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COMPLEX-ANALYTIC METHODS FOR THE STUDY OF THE EFFECTIVE CONDUCTIVITY OF COMPOSITE MATERIALS

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Abstract

The effective conductivity of 2D composite materials is discussed on the base of the complex-analytic methods. Main attention is paid to the application of the methods of boundary value problems for harmonic and analytic functions and functional equations for composite materials having multiply connected circular structure. The problems of new type are posed.

Keywords: 2D composite material; Effective conductivity; Analytic functions; Boundary Value Problems; Circular multiply connected domains; Functional equations

1 Introduction

In this report we describe a number of constructive methods for analytic functions following the ideas of the book [1] and recent papers. These methods extend the possibility of using mathematical constructions within different directions of mathematics, as well as mechanics, physics, chemistry, biology, economics etc. This field is too wide to be presented in a short paper. Thus we restrict ourselves by boundary value problems for analytic functions and some related problems of porous media mechanics and composite materials. As illustration of the mathematical constructions we present certain models for composite materials.

What is for us the meaning of the expression *explicit* or *closed form solution*? To get the closed form solution one have to construct the formula which contains a finite set of elementary and special functions, arithmetic operations, compositions, integrals, derivatives and even series. Besides all the objects in the formula ought to have precise meaning. At last, domains for parameters, as well as all functions, integrals, derivatives etc. have to be explicitly determined. It should be shown also that these domains and (if necessary) their intersections are non-empty.

This approach seems to be slightly nontraditional. For instance, in the classical books a form in series is not supposed to be a closed one. At the same time these books allow to have special functions in the solutions' formulas. But not all such functions has integral representations.

2 Method of functional equations

We describe first the main idea of the method of functional equations on the base of its application to the Dirichlet problem for a doubly connected domains.

Let us consider two discs $\mathbb{D}_k := \mathbb{D}(a_k, r_k) = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1, 2)$, $cl\mathbb{D}_1 \cap cl\mathbb{D}_2 = \emptyset$, and the doubly connected domain $\mathbb{D} := \widehat{\mathbb{C}} \backslash (cl\mathbb{D}_1 \cup cl\mathbb{D}_2)$ on the extended complex plane $\widehat{\mathbb{C}}$. The orientation chosen on $\mathbb{T}_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$ leaves $\mathbb{D}$ to the left. The Dirichlet problem for the domain $\mathbb{D}$ consists in the following. Given Hölder continuous functions $f_k(t)$ on $\mathbb{T}_k$ $(f_k(t) \in \mathcal{H}(\mathbb{T}_k),\ k = 1, 2)$, find a function $u(z)$ harmonic in $\mathbb{D}$ and Hölder continuous on $cl\mathbb{D}$

satisfying the boundary condition

$$u(t) = f_k(t), \ t \in \mathbb{T}_k, \ k = 1, 2. \qquad (1)$$

Using the Logarithmic Conjugation Theorem for a doubly connected domain we look for $u(z)$ in the following form

$$u(z) = \Re\left(\phi(z) + A \log \frac{z - a_1}{z - a_2}\right), \qquad (2)$$

where the function $\phi(z)$ is analytic in $\mathbb{D}$ and Hölder continuous on $cl\mathbb{D}$, i.e., $\phi \in \mathcal{H}_{\mathcal{A}}(\mathbb{D})$, A is an unknown real constant. Let us choose a curve L_1 connecting the points $z = a_1$ and infinity, which has no common points with $\mathbb{T}_1$. Suppose that $\log(z - a_1)$ is a fixed analytic branch of the multi-valued logarithm in $\widehat{\mathbb{C}} \backslash L_1$. The function $\log(z - a_2)$ is defined in a similar way. Let us denote

$$z^*_{(k)} := z^*_{\mathbb{T}_k} = \frac{r_k^2}{\overline{z - a_k}} + a_k$$

the inversion with respect to the circle $\mathbb{T}_k$. We have $t^*_{(k)} = t$ on $|t - a_k| = r_k$. A function $\varphi(z)$ is analytic in $|z - a_k| < r_k$ if and only if $\overline{\varphi\left(z^*_{(k)}\right)}$ is analytic in $|z - a_k| > r_k$ (see Section 2.1). Using the Decomposition Theorem we find the function $\phi(z)$ in the form

