МИНИСТЕРСТВО ОБРАЗОВАНИЯ РЕСПУБЛИКИ БЕЛАРУСЬ

УЧРЕЖДЕНИЕ ОБРАЗОВАНИЯ
«БРЕСТСКИЙ ГОСУДАРСТВЕННЫЙ ТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ»

КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

Functions of a Single and Several Variables

методические указания на английском языке
по дисциплине **«Математика»**

Брест 2019

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Данные методические указания адресованы преподавателям и студентам технических ВУЗов для проведения аудиторных занятий и организации самостоятельной работы студентов при изучении материала из рассматриваемых разделов. Методические указания на английском языке «Functions of a Single and Several Variables» содержат необходимый материал по темам «Основы математического анализа», «Дифференциальное исчисление функций одной переменной», «Функции нескольких переменных» изучаемым студентами БрГТУ технических специальностей в курсе дисциплины «Математика». Теоретический материал сопровождается рассмотрением достаточного количества примеров и задач, при необходимости приводятся соответствующие иллюстрации. Для удобства пользования каждая тема разделена на три части: краткие теоретические сведения (определения, основные теоремы, формулы для расчетов); задания для аудиторной работы и задания для индивидуальной работы.

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CONTENT

I FUNCTIONS OF A SINGLE VARIABLE

1.1 Functions and limit

Definition Let X and Y be sets. A ***function*** from X to Y is a rule or method for assigning to each element in X a unique element in Y. (See Figure 1.1)

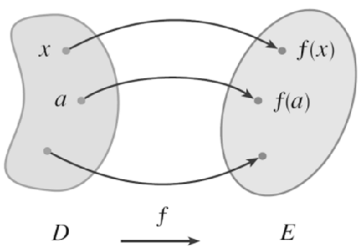


Figure 1.1

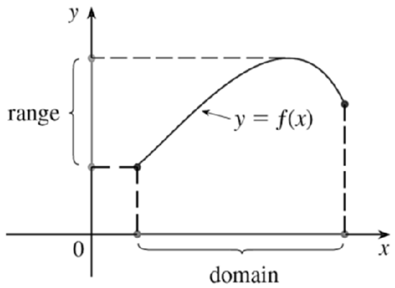


Figure 1.2

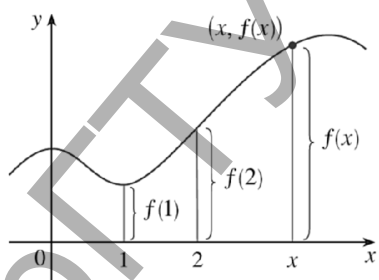


Figure 1.3

There are four possible ways to represent a function: ***verbally*** (by a description in words); ***numerically*** (by a table of values); ***visually*** (by a graph); ***algebraically*** (by an explicit formula). A function is often denoted by the symbol f. The element that the function assigns to the element x is denoted $y = f(x)$ (read f of x). In practice, though, almost everyone speaks interchangeably of the function f or the function $f(x)$.

Definition Let X and Y be sets and let f be a function from X to Y. The set X is called the ***domain*** of the function. If $y = f(x)$, y is called the ***value*** of f at x. The set of all values of the function is called the ***range*** of the function (see Figures 1.2, 1.3).

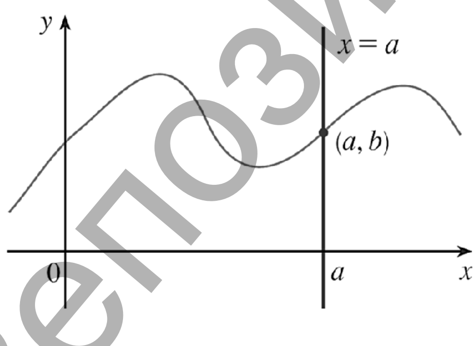


Figure 1.4

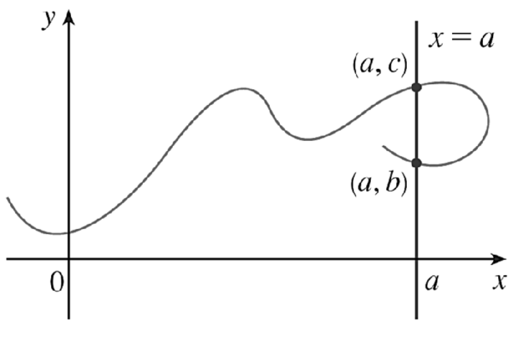


Figure 1.5

When the function is given by a formula, the domain is usually understood to consist of all the numbers for which the formula is defined. The value $f(x)$ of a function f at x is also called the ***output***; x is called the ***input*** or ***argument***. If $y = f(x)$, the symbol x is called the ***independent variable*** and the symbol y is called the ***dependent variable***. If both the inputs and outputs of a function are numbers, we shall call the ***function numerical***. In some more advanced courses such a function is also called a real function of a real variable.

If both the domain and the range of a function consist of real numbers, it is possible to draw a picture that displays the behavior of the function.

Definition *(graph of a numerical function)* Let f be a numerical function. The ***graph*** of f consists of those points $(x; y)$ such that $y = f(x)$.

If some line parallel to the y axis meets the curve more than once, then the curve is not the graph of a function. Otherwise it is the graph of a function. The curve in Figure 1.4 is the graph of a function, the curve in Figure 1.5 is not the graph of a function.

Basic characteristics of functions

Definition *(composition of functions)* Let f and g be functions. Suppose that x is such that $g(x)$ is in the domain of f. Then the function that assigns to x the value $f(g(x))$ is called the ***composition*** of f and g.

Definition *(even function)* A function f such that $f(-x) = f(x)$ is called an ***even function*** (see Figure 1.6).

Definition *(odd function)* A function f such that $f(-x) = -f(x)$ is called an ***odd function*** (see Figure 1.7).

