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Estimation of the Lyapunov Spectrum from One-Dimensional Observations Using Neural Networks

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Abstract: *This paper discusses the neural network approach for computing of Lyapunov spectrum using one dimensional time series from unknown dynamical system. Such an approach is based on the reconstruction of attractor dynamics and applying of multilayer perceptron (MLP) for forecasting the next state of dynamical system from the previous one. It allows for evaluating the Lyapunov spectrum of unknown dynamical system accurately and efficiently only by using one observation. The results of experiments are discussed.*

Keywords: - Lyapunov spectrum, Multilayer Neural Networks, Chaotic processes, Dynamical system

1. INTRODUCTION

Processing of time series often turns out to be insufficient when the data irregular and this inadequacy has often been assigned to noise and randomness. Many real dynamical systems (e.g., compound pendula, dripping faucets, chemical reactions, stock market, EEG patterns of brainwave activity, social behaviour) are believed to be nonlinear. In many such systems, chaotic behaviour has been observed. Chaos theory is nowadays widely studied and applied in various areas in order to describe, characterize, and possibly predict the system behaviour when such kind of complexity occurs [1]. Therefore nonlinear signal processing has become an inevitable and essential tool for the study of complicated systems. The techniques of nonlinear signal processing are summarized in [2]

The chaotic behaviour of a dynamical system has been manifested by the study of nonlinear mathematical equations and it has been observed on experimental data. Unfortunately, in typical practical problems, we do not know the nonlinear equations that describe the underlying dynamical system of an observed process. The problem consist of identifying the chaotic behaviour and building a model that captures the important properties of the unknown system by using only experimental data. In order to determine the main properties of the model, we must estimate dynamic invariants of the underlying system, such as the correlation dimension, the Lyapunov exponents and the Kolmogorov entropy. However, in practice, the existing approaches for the estimation of the Lyapunov exponents from experimental data are characterized by computational complexity, require a large data length and applied only when we have all observations of dynamical system. Working on real world data, it is often difficult to obtain a reliable estimate with these approaches and thus their applicability is limited.

A proposed solution to this is the use of neural networks. As was shown in [3,4] the multilayer perceptron has been used successfully for the estimation of the largest Lyapunov exponent and Lyapunov spectrum from scalar time series. In this paper is proposed new approach for computing of Lyapunov spectrum using only one-dimensional observations from unknown dynamical system.

The rest of the paper is organized as follows. Section 2 presents the new technique for computing the Lyapunov exponents using single time series. The respectively algorithm for that is described in Section 3. Section 4 discusses the results of experiments. To end, Section 5 gives conclusions.

2. A NEURAL NETWORK APPROACH TO COMPUTE THE LYAPUNOV EXPONENTS

Let's consider a dynamical system described by n differential or difference equations. This system has n Lyapunov's exponents λ_i (i=1,2,...,n), that are globally called Lyapunov's spectrum. The Lyapunov's spectrum describes the system dynamics by defining the evolution of the attractor's trajectories and characterizes the sensitive dependence on the initial conditions. These exponents are the average exponential rates of convergence (divergence) of nearby trajectories in the phase space. The largest Lyapunov's exponent is the statistical measure of the divergence between two orbits starting from slightly different initial conditions. In a chaotic system the largest Lyapunov's exponent is positive.

Let's consider a small sphere at the initial condition in the n-dimensional phase space. Through the time this sphere is transformed into an ellipsoid with n principal axes: the Lyapunov's spectrum measures the exponential growth for the principal axes of the evolving ellipsoid. In fact, let's consider the following Lyapunov's spectrum:

$$\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \qquad (1)$$

and let's order the axis of the ellipsoid by decreasing length; λ_1 corresponds to the longest axis, λ_2 corresponds to the subsequent one, and so on. The Lyapunov's exponent λ_i is defined as:

$$\lambda_i = \lim_{t\to\infty}\frac{1}{t}\cdot\ln\frac{l_i(t)}{l_i(0)} \qquad (2)$$

where $l_i(0)$ and $l_i(t)$ are the lengths of i-th axis at the initial time and at a time t, respectively. Therefore every Lyapunov's exponent characterizes the modification of

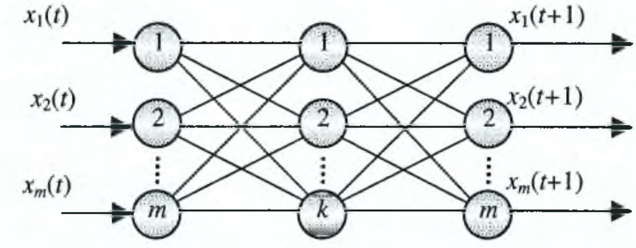

Fig.1 - Multilayer perceptron.

the principal axis of the ellipsoid. In an n-dimensional chaotic system the sum of the n Lyapunov's exponents is negative for dissipative systems. The positive exponents are responsible for the sensitivity to initial conditions. The sum of the positive Lyapunov's exponents is equal to Kolmogorov's entropy.