$$\phi(z) = \overline{\phi_1\left(z^*_{(1)}\right)} + \overline{\phi_2\left(z^*_{(2)}\right)}, \qquad (3)$$

where the function $\phi_k(z)$ is analytic in $|z - a_k| < r_k$ and Hölder continuous on $|z - a_k| \leq r_k \ (k = 1, 2)$, i.e., $\phi_k \in \mathcal{H}_{\mathcal{A}}(\mathbb{D}_k)$. Substituting (3) in (2) and using (1) we arrive at the following boundary conditions

$$\Re\overline{\phi_1\left(t^*_{(1)}\right)} + \Re\overline{\phi_2\left(t^*_{(2)}\right)} + A \log\left|\frac{t-a_1}{t-a_2}\right| = f_k(t),$$
$$t \in \mathbb{T}_k, \ k = 1, 2. \qquad (4)$$

It is easily seen that

$$\begin{aligned} \phi_1(z) + \overline{\phi_2\left(z^*_{(2)}\right)} &= g_1(z) + \\ +A\left[\log(z - a_2) - \log r_1\right], \ z &\in cl\mathbb{D}_1. \end{aligned} \qquad (5)$$

Similar arguments for the second relation (4) yield

$$\begin{aligned} \phi_2(z) + \overline{\phi_1\left(z^*_{(1)}\right)} &= g_2(z) - \\ -A\left[\log(z - a_1) - \log r_2\right], \ z &\in cl\mathbb{D}_2, \end{aligned} \qquad (6)$$

where the known function g_2 belongs to $\mathcal{H}_{\mathcal{A}}(\mathbb{D}_2)$. The equalities (5) and (6) constitute a system of two functional equations with respect to the functions $\phi_1(z)$ and $\phi_2(z)$. Excluding $\overline{\phi_2\left(z^*_{(2)}\right)}$ from equality (5) we obtain the simple functional equation

$$\phi_1(z) = \phi_1[\alpha(z)] + g(z), \ z \in cl\mathbb{D}_1, \qquad (7)$$

where the known function $g(z)$ belongs to $\mathcal{H}_{\mathcal{A}}(\mathbb{D}_1)$. The pure imaginary constant $i\gamma$ contains the additive constants $i\gamma_1$ and $i\gamma_2$ appearing in the definition of $g_1(z)$ and $g_2(z)$. The conformal mapping

$$\alpha(z) := \left(z^*_{(2)}\right)^*_{(1)} = \frac{r_1^2(z - a_2)}{r_2^2 + \overline{(a_2 - a_1)}(z - a_2)} - a_1$$

maps the closed disc $|z - a_1| \leq r_1$ into the open one $|z - a_1| < r_1$.

Now we outline some notations and facts of the general theory of the functional equation which are connected with those equations appearing at the study of boundary value problems. In particular we need the following

Theorem 1. (Denjoy–Wolff) *Let f be an analytic function in the unit disc $\mathbb{U}$ mapping $\mathbb{U}$ into itself (but not a Möbius transformation of $\mathbb{U}$ onto $\mathbb{U}$). Then there exists a point z_0, $|z_0| \leq 1$, such that the sequence of iterations $f^n(z)$ converges to z_0 uniformly on each compact subset of $\mathbb{U}$. Moreover, if $|z_0| = 1$, then $\lim_{r\to 1-0} f(rz_0) = z_0$ and $s := \lim_{r\to 1-0} f'(rz_0)$ exists and $0 < |s| \leq 1$.*

Theorem 2. *Let a function $f \in \mathcal{C}_{\mathcal{A}}(cl\mathbb{U})$ be mapping the closed unit disc $cl\mathbb{U}$ into the open unit disc $\mathbb{U}$. Then f has a unique fixed point z_0 in $\mathbb{U}$ and $|f'(z_0)| < 1$. The sequence $f^n(z)$ converges uniformly in $|z| \leq 1$ to the point z_0.*

The point z_0 is called *the attractive point* of $f(z)$.