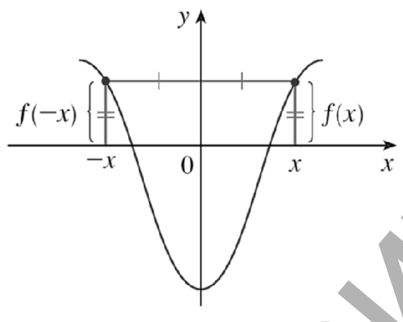


Figure 1.6

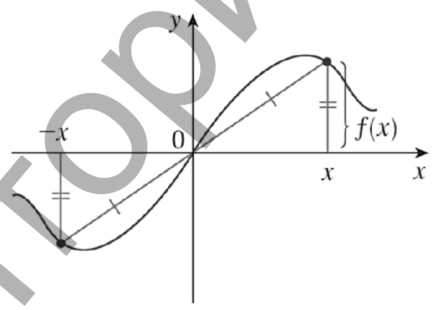


Figure 1.7

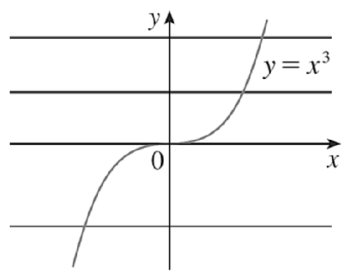


Figure 1.8

Most functions are neither even nor odd. Many functions used in calculus happen to be even or odd. The graph of such a function is symmetric with respect to the y axis or with respect to the origin, as will now be shown.

Definition A function f is called ***a one-to-one function*** if it never takes on the same value twice; that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

The graph of one-to-one numerical function has the property that every horizontal line meets it only in one point (see Figure 1.8).

Definition *(monotonic function)* If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is an ***increasing function***. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is a ***decreasing function***. These two types of functions are also called ***monotonic***.

Definition Let $y=f(x)$ be a one-to-one function. The function g that assigns to each output of f the corresponding unique input is called the ***inverse*** of f. That is, if $y=f(x)$, then $x=g(y)$.

The following types of functions are called the ***basic elementary functions***: *polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions*.

Definition *(Limit of* $f(x)$ ***at*** a***)*** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is A, and we write

$$\lim_{x\to a} f(x)=A$$

if for every number $\varepsilon>0$ there is a number $\delta>0$ such that if $0<|x-a|<\delta$ then $|f(x)-A|<\varepsilon$:

$$\lim_{x\to a} f(x)=A \Leftrightarrow \forall\varepsilon>0 \ \exists\delta(\varepsilon)>0 \ \forall x: |x-a|<\delta \Rightarrow |f(x)-A|<\varepsilon .$$

Definition *(right-hand limit of* $f(x)$ ***at*** a***)*** A number A is called ***right-hand limit of*** $f(x)$ ***at*** a if for every number $\varepsilon>0$ there is a number $\delta>0$ such that if $a<x<a+\delta$ then $|f(x)-A|<\varepsilon$. It is denoted by $\lim\limits_{x\to a^+} f(x)=\lim\limits_{x\to a+0} f(x)=A$.

Definition *(left-hand limit of* $f(x)$ ***at*** a***)*** A number A is called ***left-hand limit of*** $f(x)$ ***at*** a if for every number $\varepsilon>0$ there is a number $\delta>0$ such that if $a-\delta<x<a$ then $|f(x)-A|<\varepsilon$. It is denoted by $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a-0} f(x)=A$.

Theorem $\lim\limits_{x\to a} f(x)=A$ if and only if $\lim\limits_{x\to a+0} f(x)=\lim\limits_{x\to a-0} f(x)=A$.

Definition A function $f(x)$ is called ***infinitesimal function*** at the point a, if $\lim\limits_{x\to a} f(x)=0$. A function $f(x)$ is called ***infinitely large function*** at the point a, if $\lim\limits_{x\to a} f(x)=\infty$.

The sum and product of a finite number of infinitesimal functions as x approaches a, as well as the product of an infinitesimal function for a bounded function, are infinitesimal functions as x approaches a.

We use the following properties of limits, called the ***Limit Laws***, to calculate limits.

Theorem *(Limit Laws)* Let $u=u(x)$ and $v=v(x)$ be two functions and assume that $\lim\limits_{x\to a} u(x)=A$ and $\lim\limits_{x\to a} v(x)=B$ both exist. Then

1) $\lim\limits_{x\to a}\left(c\cdot u(x)\right)=c\lim\limits_{x\to a} u(x)=c\cdot A$, where $c-const$.

2) $\lim\limits_{x\to a}\left(u(x)\pm v(x)\right)=\lim\limits_{x\to a} u(x)\pm\lim\limits_{x\to a} v(x)=A\pm B$.

3) $\lim\limits_{x\to a}\left(u(x)\cdot v(x)\right)=\lim\limits_{x\to a} u(x)\cdot\lim\limits_{x\to a} v(x)=A\cdot B$.

4) $$\lim_{x\to a}\frac{u(x)}{v(x)}=\frac{\lim\limits_{x\to a}u(x)}{\lim\limits_{x\to a}v(x)}=\frac{A}{B},\ \lim_{x\to a}v(x)\neq 0.$$

5) $$\lim_{x\to a}u(x)^{v(x)}=\left(\lim_{x\to a}u(x)\right)^{\lim\limits_{x\to a}v(x)}=A^B.$$

If the conditions of these theorems are not satisfied, then there are the so-called indefinite expressions ***(indeterminate form)*** of types: (∞/∞), $(0/0)$, $(\infty-\infty)$, $(0\cdot\infty)$, (1^∞), (0^∞), (∞^0). To uncover indeterminate forms, additional algebraic transformations are required.

Example 1 Evaluate the limit if it exists.