Let's consider a dynamical system described by the n-dimensional observable vector $X(t)=[X_1(t),X_2(t),\ldots,X_n(t)]$ and assume that the observations $X_i(t)$ are known. In this case the Lyapunov spectrum can be computed applying MLP, by using the method given in [2].

Let's assume now, that only one observation $X_i(t)$ is known. The main goal is to compute Lyapunov exponents of unknown dynamical system using only one observation. Then the first step of proposed approach is to reconstruct the attractor dynamics from a single time series, using the method of delays [2]. After this step we can obtain the reconstructed trajectory $X(t)$, which can be presented as a matrix where each row is a phase-space vector:

$$X=[X(1)\ X(2)\ \ldots\ X(k)], \tag{3}$$

where $X(i)$ is the state of the system at discrete time i and each $X(i)$ is given by

$$X(i)=[x(i)\ x(i-\tau)\ldots x(i-(m-1)\cdot\tau)]= [x_1(i)\ x_2(i)\ldots\ x_m(i)], \tag{4}$$

where τ is the time delay and m is the embedding dimension.

It is based on the Taken's theorem [5], which states that the attractor can be reconstructed from a one dimensional observation in a phase space with dimension $m \geq 2[d]+1$, where d is the fractal dimension of the attractor and [.] is the integer part. To apply the embedding theorem it is necessary to estimate the embedding dimension, i.e. the dimension of the reconstructed state space m, and the time delay, which is the time separation of lagged samples comprising the reconstructed state vector. There exist several methods for the estimation of these parameters, e.g. mutual information for the delay time, false nearest neighbors and saturation of measures such as correlation dimension for the embedding dimension [2,4].

The second step of proposed approach is to create neural network in order to forecast the next state of dynamical system X(i) from the previous one $X(i-1)$. This network is a multilayer perceptron with m input units, k hidden units, and n output units (Fig.1). The output is defined as

$$X(t+1)=F(X(t)). \tag{5}$$

After training neural network and starting from a given initial condition, this network is able to compute the state of the dynamical system at any time, as well as to describe the evolution of the phase trajectory points. At each step the Gram-Schmidt orthogonalization procedure must be used to adjust the output vector.

3. AN ALGORITHM FOR COMPUTING OF LYAPUNOV SPECTRUM

Let $|w_i(t)|$ be the length of the i-th vector at the time t. This length characterizes the value of the vector along the i-th ellipsoid axis. Thus, the i-th Lyapunov's exponent is given by:

$$\lambda_i = \lim_{p\to\infty} \frac{1}{p} \sum_{t=1}^{p} \ln \frac{|w_i(t)|}{|w_i(t-1)|} \tag{6}$$

The correspondent length $|w_i(t)|$ can be evaluated by using a neural network and, consequently, the Lyapunov exponents can be estimated. The algorithm to compute the complete Lyapunov spectrum is as follows:

1. Take the initial point $N(0)=[x_1(0),x_2(0),\ldots,x_m(0)]$ from the basin of attraction.
2. Choose a small value $\varepsilon \approx 10^{-8}$ and define the coordinates of next n points as follows:

$$\begin{aligned} &A_1(0)=[x_1(0)+\varepsilon, x_2(t),\ldots,x_m(t)] \\ &A_2(0)=[x_1(0),x_2(t)+\varepsilon,\ldots,x_m(t)] \\ &\ldots \\ &A_m(0)=[x_1(0),x_2(t),\ldots,x_m(t)+\varepsilon] \end{aligned} \tag{7}$$

The following orthogonal vectors are obtained:

$$\begin{aligned} &NA_1(0)=[\varepsilon,0,\ldots,0] \\ &NA_2(0)=[0,\varepsilon,\ldots,0] \\ &\ldots \\ &NA_m(0)=[0,0,\ldots,\varepsilon] \end{aligned} \tag{8}$$