Theorem 3. *The operator $\varphi \mapsto \varphi[f(z)]$ is compact in the spaces $\mathcal{H}_{\mathcal{A}}(\mathbb{U})$, $\mathcal{C}_{\mathcal{A}}(\mathbb{U})$ and in the Hardy space $\mathcal{H}_p(\mathbb{U})$, $1 < p < +\infty$.*

Theorem 4. *Let $G, g \in \mathcal{C}_{\mathcal{A}}(\mathbb{U})$. If $G(z_0)\left[f'(z_0)\right]^k \neq 1$ for all $k = 0, 1, 2, \ldots$, then the functional equation*

$$\varphi(z) = G(z)\varphi[f(z)] + g(z), \ \ |z| \leq 1 \qquad (8)$$

has a unique solution $\varphi \in \mathcal{C}_{\mathcal{A}}(\mathbb{U})$. If for some k we have $G(z_0)\left[f'(z_0)\right]^k = 1$, then (8) has a solution in $\mathcal{C}_{\mathcal{A}}(\mathbb{U})$ if and only if a solvability condition on $G(z)$ and $g(z)$ is fulfilled. If so then the general solution of (8) depends on an arbitrary complex constant.

These results constitute a base of solvability theory for functional equations. The crucial idea is that if the mapping $\alpha(z)$ in the equation (7) maps its domain into itself then this equation can be solved in corresponding spaces by the method of successful approximation. In particular such situation appears at the study of several boundary value problems for analytic functions in a multiply connected circular domain.

3 Effective conductivity of composite materials

3.1 Optimal design

We refer to optimal design problems when it is necessary to determine an optimal composite in a given class of admissible composites. The investigation of optimal design problems for composite conductive materials is of mathematical and practical interest. The problems for optimal composites are usually formulated as variational problems of minimization of stored energy in the composite material (see, *e.g.* [2], [3], [6], [7] and papers cited therein). If unknown variables (*e.g.* the form of inclusions, their locations, size etc.) and/or constrains have geometric nature we deal with shape optimization problems (or optimal design problems). Typical physical constrains are the given conductivities of the components, condition of ideal contact on the boundary matrix-inclusions, the external field outside the composite. The latter two conditions can be represented in the form of the boundary value problems for certain potentials.

For periodic composites the optimal design problems coincide with the problem of optimization of effective conductivity tensor [7] in so called representative cell. We have to mention also a number of approaches which are devoted to the study of laminate composites (see e.g. [13]), fibre composites (see e.g. [9]), composites with the reach microstructure (see e.g. [16]), nanocomposites (see e.g. [14], [15]) etc.

This paper contains the description of a new approach in the study of optimal effective properties of the plane composite materials. In contrast to highly developed theory based on the weak and variational statement of the corresponding problems, this approach is oriented on the construction of the analytic solutions and even (when it is possible) the closed form solution. To show the perspectivity of this approach we use very simple examples, although several situations of more general type were considered recently as well.

We propose another statements of optimal design problems. The principal difference is that we fix shape and size of the inclusions. In particular, we consider a bounded domain occupied by a host material with N inclusions (N can be equal to unity). Hence, each inclusion has a positive concentration in the bulk material. It is known that if the characteristic size ε of the inclusion tends to zero (simultaneously N tends to infinity with fixed concentration of inclusions), the homogenization theory [8] can be applied. Such approach is not working for our model.

We also discuss optimal design problems for unbounded domains. In this case for simple external field the considered problem is equivalent to the problem of optimization of the effective conductivity of dilute composite materials, when concentration of inclusions is sufficiently small. Anyway the problem we arrive at could'not be handled via homogenization method.

The main mathematical difficulty we try to overcome is that for the moment an analytic solution of R-linear boundary value problem (Markushevich's problem in another sources) is not known. Moreover, the physically relevant statement of the optimal design problem in potential case leads to the mixed boundary conditions (different kind of boundary value problems on each component of the boundary). By using our approach we could overcome these difficulties at least in the case of very important model situations. Among the achievements presented in the paper we ought to mention the discovered phenomena of "packing" of inclusions in the optimal composites.

Our study is useful in technical applications, because this problem corresponds to the following engineering task. A designer has in his disposal a material of the given shape on the boundary of which a prescribed external field is applied. Let the designer also has inclusions of the given shape and size. It is necessary to locate these inclusions with a fixed concentration in such a way that the conductivity in the fixed direction will be maximal (minimal). So using our formulas a designer can project complex fibre composite materials to reach optimal properties.

We consider here the problems of two types.

First type problem. Bounded composite material.