(a) $\lim\limits_{x\to\infty}\frac{4x^2-3x+5}{3x^2+6x-2}$ *(b)* $\lim\limits_{x\to 1}\frac{x^2-6x+5}{x^2-5x+4}$ *(c)* $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$

Solution

(a) As x gets large, the numerator $4x^2-3x+5$ grows large, influencing the quotient to become large. On the other hand, the denominator $3x^2+6x-2$ also grows large, influencing the quotient to become small. An algebraic device will help reveal what happens to the quotient. We have

$$\lim_{x\to\infty}\frac{4x^2-3x+5}{3x^2+6x-2}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{x^2\left(\frac{4x^2}{x^2}-\frac{3x}{x^2}+\frac{5}{x^2}\right)}{x^2\left(\frac{3x^2}{x^2}+\frac{6x}{x^2}-\frac{2}{x^2}\right)}=\lim_{x\to\infty}\frac{4-\frac{3}{x}+\frac{5}{x^2}}{3+\frac{6}{x}-\frac{2}{x^2}}=\frac{4}{3}.$$

(b) The basic indeterminate of type $(0/0)$ is obtained by replacing $x=1$. We factor the numerator and the denominator:

$$x^2-6x+5=0; \qquad x^2-5x+4=0;$$
$$D=36-4\cdot1\cdot5=16>0; \qquad D=25-4\cdot1\cdot4=9>0;$$
$$x=\frac{6\pm\sqrt{16}}{2};\ x_1=5;\ x_2=1; \qquad x=\frac{5\pm\sqrt{9}}{2};\ x_1=4;\ x_2=1;$$
$$x^2-6x+5=(x-1)(x-5). \qquad x^2-5x+4=(x-1)(x-4).$$

The numerator and denominator have a common factor of $(x-1)$. Therefore we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to1}\frac{x^2-6x+5}{x^2-5x+4}=\left(\frac{0}{0}\right)=\lim_{x\to1}\frac{(x-1)(x-5)}{(x-1)(x-4)}=\lim_{x\to1}\frac{x-5}{x-4}=\frac{\lim\limits_{x\to1}(x-5)}{\lim\limits_{x\to1}(x-4)}=\frac{1-5}{1-4}=\frac{4}{3}.$$

(c) As $x\to\infty$, both $\sqrt{x^2+x}$ and x approach ∞. It is not immediately clear how their difference $\sqrt{x^2+x}-x$ behaves. It is necessary to use a little algebra and rationalize the expression:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\frac{\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}=\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}=$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(1+1/x\right)}+x}=\lim_{x\to\infty}\frac{x}{x\left(\sqrt{1+1/x}+1\right)}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{2}.$$

Exercise Set 1.1

1. Find the exact value of each expression $f(0)$, $f\left(-\frac{3}{4}\right)$, $f(-x)$, $f\left(\frac{1}{x}\right)$, $\frac{1}{f(x)}$, if $f(x)=\sqrt{1+x^2}$.

2. Find an expression for the function which graph is the given curve.

(a) The line segment joining the points $(3;-1)$ and $(2;5)$.

(b) The line segment joining the points $(-2;-1)$ and $(0;7)$.

In exercises 3 to 6 find the domain of the function

3. $y=\sqrt{-x}+\dfrac{1}{\sqrt{2+x}}$ ***4.*** $y=\lg\dfrac{2+x}{2-x}$ ***5.*** $y=\sqrt{12+x-x^2}$

6. $y=\arcsin\left(\lg\dfrac{x}{10}\right)$

In exercises 7 to 10 find a formula for the inverse of the function

7. $y=\sqrt{10-3x}$ ***8.*** $y=e^{x^3}$ ***9.*** $y=\ln(x+3)$ ***10.*** $y=2x^3+3$

In exercises 11 to 14 determine whether $f(x)$ is even, odd or neither even nor odd.

11. $f(x)=\dfrac{1}{2}\left(a^x+a^{-x}\right)$ ***12.*** $f(x)=\sqrt{1+x+x^2}$

13. $f(x)=\sin^2\left(x+\dfrac{1}{x}\right)$ ***14.*** $f(x)=\sqrt[3]{(1+x)^2}+\sqrt[3]{(x-1)^2}$

In exercises 15 to 56 find the limit, if it exists. If the limit does not exist, explain why.

15. $\lim\limits_{x\to2}(4x^2-6x+3)$ ***16.*** $\lim\limits_{x\to1}\dfrac{3x^2-4x+7}{2x^2-5x+6}$ ***17.*** $\lim\limits_{x\to2}\dfrac{x+1}{x-2}$

18. $\lim\limits_{x\to0+0}\dfrac{1}{x^4}$ ***19.*** $\lim\limits_{x\to0-0}\dfrac{1}{x^4}$ ***20.*** $\lim\limits_{x\to1+0}\dfrac{1}{x-1}$

21. $\lim\limits_{x\to-1-0}\dfrac{1}{(x+1)}$ ***22.*** $\lim\limits_{x\to-1+0}\dfrac{1}{(x+1)^2}$ ***23.*** $\lim\limits_{x\to-1}\dfrac{1}{(x+1)^2}$

24. $\lim\limits_{x\to\infty}\dfrac{1}{x+4}$ ***25.*** $\lim\limits_{x\to-1}\dfrac{x^3+1}{x^2+1}$ ***26.*** $\lim\limits_{x\to1}\dfrac{x^2-5x+10}{x^2-25}$

27. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+4}{\sqrt{x^4+1}}$ **28.** $\lim\limits_{x\to\pm\infty}\dfrac{3x^3+4x^2+2}{x^3-7x-10}$ **29.** $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

30. $\lim\limits_{n\to\infty}\dfrac{9n^2+4n-6}{2n^2+2}$ **31.** $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^4+3}-\sqrt[5]{x^3+4}}{\sqrt[3]{x^7+1}}$ **32.** $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

33. $\lim\limits_{x\to2}\dfrac{x^2-5x+6}{x^2-12x+20}$ **34.** $\lim\limits_{x\to1}\dfrac{3x^2-x-2}{4x^2-5x+1}$ **35.** $\lim\limits_{x\to2}\dfrac{x^2-7x+10}{8-x^3}$

36. $\lim\limits_{x\to1}\dfrac{x^3-x^2+x-1}{x^2-4x+3}$ **37.** $\lim\limits_{x\to2}\dfrac{x^3-8}{2x^2+x-6}$ **38.** $\lim\limits_{x\to1}\dfrac{x^3-3x^2+2}{x^2-7x+6}$

39. $\lim\limits_{x\to\frac{1}{2}}\dfrac{8x^3-1}{6x^2-5x+1}$ **40.** $\lim\limits_{x\to5}\dfrac{x^2-25}{\sqrt{x-1}-2}$ **41.** $\lim\limits_{x\to3}\dfrac{\sqrt{x+13}-4}{x^2-9}$

42. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos x-\sin x}{\cos 2x}$ **43.** $\lim\limits_{x\to0}\dfrac{\sqrt{x^2+1}-1}{x}$ **44.** $\lim\limits_{x\to-4}\left(\dfrac{1}{x+4}-\dfrac{8}{16-x^2}\right)$

45. $\lim\limits_{x\to2}\left(\dfrac{4}{x^2-4}-\dfrac{1}{x-2}\right)$ **46.** $\lim\limits_{x\to1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$ **47.** $\lim\limits_{x\to\infty}\left(\sqrt{x^2+6x+5}-x\right)$

48. $\lim\limits_{x\to\infty}\left(x\left(\sqrt{x^2+5}-\sqrt{x^2+1}\right)\right)$ **49.** $\lim\limits_{x\to0}\left(\dfrac{1}{4\sin^2 x}-\dfrac{1}{\sin^2 2x}\right)$ **50.** $\lim\limits_{x\to3}\left(\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right)$

51. $\lim\limits_{x\to\infty}\dfrac{(5+x)^2-\left(1+2x^2\right)^2}{x\left(x^2-2x^3\right)}$ **52.** $\lim\limits_{x\to5}\dfrac{x^2-6x+5}{\sqrt{x-1}-2}$ **53.** $\lim\limits_{x\to\infty}\dfrac{6x-5}{1+\sqrt{x^2+3}}$

54. $\lim\limits_{x\to9}\dfrac{\sqrt{2x+7}-5}{\sqrt{x}-3}$ **55.** $\lim\limits_{x\to4}\dfrac{2-\sqrt{x}}{\sqrt{6x+1}-5}$ **56.** $\lim\limits_{x\to3}\dfrac{x^2+x-12}{\sqrt{x-2}-\sqrt{4-x}}$

Individual Tasks 1.1

1. Find the domain of the function.
2. Find a formula for the inverse of the function.
3. Determine whether $f(x)$ is even, odd or neither even nor odd.
4. Find the limit, if it exists. If the limit does not exist, explain why.

I.	**II.**
1. $y=\lg\dfrac{x^2-3x+2}{x+1}$	*1.* $y=\arccos\dfrac{2x}{1+x}$
2. $y=\dfrac{4x-1}{2x+3}$	*2.* $y=\dfrac{e^x}{1+2e^x}$
3. $f(x)=\lg\dfrac{1+x}{1-x}$	*3.* $y=\lg\left(x+\sqrt{1+x^2}\right)$
4.	*4.*
a) $\lim\limits_{x\to\infty}\dfrac{5x^3+x^2+4}{7x^3+4x^2-x-3}$	*a)* $\lim\limits_{x\to\infty}\dfrac{x^2+4}{x^3+x-3}$

b) $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^2+1}}{x+1}$	*b)* $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^4+x}+x}$
c) $\lim\limits_{x\to-3}\dfrac{2x^2+11x+15}{3x^2+5x-12}$	*c)* $\lim\limits_{x\to-1}\dfrac{5x^2+4x-1}{3x^2+x-2}$
d) $\lim\limits_{x\to2}\dfrac{\sqrt{x+7}-3}{\sqrt{x+2}-2}$	*d)* $\lim\limits_{x\to1}\dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1}$
e) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$	*e)* $\lim\limits_{x\to\infty}\left(x\left(\sqrt{x^2+4}-x\right)\right)$

1.2 Some remarkable limits

To uncover the indeterminate of types $(0/0)$, (1^∞) the following two ***remarkable limits*** are widely used:

(1) $\lim\limits_{x\to0}\dfrac{\sin x}{x}=1$ (2) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$ or $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=e$.

The general form of these limits can be represented by the following expressions:

(1a) $\lim\limits_{f(x)\to0}\dfrac{\sin f(x)}{f(x)}=1$ (2a) $\lim\limits_{f(x)\to\infty}\left(1+\dfrac{1}{f(x)}\right)^{f(x)}=e$.

Example 1 Evaluate the limit if it exists.

(a) $\lim\limits_{x\to\pi}\dfrac{tgx}{x-\pi}$ *(b)* $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$ *(c)* $\lim\limits_{x\to0}\dfrac{x}{\arcsin 3x}$

Solution

(a) To uncover the indeterminate of type $(0/0)$, trigonometric simplifications can be used:

$$\lim_{x\to\pi}\frac{tgx}{x-\pi}=\left(\frac{0}{0}\right)=\lim_{x\to\pi}\frac{\sin(x-\pi)}{(x-\pi)\cos(x-\pi)}=\lim_{x\to\pi}\frac{1}{\cos(x-\pi)}\cdot\lim_{x\to\pi}\frac{\sin(x-\pi)}{x-\pi}=1\cdot1=1.$$

(b) The expression $\dfrac{\sin x}{x}$ is a product of a bounded function $\sin x$ and infinitesimal function $y=1/x$ at the $x\to\infty$. Then $y=\dfrac{\sin x}{x}$ is an infinitesimal function at the $x\to\infty$. Thus, $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$.

(c) To uncover the indeterminate of type $(0/0)$, following substitution can be used $\arcsin 3x=t \Rightarrow x=\dfrac{1}{3}\sin t$. If $x\to0$, then $t\to0$ and given limit can be rewritten and calculated as following

$$\lim_{x\to0}\frac{x}{\arcsin 3x}=\left(\frac{0}{0}\right)=\lim_{t\to0}\frac{\frac{1}{3}\sin t}{t}=\frac{1}{3}.$$

Example 2 Evaluate the limit if it exists.

$$(a)\ \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{x} \qquad (b)\ \lim_{x\to\infty}\left(\frac{2x-1}{2x+5}\right)^{1-3x}$$

Solution

(a) If $x\to\infty$, then $\frac{2}{x}\to 0$ and the basic indeterminate of type $\left(1^{\infty}\right)$ will be obtained. To uncover obtained indeterminate form the remarkable limit $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$ can be used:

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{x}=\left(1^{\infty}\right)=\lim_{x\to\infty}\left(1+\frac{-2}{x}\right)^{\frac{x}{-2}\cdot(-2)}=\lim_{x\to\infty}\left[\left(1+\frac{-2}{x}\right)^{\frac{x}{-2}}\right]^{-2}=e^{-2}.$$