3. Compute the length of each vector $|NA_i(0)| = |w_i(0)| = \varepsilon$, where $i=\overline{1,m}$.
4. At the time $t=0$, use the set of points $N(0)$, $A_1(0)$, $A_2(0),\ldots,$ $A_m(0)$ as the input vector of the neural network. The output produced by the predicting network is the set of the coordinates of the points at the next time $t=t+1$:

$N(1) = [x_1(1,N), x_2(1,N), \ldots, x_m(1,N)]$
$A_1(1) = [x_1(1,A_1), x_2(1,A_1), \ldots, x_m(1,A_1)]$
...
$A_2(1) = [x_1(1,A_2), x_2(1,A_2), \ldots, x_m(1,A_2)]$ (9)
...
$A_m(1) = [x_1(1,A_m), x_2(1,A_m), \ldots, x_m(1,A_m)]$,

where $x_j(1,A_j)$ is the j-th coordinate of the point A_j at the time $t=1$. This leads to the next set of vectors:

$$NA_1(1) = w_1(1) = [w_{11}, w_{21}, \ldots, w_{m1}]$$
$$NA_2(1) = w_2(1) = [w_{12}, w_{22}, \ldots, w_{m2}] \quad (10)$$
$$NA_m(1) = w_m(1) = [w_{1m}, w_{2m}, \ldots, w_{mm}]$$

where w_{ij} is the i-th coordinate of the j-th vector, having defined $w_{ij} = x_i(1, A_j) - x_i(1, N)$.

5. The basis $[w_1(1), w_2(1), \ldots, w_m(1)]$ is transformed into the orthonormal frame by using the Gram-Schmidt algorithm, as follows:
   a) The first vector of the orthonormal frame is chosen as:

$$w_1'(1) = \left[\frac{w_{11}}{|w_1(1)|}, \frac{w_{21}}{|w_1(1)|}, \ldots, \frac{w_{m1}}{|w_1(1)|}\right] \quad (11)$$

   where $|w_1(1)| = \sqrt{w_{11}^2 + w_{21}^2 + \ldots + w_{m1}^2}$.

   b) The subsequent vectors are defined by the following recurrent formulas:

$$w_i(1) = w_i(1) - \sum_{j=1}^{i-1} (w_i^T(1) \cdot w_j'(1)) \cdot w_j'(1)$$

$$|w_i(1)| = \sqrt{w_{1i}^2 + w_{2i}^2 + \ldots + w_{mi}^2} \quad (12)$$

$$w_i'(1) = \left[\frac{w_{1i}}{|w_i(1)|}, \frac{w_{2i}}{|w_i(1)|}, \ldots, \frac{w_{mi}}{|w_i(1)|}\right]$$

   where $i = \overline{2, m}$.

   c) Compute:

Table 1. Estimation of Lyapunov spectrum of Lorenz system using neural network

dt	T	Lyapunov spectrum			Absolute error	Relative error
		λ_1	λ_2	λ_3		
0,04170	0,1668	0,612978	-0,2016840	-15,0033	0,559053	3,83%
0,04200	0,1680	0,725777	-0,0211582	-14,6402	0,193839	1,33%
0,04215	0,1686	0,966544	-0,3009800	-15,9458	1,407730	9,64%
0,04220	0,1688	0,965399	-0,3006240	-15,9270	1,389170	9,51%
0,08500	0,1700	1,143851	-0,2816490	-14,9843	0,553092	3,79%
0,04260	0,1704	1,021790	-0,4326160	-15,6514	1,168620	8,00%
0,04260	0,1704	0,483841	0,0528098	-13,3949	1,251610	8,57%
0,04300	0,1720	0,742471	-0,0865899	-14,2650	0,358420	2,45%
0,08600	0,1720	0,830438	-0,3357490	-13,5627	1,066370	7,30%
0,04320	0,1728	0,570654	-0,1465600	-14,9297	0,511766	3,51%
0,08700	0,1740	1,216890	-0,6435080	-14,5374	0,715508	4,90%

