\ \frac{1}{\varepsilon}exp\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]\,\psi(\frac{x}{\varepsilon}), & |x|<\varepsilon.\end{cases}$$

and we must investigate convergence of $w_\varepsilon(x)$ in dependence on $C(\varepsilon)$.

By Proposition 2 it is necessary to proof that function

$$J(\varepsilon)=\int_{-1}^{1}w_\varepsilon(x)dx$$

has limit as $\varepsilon\to 0$.

This function can be calculated in explicit form:

$$\begin{aligned}J(\varepsilon) &= 2\ln\frac{1}{\varepsilon}+\frac{2}{\varepsilon}\int_{-\varepsilon}^{0}exp\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]\big[\frac{x}{\varepsilon}+1\big]dx=\\ &= 2\ ln\frac{1}{\varepsilon}+2\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]^{-1}\big\{exp\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]-1\big\}=\\ &= 2\ ln\frac{1}{\varepsilon}+2\big[\frac{C(\varepsilon)}{\varepsilon}+i\pi\big]^{-1}\big\{-exp\big[\frac{C(\varepsilon)}{\varepsilon}\big]-1\big\}.\end{aligned}$$

Real part is

$$\mathscr{R}e(J(\varepsilon))=2\ ln\frac{1}{\varepsilon}-2\big[\frac{C(\varepsilon)}{\varepsilon}\big]\big[(\frac{C(\varepsilon)}{\varepsilon})^2+\pi^2\big]^{-1}\big\{exp\big[\frac{C(\varepsilon)}{\varepsilon}\big]+1\big\}.$$

If $C(\varepsilon)=\varepsilon$, then

$$\mathscr{R}e(J(\varepsilon))=2\ ln\frac{1}{\varepsilon}-2\big[1+\pi^2\big]^{-1}\big\{e+1\big\}$$

and

$$\mathscr{R}e(J(\varepsilon))\to+\infty \text{ as } \ \varepsilon\to 0.$$

If $C(\varepsilon)=a\varepsilon\ln\frac{1}{\varepsilon}$, where $a>0$, then

$$\mathscr{R}e(J(\varepsilon))=2\ ln\frac{1}{\varepsilon}-2a\ln\frac{1}{\varepsilon}\big\{[a\ln\frac{1}{\varepsilon}]^2+\pi^2\big\}^{-1}\left(\frac{1}{\varepsilon^a}+1\right)$$

and

$$\mathscr{R}e(J(\varepsilon)) \to -\infty \text{ as } \varepsilon \to 0 .$$

Therefore, for a given $K \in \mathbb{R}$ and small ε exists $C(\varepsilon)$ such that

$$\varepsilon < C(\varepsilon) < a\varepsilon \ln \frac{1}{\varepsilon}$$

and $\mathscr{R}e(J(\varepsilon)) = K$. Note that

$$\frac{\varepsilon}{C(\varepsilon)} \to 0 \tag{5.5}$$

for such $C(\varepsilon)$ and from (5.5) follows that the imaginary part

$$\mathscr{J}m(J(\varepsilon)) = 2\pi \big[(\frac{C(\varepsilon)}{\varepsilon})^2 + \pi^2 \big]^{-1} \big\{ exp \big[\frac{C(\varepsilon)}{\varepsilon} \big] + 1 \big\}$$

converges to 0. The Proposition 2 can be applied and $w_\varepsilon(x) \to U = U_0 + K\delta$, where U_0 is given by (5.2) and U are the required generalized solutions. □

6 Conclusion

The results of the paper demonstrate that the properties of the equations with generalized coefficients are very differ from the case of ordinary differential equations, even for the most simple first order equations.

References

[1] I.M. GELFAND, G.E. SHILOV, *Generalized functions*, Vol.1 Academic Press Boston (1964).

[2] V.S. VLADIMIROV, *Generalized Functions in Mathematical Physics*, URSS (1994) [in Russian]

[3] S. ALBEVERIO, F.GESTEZY, R. HOEGH-KROHN, H. HOLDEN, *Solvable Models in Quantum Mechanics,* Springer, New York (1988) .

[4] S.T. ZAVALISHIN,A.N. SESEKIN, *Pulse Processes: Models and Applications,* Moscow, Nauka (1991). [in Russian]

[5] H. BREMERMANN, *Distributions, complex variabes, and Fourier Transforms,* Addison–Wesley, 1965

[6] A. B. ANTONEVICH, T.G. SHAGOVA, *On the generalized sulutions of a differential equations with rational coefficients*, Taurida Journal of computer science and mathematics, 3 (2019) 23–36. [in Russian]

[7] A.B. ANTONEVICH, E.V.KUZMINA, *The sulutions of differential equation $u' + \frac{s}{x}u = 0$ in the distributions space,* Vesnik of Yanka Kupala State University of Grodno, Series 2, 10 (2020) 56–66. [in Russian]