(b) If $x\to\infty$, then $\frac{2x-1}{2x+5}\to 1$, $(1-3x)\to\infty$ and to uncover the indeterminate of type $\left(1^{\infty}\right)$, the following algebraic simplifications can be used:

$$\lim_{x\to\infty}\left(\frac{2x-1}{2x+5}\right)^{1-3x}=\left(1^{\infty}\right)=\lim_{x\to\infty}\left(1+\frac{2x-1}{2x+5}-1\right)^{1-3x}=\lim_{x\to\infty}\left(1+\frac{2x-1-2x-5}{2x+5}\right)^{1-3x}=$$

$$=\lim_{x\to\infty}\left(1+\frac{-6}{2x+5}\right)^{1-3x}=\lim_{x\to\infty}\left(1+\frac{-6}{2x+5}\right)^{\frac{2x+5}{-6}\cdot\frac{-6}{2x+5}\cdot(1-3x)}=\lim_{x\to\infty}e^{\frac{-6(1-3x)}{2x+5}}=\lim_{x\to\infty}e^{\frac{18x-6}{2x+5}}=e^{9}.$$

Exercise Set 1.2

In exercises 1 to 48 find the limit if it exists.

1. $\lim_{x\to 0}\frac{tgx}{x}$ ***2.*** $\lim_{x\to 0}\frac{5x}{\sin 3x}$ ***3.*** $\lim_{x\to 0}\frac{\sin 7x}{\sin 3x}$

4. $\lim_{x\to\pi}\frac{\sin 5x}{\sin 6x}$ ***5.*** $\lim_{x\to 0}\frac{2\arcsin x}{3x}$ ***6.*** $\lim_{x\to 0}\frac{\sin 3x-\sin x}{5x}$

7. $\lim_{x\to 0}\frac{\arcsin 5x}{\sin x}$ ***8.*** $\lim_{x\to 0}\frac{\ln(1+4x)}{5x}$ ***9.*** $\lim_{x\to 0}\frac{3^{x}-1}{x}$

10. $\lim_{x\to 0}\frac{\cos x-\cos 5x}{2x^{2}}$ ***11.*** $\lim_{x\to 0}x\cdot ctg\frac{x}{3}$ ***12.*** $\lim_{x\to 1}\frac{\sin(5x-5)}{x^{2}+4x-5}$

13. $\lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1}$ ***14.*** $\lim_{x\to 0}\frac{\sin^{2}3x-\sin^{2}x}{x^{2}}$ ***15.*** $\lim_{x\to\infty}x\left(arctgx-\frac{\pi}{2}\right)$

16. $\lim_{x\to\infty}\left(\frac{3x+2}{3x-1}\right)^{4x-1}$ ***17.*** $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$ ***18.*** $\lim_{x\to\pm\infty}\left(\frac{2x-1}{x+3}\right)^{-x}$

19. $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$ ***20.*** $\lim_{x\to\infty}\left(\frac{5x+1}{5x-1}\right)^{2x+1}$ ***21.*** $\lim_{x\to-\infty}\left(\frac{3x+1}{4x-3}\right)^{-2x-1}$

22. $\lim\limits_{x\to\infty}\left(\dfrac{5x-7}{3x+4}\right)^{3x^2}$ **23.** $\lim\limits_{x\to\infty}\left(\dfrac{3x-1}{2x+5}\right)^{3x}$ **24.** $\lim\limits_{x\to\pm\infty}\left(1-\dfrac{4}{5x}\right)^{3x}$

25. $\lim\limits_{x\to\infty}\left(\dfrac{5x-4}{5x+2}\right)^{\frac{2x+1}{3}}$ **26.** $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+4}\right)^{x}$ **27.** $\lim\limits_{x\to0}(1+tgx)^{ctgx}$

28. $\lim\limits_{x\to2}(2-x)tg\dfrac{\pi x}{4}$ **29.** $\lim\limits_{x\to0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$ **30.** $\lim\limits_{x\to\pi}\dfrac{\sin x}{\pi^2-x^2}$

31. $\lim\limits_{x\to0}(1+\sin x)^{\frac{1}{x}}$ **32.** $\lim\limits_{x\to0}\left(\dfrac{\sin 3x}{x}\right)^{x+2}$ **33.** $\lim\limits_{x\to0}(1-3x)^{\frac{1}{x}}$

34. $\lim\limits_{x\to\infty}\left(x\cdot\sin\dfrac{1}{x}\right)$ **35.** $\lim\limits_{x\to0}\dfrac{1+x^2-\cos x}{\sin^2 x}$ **36.** $\lim\limits_{x\to0}\left(1+tg^2x\right)^{2ctg^2x}$

37. $\lim\limits_{x\to0}\left(\sqrt{1+x}-x\right)^{1/x}$ **38.** $\lim\limits_{x\to0}(\cos x)^{1/x}$ **39.** $\lim\limits_{x\to0}(\cos 3x)^{\frac{2}{x^2}}$

40. $\lim\limits_{x\to\pi/2}\left(tg\dfrac{x}{2}\right)^{tg3x}$ **41.** $\lim\limits_{x\to2}(5-2x)^{\frac{1}{4-x^2}}$ **42.** $\lim\limits_{x\to0}(\cos 2x)^{ctg^2 2x}$

43. $\lim\limits_{x\to\pi}\dfrac{\sin^2 x}{1+\cos^3 x}$ **44.** $\lim\limits_{x\to\pi}\dfrac{\sqrt{1-tgx}-\sqrt{1+tgx}}{\sin 2x}$ **45.** $\lim\limits_{x\to0}\left(1+tg^2\sqrt{x}\right)^{\frac{1}{2x}}$

46. $\lim\limits_{x\to\pm\infty}\dfrac{\ln\left(1+e^x\right)}{x}$ **47.** $\lim\limits_{x\to\pm0}\dfrac{|\sin x|}{x}$ **48.** $\lim\limits_{x\to1\pm0}\dfrac{x-1}{|x-1|}$

Individual Tasks 1.2

1-5. Find the limit if it exists.

