Decomposition Approach to Minimize a Class of Superposition of Recurrent Monotone Functions on a Set of Parameterized Paths in Digraph

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Abstract

A problem of finding optimal parameterized path in digraph is considered. A parameterized path is a path each arc of which is assigned a parameter from a given set. A superposition of recurrent-monotone functions is accepted as an objective function where one of the functions is defined by using max operation. A two-level decomposition scheme for solving an initial problem is proposed. Methods for solving the obtained after decomposition subproblems are developed.

1. Introduction

Problems of finding optimal paths often arise in design and analysis of various networks. Shortest path problem is one of the most well-known such problems. A lot of papers is devoted to this problem (see, for instance, bibliography in [1, 2]). Last decades formulations of these problems with complex objective functions are intensively studied. Sometimes such problems arise when reducing multicriteria problems to a problem with one so-called generalized criterion which is in fact a superposition of partial criteria. This paper focuses on a class of such problems.

In the section 2, the considered problem is formulated. In the section 3, a twolevel decomposition scheme for solving the initial problem is proposed. In the following sections, methods for solving subproblems obtained after decomposition are developed taking into account their features.

2. Statement of the problem

Let $G=(V,E)$ be a finite directed graph with distinguished vertices s and t without multiple arcs. Each arc $(v,p)\in E$ is assigned a set Γ_{vp} and a vector-function $c_{vp}(\alpha)=(c^1_{vp}(\alpha),c^2_{vp}(\alpha),c^3_{vp}(\alpha))$ where $\alpha\in\Gamma_{vp}$ and $c^r_{vp}(\alpha)$, $r=1,2,3$, are non-negative real-valued functions.

A pair $x=(w,\gamma)$ is a parameterized path in graph G, if $w=(i_0,\ldots,i_l)$ is a path in graph G and $\gamma=(\gamma_1,\ldots,\gamma_l)\in\Gamma(w)=\prod_{k=1}^{l}\Gamma_{i_{k-1}i_k}$.

On a set X_v of parameterized paths in graph G from the vertex s to a vertex $v\in V$, the following functions are defined:

$$f^1(x)=\max\{c^1_{i_{k-1}i_k}(\gamma_k)\mid 1\le k\le l\},$$

$$f^r(x)=\sum_{k=1}^{l}c^r_{i_{k-1}i_k}(\gamma_k),\ r=2,3.$$

An initial problem **A** is to find a parameterized path $x^*=(w^*,\gamma^*)\in X=X_t$, which minimizes a function $g(x)=f^1(x)\cdot f^2(x)+f^3(x)$.

3. Decomposition scheme

For solving the problem **A**, the following decomposition scheme can be used. It is a concrete definition of a general scheme for solving similar class of problems [3, 4, 5].

Let us introduce a set $Y\subset R$ such that

$$Y\cap\{f^1(x)\mid x\in X^*\}\ne\varnothing$$

and

$$[\min\{f^1(x)\mid x\in X\},\max\{f^1(x)\mid x\in X\}]\supseteq Y$$

where X^* is a set of solutions of the problem **A**, and a function $g^0(x,y)=y\cdot f^2(x)+f^3(x)$ which is defined on $X\times Y$.