Let Q be a bounded domain in the complex plane $\mathbb{C}$ encircled by a simple piece-wise smooth Lyapunov curve $\Gamma = \partial Q$. Let $h(t)$, $q(t)$ be given continuous functions on Γ. Let $g(z)$ be a given continuous function in the domain Q.

The question is to find a piece-wise smooth curve L, consisting of a finite number of connected components $L = \bigcup\limits_{k=1}^{n} L_k$, $L \subset \overline{\operatorname{int}\Gamma}$ (Γ è L_k, which have at most finite number of common points, and piece-wise harmonic function u,

$$\Delta u(z) = 0,\ z \in D^+ \cup D_k, \tag{9}$$

in $D_k = \mathrm{int} L_k$, $D^+ = Q \backslash \overline{\mathrm{int} \bigcup_{k=1}^{n} L_k}$, boundary conditions on the curve $L = \bigcup_{k=1}^{n} L_k$:

$$u^+(t) = u^-(t),\ \lambda_m \frac{\partial u^+}{\partial n}(t) = \lambda_i \frac{\partial u^-}{\partial n}(t), \tag{10}$$

or

$$u^+(t) - u^-(t) = g(t),\ \lambda_m \frac{\partial u^+}{\partial n}(t) = \lambda_i \frac{\partial u^-}{\partial n}(t), \tag{11}$$

and one of the following boundary conditions

$$\frac{\partial u^+}{\partial n}(t) = 0,\ t \in \Gamma, \tag{12}$$

$$u^+(t) = h(t),\ t \in \Gamma, \tag{13}$$

$$\frac{\partial u^+}{\partial n}(t) = q(t),\ t \in \Gamma, \tag{14}$$

on Γ, such that the effective conductivity functional λ_e possesses the maximal or minimal value.

Second type problem. Unbounded composite material.
$Q = \mathbb{C}$, thus the external boundary Γ is absent. Condition on Γ is replaced by certain condition at ∞. It is described by a single-valued analytic function $f(z)$ in $\mathbb{C}$, having isolated singularity at ∞.
The problem is to find a piece-wise smooth curve $L = \bigcup_{k=1}^{n} L_k$, having at most finite number of common points, and the potential u, satisfying the Laplace equation in D_k and in $D^+ = \mathbb{C} \setminus \bigcup_{k=1}^{n} \overline{D_k}$, which has the singularity of a prescribed type at ∞, and satisfies boundary condition (10) (or (11)) on L, such that the effective conductivity functional λ_e possesses the maximal or minimal value.

3.2 *The Schwarz boundary value problem for a pseudo-fractal domain*

One of the suitable methods for the study of 2D composite materials is the complex analytic approach. In the potential case the effective conductivity of the composite is described by the boundary value problems for harmonic or analytic function. We have to study the boundary value value problems (usually of mixed type) for multiply connected domains on the boundary of inclusions (and on the outer boundary of matrix if the composite is bounded). The model problem we start with is the so called Schwarz boundary value problem. The solvability of this problem was studied intensively. But the question of the exact analytic representation of the solution remains to be a difficult problem far from the finite description.

Among the investigations in this area we have to point out the work by Villat (1916, an analytic solution to the Schwarz problem for concentric annulus), Aksent'ev (1963-1967), Aleksandrov & Sorokin (1972), Mityushev (1985-1998). All these studies deal with multiply connected circular domains.

In our work we use the formula for the solution of the Schwarz problem for the multiply connected circular domains presented e.g. in [1].

This formula is a sum of certain series. Convergence of this series in the spaces of functions analytic on a bounded domains and continuous up to their boundaries was established earlier. Our main interest is in the description of components of this series in certain concrete cases.

Such aim follows from the possible applications of this construction. In particular, we use it in the study of the effective conductivity of composite materials having so called pseudo-fractal structure. Pseudo-fractal and fractal structures are of great importance for the composite materials with compound structure. The most investigated cases of composite materials of such a type are those periodic or doubly periodic (see e.g. contributions by Adler, Mityushev, Grigolyuk & Filshtinskii and others).

The problem we stated is based on the following ideas:

1) the most natural model for the composite with a compound structure could be a model which leads to the boundary value problem on a general infinitely connected domains. The theory of such problems is not too developed at the moment. Pseudo-fractals constitute a good approximation of the above said domains.