I.	**II.**
1. $\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}$	*1.* $\lim\limits_{x\to0}\dfrac{1-\cos 6x}{x\sin 3x}$
2. $\lim\limits_{x\to1}\dfrac{\sin(2(x-1))}{x^2-7x+6}$	*2.* $\lim\limits_{x\to-1}\dfrac{\sin(3x+3)}{x^2-4x-5}$
3. $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{2x-1}\right)^{3x+1}$	*3.* $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$
4. $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{3x+4}\right)^{x^2}$	*4.* $\lim\limits_{x\to-\infty}\left(\dfrac{2x+1}{4x-3}\right)^{-2x}$
5. $\lim\limits_{x\to0}(\cos x)^{\frac{1}{x^2}}$	*5.* $\lim\limits_{x\to\pi/4}(tgx)^{tg2x}$

1.3 Comparison of infinitesimal functions

Let $\alpha(x)$, $\beta(x)$ are two infinitesimal functions at the point x_0 and the following limit can be evaluated $\lim\limits_{x\to x_0}\dfrac{\alpha(x)}{\beta(x)}=A$.

1. If $A \neq \infty$ and $A \neq 0$, then $\alpha(x)$ and $\beta(x)$ are called *infinitesimal functions of the same order* at the point x_0.

2. If $A = 1$, then $\alpha(x)$ and $\beta(x)$ are called *equivalent infinitesimal functions* at the point x_0 and denoted by $\alpha(x) \sim \beta(x)$.

3. If $A = 0$, then $\alpha(x)$ is called the *infinitesimal function of a higher order of smallness than* $\beta(x)$ at the point x_0 and denoted by $\alpha(x) = o(\beta(x))$.

4. If $A = \infty$, then $\alpha(x)$ is called the *infinitesimal function of a lower order of smallness than* $\beta(x)$ at the point x_0 and denoted by $\beta(x) = o(\alpha(x))$.

5. If limit $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)}$ does not exit then $\alpha(x)$ and $\beta(x)$ are called *incomparable infinitesimal functions*.

Theorem Let $\alpha(x) \sim \alpha_1(x)$ and $\beta(x) \sim \beta_1(x)$ at the point x_0.

$$\text{If } \lim_{x \to x_0} \frac{\alpha_1(x)}{\beta_1(x)} = A, \text{ then } \lim_{x \to x_0} \frac{\alpha(x)}{\beta(x)} = A.$$

Let $\alpha(x)$ is be the infinitesimal function at the point x_0 and the following table of Equivalent infinitesimal functions can be used in solving problems.

Table of Equivalent infinitesimal functions

$\sin\alpha(x) \sim \alpha(x)$	$tg\alpha(x) \sim \alpha(x)$
$\arcsin\alpha(x) \sim \alpha(x)$	$arctg\alpha(x) \sim \alpha(x)$
$(1-\cos\alpha(x)) \sim \dfrac{\alpha^2(x)}{2}$	$\left(a^{\alpha(x)} - 1\right) \sim \alpha(x)\ln a$
$\left(e^{\alpha(x)} - 1\right) \sim \alpha(x)$	$\log_a(1+\alpha(x)) \sim \dfrac{\alpha(x)}{\ln a}$
$\ln(1+\alpha(x)) \sim \alpha(x)$	$(1+\alpha(x))^k - 1 \sim k \cdot \alpha(x)$

Example 1 Evaluate the limit if it exists.

(a) $\lim\limits_{x \to 0} \dfrac{2x\sin 3x}{1-\cos x}$ *(b)* $\lim\limits_{x \to 0} \dfrac{\ln\cos x}{x^2}$

Solution

(a) If $x \to 0$, then the basic uncertainty $(0/0)$ will be obtained. To uncover obtained uncertainty the algebraic simplifications and the table of Equivalent infinitesimal functions can be used:

$$\lim_{x \to 0} \frac{2x\sin 3x}{1-\cos x} = \left(\frac{0}{0}\right) = \begin{bmatrix} 1-\cos x \sim \dfrac{x^2}{2} \\ \sin 3x \sim 3x \end{bmatrix} = \lim_{x \to 0} \frac{2x \cdot 3x}{\dfrac{x^2}{2}} = 12.$$

(b) Like in the previous example, the algebraic simplifications and the table of Equivalent infinitesimal functions can be used:

$$\lim_{x\to 0}\frac{\ln\cos x}{x^2}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\ln\left(1+(\cos x-1)\right)}{x^2}=\begin{bmatrix}\ln(1+t)\sim t\\ \lim\limits_{x\to 0}(\cos x-1)=0\\ t=\cos x-1\end{bmatrix}=$$

$$=\lim_{x\to 0}\frac{\cos x-1}{x^2}=\left[1-\cos x\sim\frac{x^2}{2}\right]=\lim_{x\to 0}\frac{-x^2/2}{x^2}=-\frac{1}{2}.$$

Exercise Set 1.3

In exercises 1 to 8 compare infinitesimal functions at the given point.

1. $\alpha(x)=\dfrac{3x^4-4}{x+1},\ \beta(x)=x^3,\ x_0=0$

2. $f(x)=1-\cos x,\ \varphi(x)=3x^2,\ x_0=0$

3. $\alpha(x)=\sqrt[3]{x^4+2x^3},\ \beta(x)=\ln(1+x),\ x_0=0$

4. $\alpha(x)=\dfrac{x+1}{x^2+1},\ \beta(x)=\dfrac{1}{x},\ x_0=\infty$

5. $\alpha(x)=1-\cos^3 x,\ \beta(x)=\sin^2 x,\ x_0=0$

6. $\alpha(x)=\dfrac{arctgx}{x^2+1},\ \beta(x)=\dfrac{1}{x^2},\ x_0=\infty$

7. $\alpha(x)=1+\sin^3 x,\ \beta(x)=\cos^2 x,\ x_0=\pi/2$

8. $f(x)=tgx,\ \varphi(x)=\arcsin x,\ x_0=0$

In exercises 9 to 34 find the limit if it exists.