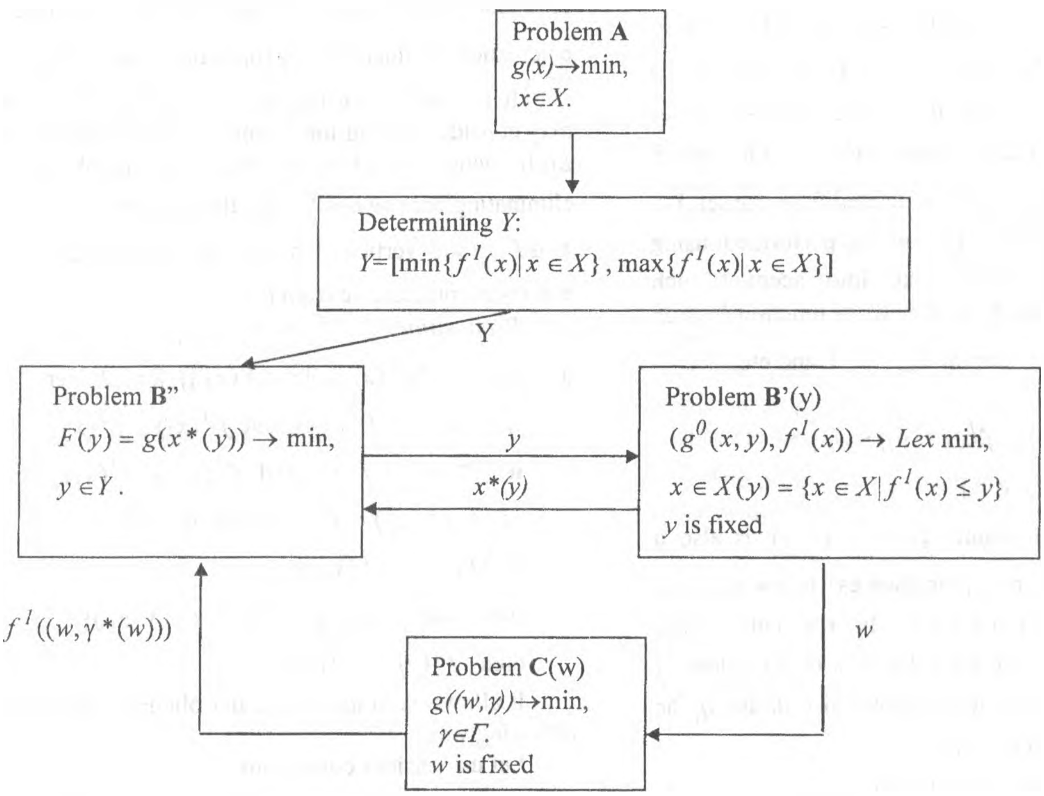

Fig.1. Decomposition scheme for solving the initial problem **A**

At a lower level, for fixed value of parameter $y \in Y$ we solve (possible approximate to a criterion $f^1(x)$) local problems **B'**(y) of lexicographic minimization

$$(g^0(x,y), f^1(x)) \to Lex\min,$$

$$x \in X(y) = \{x \in X | f^1(x) \leq y\}$$

and at upper level, we solve problem **B"**:

$$F(y) = g(x^*(y)) \to \min,$$

$$y \in Y,$$

where x*(y) is an obtained solution of the problem **B'**(y),

Solution $x^*(y)=(w,\gamma)$ can be "improved" after solving an auxiliary problem **C**(w):

$$g((w,\gamma)) \to \min,$$

$$\gamma \in \Gamma.$$

Diagram of this procedure is shown in Fig. 1.

Proposition 1. If y is a ε-approximate solution of the problem **B"**, then x*(y) is a ε-approximate solution of the problem **A**.

It should be noted that if sets Γ_{vp} are finite for all $(v,p) \in E$, then the problem **A** is polynomially (with regard to dimensions of graph *G*) solvable.

4. Solving the problem B"

For solving the problem **B"** we propose to use the following modification of "branch and bound method" which takes into account non-unimodality of function F(y) on Y.

To a current step of an algorithm, a value y^0 and a current record $F^0 = F(y^0)$ of function $F(y)$ on Y are known as well as from the initial set Y a set Y^0 is extracted such that $\min\{F(y) | y \in Y^0 \cup \{y^0\}\} - F^* \leq \varepsilon$ where $F^* = \min\{F(y) | y \in Y\}$ and ε is a required accuracy (with regard to objective function) for solution of problem **A**. The set Y^0 is divided into subsets Y_i contained in disjoint intervals (y_i^-, y_i^+).

At the current step, a value y' is chosen in a set Y_i for which problem **B'**(y') is solved and parameterized path $x^*(y')=(w^*(y'), \gamma^*(y'))$ is found. The problem **C**(w) for $w=w^*(y')$ is solved (if necessary). It results in correction of values y^0 and F^0, replacing the set $Y_{i'}$ by its subsets $Y_{i1} = \{y \in Y_{i'} | y < y'\}$ and $Y_{i2} = \{y \in Y_{i'} | y > y'\}$. After that for each subset Y_i of Y^0, a value z_i is defined in such a way that either $F(y) \geq F^0 - \varepsilon$ for $z_i \leq y \leq y_i^+$ or there exists $y' < z_i$ for which $F(y') \leq F(y)$. Subset $[z_i, y_i^+]$ is deleted from the set Y_i.

A choice of next Y_i can be performed using various heuristics which take into account such characteristics of sets Y_i as bounds of function $F(y)$ on the set Y_i, length of interval (y_i^-, y_i^+) and etc.

