UDK 517.9

ON THE FUNDAMENTAL SOLUTION OF THE BIHARMONIC EQUATION IN $\mathbf{R}^4$

D. A. Basik, D. V. Haluts, R. N. Kozinets
Brest State A. S. Pushkin University, Brest
Scientific adviser: A. I. Basik, PhD in Physics and Mathematics, Assistant Professor

In this paper we prove some integral equalities for the operator Δ^2 in $\mathbf{R}^d$, where

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

In the four-dimensional case ($d = 4$), we find radial solutions of the biharmonic equation. Using the solutions we have found, a fundamental solution of the operator Δ^2 in $\mathbf{R}^4$ is constructed.

Theorem 1. *Let $\Omega \subset \mathbf{R}^d$ ($d \in \mathbf{N}$) be a bounded domain whose boundary $\partial\Omega$ is a sufficiently smooth surface, let $n = (n_1, ..., n_d)$ be the unit outward normal field on $\partial\Omega$ and let $u, v \in C^4(\overline{\Omega})$. Then we have*

$$\int_{\Omega} u(x)\Delta^2 v(x)dx = \int_{\Omega} \Delta u(x)\Delta v(x)dx + \int_{\partial\Omega} \left(u(x)\frac{\partial(\Delta v(x))}{\partial n} - \frac{\partial u(x)}{\partial n}\Delta v(x) \right) dS(x) \qquad (1)$$

and

$$\int_{\Omega} \left(u(x)\Delta^2 v(x) - v(x)\Delta^2 u(x) \right) dx = \int_{\partial\Omega} \left(u(x)\frac{\partial(\Delta v(x))}{\partial n} + \Delta u(x)\frac{\partial v(x)}{\partial n} \right) dS(x) - \qquad (2)$$

$$- \int_{\partial\Omega} \left(v(x)\frac{\partial(\Delta u(x))}{\partial n} + \Delta v(x)\frac{\partial u(x)}{\partial n} \right) dS(x).$$

In the equations (1) *and* (2) *above, $\partial/\partial n$ is the directional derivative in the direction of the normal n of the surface element $dS(x)$.*

Let's call equality (1) an analogue of Green's first identity, (2) – an analogue of Green's second identity for the operator Δ^2.

Proof. Integrated by parts [1, p. 302], we have

$$\int_{\Omega} u(x)\Delta^2 v(x)dx = \sum_{j=1}^{d} \int_{\Omega} u(x)\frac{\partial^2}{\partial x_j^2}(\Delta v(x))dx =$$

$$= -\sum_{j=1}^{d} \int_{\Omega} \frac{\partial}{\partial x_j}(u(x))\frac{\partial}{\partial x_j}(\Delta v(x))dx + \sum_{j=1}^{d} \int_{\partial\Omega} u(x)\frac{\partial}{\partial x_j}(\Delta v(x))n_j(x)dS(x) =$$

$$= \sum_{j=1}^{d} \int_{\Omega} \frac{\partial^2}{\partial x_j^2}(u(x))\Delta v(x)dx - \sum_{j=1}^{d} \int_{\partial\Omega} \frac{\partial}{\partial x_j}(u(x))\Delta v(x) n_j(x) dS(x) + \int_{\partial\Omega} u(x)\frac{\partial}{\partial n}(\Delta v(x))dS(x) =$$

$$= \int_{\Omega} \Delta u(x)\Delta v(x)dx - \int_{\partial\Omega} \frac{\partial}{\partial n}(u(x))\Delta v(x)dS(x) + \int_{\partial\Omega} u(x)\frac{\partial}{\partial n}(\Delta v(x))dS(x).$$

This completes the proof of formula (1).

Applying formula (1) to integrals $\int_{\Omega} u(x)\Delta^2 v(x)dx$ and $\int_{\Omega} v(x)\Delta^2 u(x)dx$, we obtain formula (2). Theorem 1 has been proven.