Table 2. Estimation of Lyapunov spectrum of Roessler system using neural network

dt	τ	Lyapunov spectrum			Absolute error	Relative error
		λ_1	λ_2	λ_3		
0,04	0,04	0,173003	-0,0821049	-5,47571	0,154879	2,87%
0,07	0,07	0,060350	-0,3888620	-5,18352	0,441825	8,19%
0,06	0,12	0,090696	0,0030709	-5,02998	0,363565	6,74%
0,06	0,12	0,106080	-0,0358488	-5,79224	0,402378	7,46%
0,06	0,12	0,077922	-0,0187908	-5,93021	0,537581	9,97%
0,07	0,14	0,129117	-0,1092460	-4,93167	0,477637	8,86%
0,08	0,16	0,106981	-0,0449128	-5,36074	0,065971	1,22%
0,08	0,16	0,085461	-0,0282390	-5,31476	0,084427	1,57%
0,04	0,16	0,119605	-0,2027930	-5,56896	0,272851	5,06%
0,06	0,18	0,141245	-0,0751598	-5,48983	0,141277	2,62%
0,08	0,48	0,078753	-0,0144016	-5,24691	0,147000	2,73%

$$s_i(1) = \ln\frac{|w_i(1)|}{|w_i(0)|} \quad (13)$$

where $i = \overline{1,n}$.

The result is the new set of points:

$$N(1) = [x_1(1,N), x_2(1,N), ..., x_m(1,N)]$$
$$A_1(1) = [\bar{x}_1(1,A_1), \bar{x}_2(1,A_1), ..., \bar{x}_m(1,A_1)] \quad (14)$$
$$A_2(1) = [\bar{x}_1(1,A_2), \bar{x}_2(1,A_2), ..., \bar{x}_m(1,A_2)]$$
$$...$$
$$A_m(1) = [\bar{x}_1(1,A_m), \bar{x}_2(1,A_m), ..., \bar{x}_m(1,A_m)],$$

where $\bar{x}_j(1,A_j) = w_{ij} + x_i(1,N)$.

6. Repeat from step 3 to step 5 for $t = \overline{1,p}$, where $p \approx 1000$.
7. Define the Lyapunov spectrum as:

$$\lambda_i = \frac{1}{p}\sum_{t=1}^{p} s_i(t) \quad (15)$$

where $i = \overline{1,n}$. The following Lyapunov exponents are therefore obtained:

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n. \quad (16)$$

Thus the proposed approach permits to estimate Lyapunov exponents using only single time series.

4. EXPERIMENTAL RESULTS

Let's examine proposed approach for estimation of Lyapunov spectrum. As the chaotic systems, which we want to model are the Lorenz and Roessler attractors. The Lorenz attractor is described by the following three coupled nonlinear differential equations:

$$\begin{cases} \frac{dx}{dt} = G(y-x) \\ \frac{dy}{dt} = -xz + rx - y \\ \frac{dz}{dt} = xy - bz \end{cases} \quad (17)$$

where G=10, r=28, and b=8/3 for chaotic behavior. Lorenz proposed this model for the atmospheric turbulence. For such a system actual values of Lyapunov exponents are 0.906, 0, and –14,472, respectively. The value of fractal dimension is 2.06. The Roessler attractor can be described by the following equations:

$$\begin{cases} \frac{dx}{dt} = -y - z, \\ \frac{dy}{dt} = x + ay, \\ \frac{dz}{dt} = b + (x-r)z, \end{cases} \quad (18)$$

where $a = b = 0.2$ and $r = 5.7$ for chaotic behavior. The actual values of Lyapunov exponent are 0.07, 0, and –5.39 respictevely. The value of fractal dimension is 2.03. Only the X-series has been used in both cases; the size of the data set was 400 points. We have been choose the embedding dimension $m=3$ less than in accordance with Takens criterion. A multilayer perceptron with 3 input units, 10 hidden units, and 3 output units has been used. By using this technique we obtained the Lyapunov exponents for Lorenz and Roessler dynamical system, as it is shown in tables 1 and 2. During the experiments the time step dt between points and time delay τ have been changed. As can be seen from tables the proposed technique permits to predict the Luapunov exponents for Lorenz and Roessler dynamical systems accurately and efficiently, using only one dimensional observations.

5. CONCLUSION

In this paper some aspects of chaotic time series processing have been addressed, namely estimation of the Lyapunov spectrum from the single time series. The proposed approach based on the reconstruction of the attractor dynamics and applying of the multilayer perceptron for estimation of Lyapunov exponents. The neural network technique allow for evaluating the Luapunov spectrum even on small data set, that permits both for reducing the computationally complexity and for limit the observation time. Finally I would like to thank Nikolay Chumerin for performing some experiments.

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