Since

$$\min\{F(y) | y \in Y_i, f^1(x^*(y)) > y_i^-\} \geq \min\{g^0(x,y) | y \in Y_i\}$$

then either a lower bound q_i of $g^0(x,y)$ is also a lower bound of $F(y)$ on Y_i or there exists $z < y_i^-$ such that $F(z) \leq \min\{F(y) | y \in Y_i\}$. In the latter case, eliminating Y_i from the set Y does not result in loss of solution to the problem **B**″. It allows also to use q_i as lower bound of $F(y)$ on Y_i.

In particular, we can consider

$$q_i = \max[g' - (y' - y_i^-)\beta_i^2, g' y_i^- / y' + \beta_i^3 (1 - y_i^- / y)]$$

where β_i^2 is an upper bound of function $f^2(x)$ and β_i^3 is a lower bound of function $f^3(x)$ on the set $\{x \in X_t | y_i^- < f^1(x) < y_i^+\}$. The eliminated set $[z_i, y_i^+]$ can be defined as

$$z_i = \min[y' - (g' - F^0)/\beta_i^2, y'(g' - F^0)/(g' - \beta_i^3)].$$

It should be noted that if the bound β_i^3 is accessible then its use is preferable in comparison with β_i^2.

5. Solving the problem B′(y)

A general scheme for solving the problem **A** requires to solve a sequence of problem **B**′(y) for some sequence $\{y_i\}$ of parameters $y \in Y$, which is generated during solving the problem **B**″. It results in a development of special methods for solving problems **B**′(y) which take into account their parametric properties. Methods proposed hereafter allow to use for solving the current problem **B**′(y) data which have already been obtained during solving problems **B**′(y) for the nearest lesser and greater values to y from the sequence $\{y_i\}$. Such approach is very effective if calculation of functions $c_{vp}^r(\alpha)$ is time consuming.

It is easy to see that for the problem **B**′(y) one may consider instead the graph G a graph $G(y) = (V(y), E(y))$ which is obtained from the graph G by eliminating arcs $(v,p) \in E$ such that $c_{vp}^1(\alpha) > y$ for all $\alpha \in \Gamma_{vp}$ and vertices that are not accessible from s and counteraccessible from t.

Proposition 2. If $z_1, z_2 \in \{y_i\}$, $z_1 < z_2$ and $x_k = \arg\min\{g^0(x, z_k) | x \in X(z_k)\}$, $k=1,2$, then

i) $f^2(x_1) \geq f^2(x_2)$ and $f^3(x_1) \leq f^3(x_2)$;

ii) $f^2(x_1) = f^2(x_2)$ iff $f^3(x_1) = f^3(x_2)$;

iii) if $f^2(x_1) = f^2(x_2)$ and for $k=1,2$,

$(g^0(x, z_k), f^1(x_k)) =$

$Lex \min\{(g^0(x, z_k), f^1(x)) | x \in X(z_k)\}$,

then $f^1(x_1) = f^1(x_2)$.

It allows us to use the earlier obtained data in the following way.

Let us consider collections

$H^n(y) = \{h_p^n(y) | p \in V(y)\}$ of vectors

$h_p^n(y) = (h_p^{n,1}(y), h_p^{n,2}(y), h_p^{n,3}(y))$, $n=1,2,3$

such that $h_s^n(y) = (0,0,0)$ and for all $p \in V(y) \backslash \{s\}$ the following recurrent relationships are satisfied:

$$h_p^n(y) = \arg lex \min\{(y \cdot \lambda^2 + \lambda^3, \lambda^1) | (\lambda^1, \lambda^2, \lambda^3) \in Z_p^n(y)\} \quad (1)$$

moreover if $h_p^n(y) \neq h_p^n(y_p^n(y))$, then there exists a parameterized path

$$x_p^n(H_p^n(y)) = ((v = i_0, \ldots, i_l), (\gamma_1, \ldots, \gamma_l))$$

such that $h_v^n(y) = h_v^n(y_v^n(y))$ and for $k=1,\ldots,l$

$$h_{i_k}^{n,1}(y) = \max(h_{i_{k-1}}^{n,1}(y), c_{i_{k-1}i_k}^1(\gamma_k)) \neq h_{i_k}^{n,1}(y_{i_k}^n(y))$$

$$h_{i_k}^{n,r}(y) = h_{i_{k-1}}^{n,r}(y) + c_{i_{k-1}i_k}^r(\gamma_k)) \neq h_{i_k}^{n,r}(y_{i_k}^n(y)),$$