Let $y \in \mathbf{R}^4$ be a fix point and let r be a distance between points $x \in \mathbf{R}^4$ and y, i.e. $r = |x - y| = ((x_1 - y_1)^2 + ... + (x_4 - y_4)^2)^{0,5}$. Let's find radial solutions of the biharmonic equation in $\mathbf{R}^4$

$$\Delta^2 u = 0, \tag{3}$$

i.e. solutions of equation (3) depending only on r. Thus $u = f(r)$, where the function $f(r)$ satisfies the ordinary differential equation

$$\left(f''(r) + \frac{3}{r}f'(r)\right)'' + \frac{3}{r}\left(f''(r) + \frac{3}{r}f'(r)\right)' = 0. \tag{4}$$

We define a new function $g(r) = f''(r) + 3f'(r)/r$ for $r > 0$. Then we obtain

$$g''(r) + \frac{3}{r}g'(r) = 0 \Leftrightarrow (r^3 g'(r))' = 0 \Leftrightarrow g'(r) = \frac{C_1}{r^3} \Leftrightarrow g(r) = -\frac{C_1}{2r^2} + C_2,$$

where C_1, C_2 are constants of integration. Returning to change of the variable, we have

$$f''(r) + \frac{3}{r}f'(r) = -\frac{C_1}{2r^2} + C_2 \Leftrightarrow (r^3 f'(r))' = -\frac{C_1 r}{2} + C_2 r^3 \Leftrightarrow$$

$$\Leftrightarrow r^3 f'(r) = -\frac{C_1 r^2}{4} + C_2\frac{r^4}{4} + C_3 \Leftrightarrow f(r) = -\frac{C_1}{4}\ln r + \frac{r^2 C_2}{8} - \frac{C_3}{2r^2} + C_4.$$

Let's define the function Φ as follows. For $x \in \mathbf{R}^4 \setminus \{0\}$, let

$$\Phi(x) = C\ln|x|, \tag{5}$$

where $C \in \mathbf{R}$. The function Φ is locally integrable function in $\mathbf{R}^4$ and we see that Φ satisfies biharmonic equation (3) on $\mathbf{R}^4 \setminus \{0\}$. Let's find C so that the function Φ satisfies the equation $\Delta^2 u = \delta$ in the sense of distributions [1, p. 542]. That is, for all $\varphi \in D(\mathbf{R}^4)$ we have $\int_{\mathbf{R}^4} \Phi(x)\Delta^2\varphi(x)dx = \varphi(0)$.

For this reason, we call Φ the fundamental solution of the biharmonic equation.

So, let $\varphi \in D(\mathbf{R}^4)$ and $\operatorname{supp}\varphi$ lie in an open ball $B(0;R)$. Then we have

$$\int_{\mathbf{R}^4} \Phi(x)\Delta^2\varphi(x)dx = \lim_{\varepsilon\to+0} \int_{\varepsilon<|x|<R} \Phi(x)\Delta^2\varphi(x)dx.$$

Applying theorem 1 and taking into account the fact that $\varphi(x)$ vanishes as $x \notin \operatorname{supp}\varphi$, we have

$$\int_{\mathbf{R}^4} \Phi(x)\Delta^2\varphi(x)dx = \lim_{\varepsilon\to+0} \int_{|x|=\varepsilon} \left(\Phi(x)\frac{\partial(\Delta\varphi(x))}{\partial n} + \Delta\Phi(x)\frac{\partial\varphi(x)}{\partial n} \right) dS(x) -$$

$$-\lim_{\varepsilon\to 0} \int_{|x|=\varepsilon} \left(\varphi(x)\frac{\partial(\Delta\Phi(x))}{\partial n} + \Delta\varphi(x)\frac{\partial\Phi(x)}{\partial n} \right) dS(x),$$

where $n = -x/\varepsilon$.

Considering the fact that $\varphi \in D(\mathbf{R}^4)$, and, therefore, infinitely differentiable, we have

$$\left| \int_{|x|=\varepsilon} \Phi(x)\frac{\partial(\Delta\varphi(x))}{\partial n} dS(x) \right| \le |\, 2\pi^2 C\varepsilon^3 \ln\varepsilon \,| \cdot \sup_{x\in\mathbf{R}^4} \sum_{j=1}^{4} \left| \frac{\partial(\Delta\varphi(x))}{\partial x_j} \right| \to 0 \text{ as } \varepsilon \to +0;$$