2) nowadays there are stated the problems of constructing composites for which the mixing is performed on the molecular level. It is known that such structure usually has the self-similarity property.

We consider on the complex plane $\mathbb{C}$ a multiply connected domain D whose boundary consists of a family of circles

$$|z| = \frac{1}{3}, \quad \left| z - 2\sum_{j=1}^{m} \frac{e^{\frac{\pi i}{3}k_j}}{3^j} \right| = \frac{1}{3^{m+1}}, \quad m \in \mathbb{N}.$$

This family has the self-similarity property (fractal structure). If we put $m = 1, 2, \ldots, P$, where $P \in \mathbb{N}$ is a fixed positive integer number, then it is said that the considered family has the pseudo-fractal structure.

In our work [17] we construct an approximate solution to the Schwarz problem in a pseudo-fractal domain D. This problem is to find a function Ψ analytic in the domain D continuous up to the boundary of this domain, satisfying the boundary condition

$$\begin{cases} \operatorname{Re} \Psi(t) = q(t), \quad t \in \partial D, \\ \\ \operatorname{Im} \Psi(z_0) = 0, \end{cases} \tag{15}$$

where f is a given on ∂D Hölder-continuous function, z_0 is a given point in D.

Solution to the problem (15) for multiply connected circular domains can be represented in the form of series in the Schottky group generated by symmetries with respect to the boundary curves. In our work we describe the structure of the group for the domain D. Its lower elements are given in the explicit form.

Similar problems for another types of fractal and pseudo-fractal domains were studied by Adler $\&$ Mityushev in connection with the calculation of the effective conductivity of composite materials possessing self-similarity property.

Let $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ be a multiply connected circular domain, $\mathbb{D}_k \equiv \{z \in \mathbb{C} : |z - a_k| < r_k\}$. Let the following formula

$$z^*_{(k_m,k_{m-1},\ldots,k_1)} = \left(z^*_{(k_{m-1},\ldots,k_1)} \right)^*_{(k_m)} \tag{16}$$

be determined the successful symmetries with respect to the circles $\mathbb{T}_{k_1} = \partial \mathbb{D}_{k_1}, \ldots, \mathbb{T}_{k_m} = \partial \mathbb{D}_{k_m}$. If in the sequence $k_m, k_{m-1}, \ldots, k_1$ no two neighboring indexes are equal, then the number m is called a *level* of the mapping $z \mapsto z^*_{(k_m,k_{m-1},\ldots,k_1)}$. If m is even then the corresponding mappings are simply Möbius transformations. If m is odd then the corresponding mappings are Möbius transformations with respect to $\overline{z}$. For each fixed domain $\mathbb{D}$ the family of successful symmetries generates so called Schottky group of symmetries $\mathcal{K}$. Denote by $\mathcal{G}$ the subgroup of all even elements of the group $\mathcal{K}$, and by $\mathcal{F}$ - the family of all elements of odd order (a conjugate class).

The solution $\Psi(z) \equiv \mathbf{T}(\mathbb{D}, \mathbf{q})(z)$ to the Schwarz problem (15) for a multiply connected circular domain $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ can be represented in the form of series in the above described Schottky group (see [1, Thms. 4.11, 4.12]).

The series converge on each compact subset $\operatorname{cl} \mathbb{D} \setminus \{\infty\}$.

3.3 *Random composite materials*

We investigate also the effective conductivity of two-dimensional composite materials with the quasi periodic structure, i.e. it consists of periodically located cells occupied by a finite number of circular disjoint inclusions generating the quasi pseudofractal structure inside each cell. Under the quasi pseudofractal structure we understand a random "shaking" of inclusions about the pseudofractal array.

To study the effective conductivity of such composite we should average characteristics describing the material as a whole. This idea is realized by applying of the homogenization method [8].

The effective conductivity of composites with the pseudofractal structure was investigated by Adler and Mityushev [18], [19]. The "shaking" idea was first used by Berlyand and Mityushev and applied to circular inclusions which form the periodic square array [5]. In this paper we assume that the presented fundamental unit cell is divided into four parts each of which has the pseudofractal structure. We apply the "shaking" method to the given array, where each inclusion of certain level shakes in its own zone, the size of which proportionally depends on the level.

This approach is described in the papers [20], [21].

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