$r=2,3$. Here

$$Z_p^n(y) = \{h_p^n(y_p^n(y))\} \cup \{(\max(h_v^{n,1}(y), c_{vp}^1(\alpha)), h_v^{n,2}(y) + c_{vp}^2(\alpha), h_v^{n,3}(y) + c_{vp}^3(\alpha)) | (v,\alpha) \in Q_p^n(y)\}$$

$$Q_p^n(y) \subseteq Q_p(y) = \{(v,\alpha) | (v,p) \in E(y),$$
$$\alpha \in \Gamma_{vp}, c_{vp}^1(\alpha) \le y\}.$$

The value $y_p^n(y)$ and the sets $Q_p^n(y)$, $n=1,2,3$, $p \in V(y)$, are defined as:

a) for $n=2,3$ and $h_v^{n,1}(y^r(y)) \le y$,

$y_p^n(y) = y^r(y)$,

$\{(v,\alpha) \in Q_p(y) | c_{vp}^2(\alpha) < h_p^{n,2}(y^r(y)) - h_v^{n,2}(y),$

$c_{vp}^3(\alpha) > h_p^{n,3}(y^r(y)) - h_v^{n,3}(y)\} \subseteq Q_p^n(y)$;

b) for $n=3$ and $h_v^{3,1}(y^r(y)) > y$ or $n=1$

$y_p^n(y) = y^1(y)$,

$\{(v,\alpha) \in Q_p(y) | c_{vp}^2(\alpha) > h_p^{n,2}(y^1(y)) - h_v^{n,2}(y),$

$c_{vp}^3(\alpha) < h_p^{n,3}(y^1(y)) - h_v^{n,3}(y)$ or

$c_{vp}^1(\alpha) > y$ or $h_v^{n,1}(y) > y^1(y)\} \subseteq Q_p^n(y)$;

c) otherwise $y_p^n(y)$ is not defined and $Q_p^n(y) = Q_p(y)$.

It is easy to prove

Proposition 3. For all $y \in \{y_t\}$, $n=1,2,3$, and $p \in V(y)$

$$(y\ h_p^{n,2}(y) + h_p^{n,3}(y), h_p^{n,1}(y)) =$$
$$Lex \min\{(y \cdot f^2(x) + f^3(x), f^1(x) | x \in X, f^1(x) \le y\}.$$

For any vertex $p \in V(y)$, $n=1,2,3$, we can construct a parameterized route $x_p^n(y)$, which consists of d ($d \ge 1$) parameterized paths

$x_{p_m}^n = x_{p_m}^n (H_p^n(y_{u_m})) = ((v_m = i_{m0}, K, i_{ml_m}),$ $(\gamma_{m1}, K, \gamma_{ml_m}))$, obtained for some $y_{u_m} \in \{y_t\}$ such that $v_1 = s$, $p_{u_d} = p$, $h_p^n(y_{u_d}) = h_p^n(y)$ and $u_{m-1} < u_m$, $p_{m-1} = v_m$, $h_{v_m}^n(y_{u_{m-1}}) = h_{v_m}^n(y_{u_m})$ for $m=2,\ldots,d$.

It is easy to prove that $x_p^n(y)$ is a parameterized path if in relationships (1) we accept $h_p^n(y) = h_p^n(y_p^n(y))$ iff $Z_p^n(y)$ does not contain a vector $(\lambda^1, \lambda^2, \lambda^3)$ such that

$$(y \cdot h_p^{n,2}(y_p^n(y)) + h_p^{n,3}(y_p^n(y)), h_p^{n,1}(y_p^n(y))) =$$
$$(y \cdot \lambda^2 + \lambda^3, \lambda^1)$$

6. Conclusions

For solving the problem **C**(w), a twolevel procedure can be used, which is similar to the proposed one for the initial procedure **A**. Inclusion the problem **C**(w) into general procedure for solving the initial problem **A** allows us to obtain "good" upper bounds of function *F(y)*. Taking into account such role of the problem **C**(w), it is sufficient to obtain its approximate solution. Furthermore, we can solve it even for some subpath containing vertex t.

The proposed approach can be improved for particular applications. For instance, it is possible to reduce the sets Γ_{vp} during solving problems **B**′(y).

The proposed approach can also be extended for more wide class of such problems. For example, a problem when function $f^1(x)$ is defined both operation "max" and "min" can be reduced [3] to the considered problem.

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