9. $\lim\limits_{x\to 2}\dfrac{\sin\left(3(x-2)\right)}{x^2-3x+2}$

10. $\lim\limits_{x\to 0}\dfrac{x\sin 6x}{(arctg2x)^2}$

11. $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin 5x}{2x}$

12. $\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{\sin 10x}$

13. $\lim\limits_{x\to 0}\dfrac{e^{\sin 2x}-1}{x^2+4x}$

14. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x^2)}{\sin^2 7x}$

15. $\lim\limits_{x\to e}\dfrac{\ln x^3-3}{x-e}$

16. $\lim\limits_{x\to 3}\dfrac{\ln(x^2-5x+7)}{x-3}$

17. $\lim\limits_{x\to 0}(\cos x)^{1/\sin^2 x}$

18. $\lim\limits_{x\to 0}\dfrac{\arcsin 8x}{\ln(1+4x)}$

19. $\lim\limits_{x\to 0}\dfrac{tg3x}{tg8x}$

20. $\lim\limits_{x\to 0}\dfrac{tg^3 4x}{\sin^3 10x}$

21. $\lim\limits_{x\to 0}\dfrac{\sqrt[4]{1+2x}-1}{5x}$

22. $\lim\limits_{x\to 0}\dfrac{\dfrac{1}{\sqrt[5]{1-3x}}-1}{2x}$

23. $\lim\limits_{x\to 0}\dfrac{\ln(1+5x)}{x}$

24. $\lim\limits_{x\to -1}\dfrac{\sin 3(x+1)}{x^2+4x-5}$

25. $\lim\limits_{x\to 2}\dfrac{tg(x^2-3x+2)}{x^2-4}$

26. $\lim\limits_{x\to 0}\dfrac{\ln(1-2x)}{\sin\pi(x+4)}$

27. $\lim\limits_{x\to\pi}\left(ctg\dfrac{x}{4}\right)^{1/\cos\frac{x}{2}}$

28. $\lim\limits_{x\to 0}\dfrac{2^{3x}-3^{2x}}{x+\arcsin x^3}$

29. $\lim\limits_{x\to 1}\dfrac{e^x-e}{\ln x}$

30. $\lim\limits_{x\to 0}\dfrac{1-x^2}{\sin\pi x}$

31. $\lim\limits_{x\to 0}\dfrac{3^{x+1}-3}{\ln\left(1+x\sqrt{1+xe^x}\right)}$

32. $\lim\limits_{x\to 1}\dfrac{arctg\left(x^2-2x\right)}{\sin\pi x}$

33. $\lim\limits_{x\to 0}\dfrac{\ln\left(1+\sqrt{x^3}\right)}{e^{x^2}-1}$ **34.** $\lim\limits_{x\to 0}\dfrac{2\left(e^{\pi x}-1\right)}{3\left(\sqrt[3]{1+x}-1\right)}$

Individual Tasks 1.3

1. Compare infinitesimal functions at the given point.
2-5. Find the limit if it exists.

I.	**II.**
1. $y=\dfrac{1-x}{1+x},\ y=1-\sqrt[3]{x},\ x_0=1$	*1.* $y=\dfrac{2x}{1+x},\ y=x,\ x_0=0$
2. $\lim\limits_{x\to 0}\dfrac{e^{\sin 7x}-1}{x^2+3x}$	*2.* $\lim\limits_{x\to 0}\dfrac{\ln(1+7x)}{\sin 7x}$
3. $\lim\limits_{x\to 0}\dfrac{\sqrt[5]{1+x}-1}{x}$	*3.* $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\ln tgx}{\cos 2x}$
4. $\lim\limits_{x\to 0}\dfrac{\dfrac{1}{\sqrt[3]{1+x}}-1}{x}$	*4.* $\lim\limits_{x\to 1}\dfrac{\cos\dfrac{\pi x}{2}}{1-\sqrt{x}}$
5. $\lim\limits_{x\to 2\pi}\dfrac{\left(x-2\pi\right)^2}{tg\left(\cos x-1\right)}$	*5.* $\lim\limits_{x\to \pi}\dfrac{tg\left(3^{\pi/x}-3\right)}{3^{\cos\frac{3x}{2}}-1}$

1.4 Continuity. Asymptotes

Definition *(Continuity at the point* x_0***)*** Assume that $f(x)$ is defined in some open interval $(a;b)$ that contains the point x_0. Then function $f(x)$ is ***continuous at the point*** x_0, if $\lim\limits_{x\to x_0} f\left(x\right)=f\left(x_0\right)$.

It means that function $f(x)$ satisfies the following conditions:

$$f\left(x_0\right)=\lim_{x\to x_0} f\left(x\right)=\lim_{x\to x_0-0} f\left(x\right)=\lim_{x\to x_0+0} f\left(x\right).$$

Definition *(Continuous function)* Let $f(x)$ be a function which domain is the x axis or is made up of open intervals. Then $f(x)$ is a ***continuous function*** if it is continuous at each point x_0 in its domain.

A function obtained by the sum, difference, product, and composition of continuous functions is also continuous. The following theorem can be proved.

Theorem The *basic elementary functions* are continuous at every point in their domains.

If $f(x)$ is defined near x_0 (in other words, $f(x)$ is defined on an open interval containing x_0, except perhaps at x_0), we say that it is ***discontinuous at*** x_0 (or has a ***discontinuity at*** x_0) if $f(x)$ does not satisfy the conditions of the previous definition.

Geometrically, you can think of a function that is continuous at every number within in an interval as the function which graph has no break in it. The graph can be drawn without removing your pen from the paper.

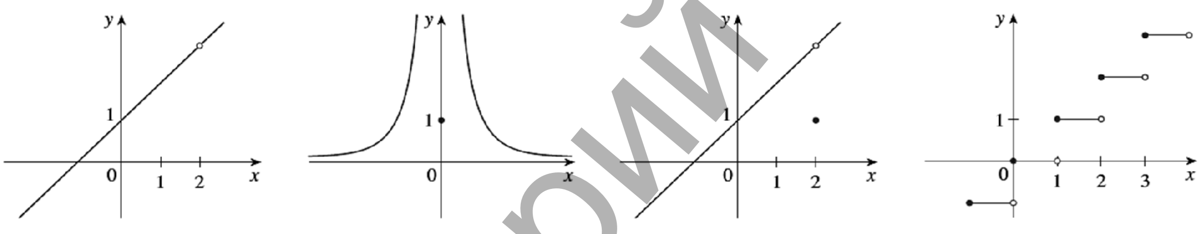
Let $f(x)$ has discontinuity at the point x_0.