$$\left| \int_{|x|=\varepsilon} \Delta\Phi(x)\frac{\partial(\varphi(x))}{\partial n} dS(x) \right| \le |\, 4\pi^2 C\varepsilon \,| \cdot \sup_{x\in\mathbf{R}^4} \sum_{j=1}^{4} \left| \frac{\partial(\varphi(x))}{\partial x_j} \right| \to 0 \text{ as } \varepsilon \to +0;$$

$$\left| \int_{|x|=\varepsilon} \Delta\varphi(x)\frac{\partial(\Phi(x))}{\partial n} dS(x) \right| \le |\, 2\pi^2 C\varepsilon^2 \,| \cdot \sup_{x\in\mathbf{R}^4} \left| \Delta\varphi(x) \right| \to 0 \text{ as } \varepsilon \to +0.$$

Hence

$$\int_{\mathbf{R}^4} \Phi(x)\Delta^2\varphi(x)dx = -\lim_{\varepsilon\to+0} \int_{|x|=\varepsilon} \varphi(x)\frac{\partial(\Delta\Phi(x))}{\partial n} dS(x) = -4C \lim_{\varepsilon\to+0} \varepsilon^{-3} \int_{|x|=\varepsilon} \varphi(x)dS(x).$$

Applying the first mean-value theorem [1, p. 338], for $\varepsilon > 0$ there exists point x_ε such that $|x_\varepsilon| = \varepsilon$ and

$$\int_{|x|=\varepsilon} \varphi(x)dS(x) = 2\pi^2\varepsilon^3\varphi(x_\varepsilon).$$

Finally

$$\int_{\mathbf{R}^4} \Phi(x)\Delta^2\varphi(x)dx = -8\pi^2 C \lim_{\varepsilon\to+0} \varphi(x_\varepsilon) = -8\pi^2 C\varphi(0).$$

By taking $C = -1/(8\pi^2)$, we proved the following statement.

Theorem 2. *The function* $\Phi(x) = -(8\pi^2)^{-1} \ln |x|$ *is the fundamental solution of the biharmonic equation in* $\mathbf{R}^4$.

We express our gratitude to Kovalenko O. N., Senior Lecturer of the Department of Foreign Languages of Brest State A. S. Pushkin University, for helping us in translating.

References

1. Zorich, V. A. Matematicheskij analiz. CHast' II / V. A. Zorich. – M. : MCNMO. – 2002. – 794 s.

УДК 519.6+517.983

АПРИОРНЫЕ ОЦЕНКИ ПОГРЕШНОСТИ ДЛЯ ЯВНОГО ИТЕРАЦИОННОГО МЕТОДА РЕШЕНИЯ НЕКОРРЕКТНЫХ ЗАДАЧ В ПОЛУНОРМЕ ГИЛЬБЕРТОВА ПРОСТРАНСТВА

М. С. Горбач,
БрГУ имени А.С. Пушкина, Брест
Научный руководитель:
О. В. Матысик, кандидат физико-математических наук, доцент

Как известно, погрешность метода простой итерации Ландвебера $x_{n+1,\delta} = x_{n,\delta} + \alpha\left(y_\delta - Ax_{n,\delta}\right)$, $x_{0,\delta} = 0$ [1–2] с постоянным шагом зависит от шага по антиградиенту, и притом так, что для сокращения числа операций желательно, чтобы шаг по антиградиенту был как можно большим. Однако на этот шаг накладывается ограничение сверху [2]: $0 < \alpha \le \frac{5}{4\|A\|}$. Возникает идея попытаться ослабить эти ограничения. Это удается сделать, выбирая для шага три значения α, β, γ попеременно, где γ уже не обязано удовлетворять прежним требованиям.

В гильбертовом пространстве H решается уравнение первого рода $Ax = y$ с положительным ограниченным самосопряженным оператором A, для которого нуль не является собственным значением. Однако нуль принадлежит спектру оператора A, следовательно, задача некорректна. Предполагая существование единственного точного решения x при точной правой части y, будем искать его приближенное значение x_n, используя метод

$$x_{n+1} = x_n - \alpha_{n+1}(Ax_n - y), \quad x_0 = 0,$$
$$\alpha_{3n+1} = \alpha, \quad \alpha_{3n+2} = \beta, \quad \alpha_{3n+3} = \gamma, \quad n = 0, 1, 2, \dots . \tag{1}$$