1. If $\lim\limits_{x\to x_0-0} f(x) = \lim\limits_{x\to x_0+0} f(x) \neq f(x_0)$, then x_0 is called a point of ***removable discontinuity***.

2. If $\lim\limits_{x\to x_0-0} f(x) = A_1$, $\lim\limits_{x\to x_0+0} f(x) = A_2$ and $A_1 \neq A_2$, then x_0 is called a point of ***jump discontinuity***.

3. If either $\lim\limits_{x\to x_0-0} f(x) = \infty$ or $\lim\limits_{x\to x_0+0} f(x) = \infty$, or at least one of these limits does not exist, then x_0 is called a point of ***infinite discontinuity***.

Figure 1.9 shows the graphs of the different types of functions. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The discontinuity illustrated in parts (a) and (c) is removable discontinuity because we could remove the discontinuity by redefining $f(x)$ at just the single number 2. The discontinuity in part (b) is infinite discontinuity. The discontinuities in part (d) are jump discontinuities because the function "jumps" from one value to another.



(a) $f(x) = \frac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

Figure 1.9

Example 1 Locate the discontinuities of the function

$$f(x) = \begin{cases} x+2, \ if \quad x < -2; \\ \dfrac{x^2-4}{2}, if \ -2 \le x < 0; \\ \sin x, \ if \quad x \ge 0. \end{cases}.$$

Solution

The function is defined on the entire numerical axis and continuous on the intervals $(-\infty; -2), (-2; 0), (0; +\infty)$, since it is represented on them by elementary functions. Let us investigate the function at points $x = -2$ and $x = 0$, passing through which the analytic formula of the function changes.

If $x = -2$, then $f(-2) = \dfrac{(-2)^2 - 4}{2} = 0$. Right-hand and left-hand limits can be calculated as follows: $\lim\limits_{x\to -2-0}(x+2) = 0$, $\lim\limits_{x\to -2+0} \dfrac{x^2-4}{2} = 0$.

According to the definition of continuity at the point x_0, the given function is continuous at the point $x=-2$ because $\lim\limits_{x\to -2-0} f(x) = \lim\limits_{x\to -2+0} f(x) = f(-2) = 0$.

If $x=0$, then $f(0)=\sin 0 = 0$. Right-hand and left-hand limits can be calculated as follows:

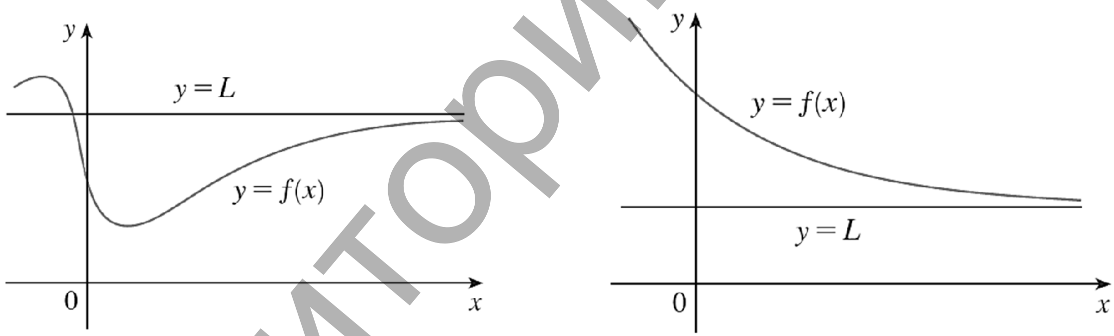
$$\lim_{x\to 0-0} \frac{x^2-4}{2} = -2, \quad \lim_{x\to 0+0} \sin x = 0.$$

According to the definition of discontinuity at the point x_0, the given function has jump discontinuity $x=0$, because $\lim\limits_{x\to 0-0} f(x) \neq \lim\limits_{x\to 0+0} f(x) = f(0)$.

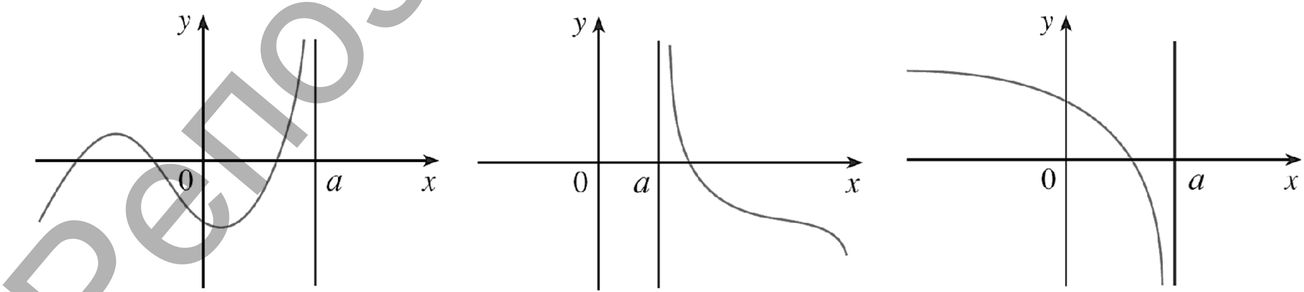
Definition The line $y=A$ is called a ***horizontal asymptote*** of the graph of $f(x)$ if $\lim\limits_{x\to +\infty} f(x) = A$, where A is a real number. An asymptote is defined similarly if $f(x)\to A$ as $x\to -\infty$.

Definition The line $x=x_0$ is called a ***vertical asymptote*** of the graph of $f(x)$ if $\lim\limits_{x\to x_0+0} f(x) = +\infty$ or $\lim\limits_{x\to x_0-0} f(x) = +\infty$. A similar definition is used if $\lim\limits_{x\to x_0-0} f(x) = -\infty$ or $\lim\limits_{x\to x_0+0} f(x) = -\infty$.

Figures 1.10 and 1.11 show some of these asymptotes.



Figures 1.10



(a) $\lim\limits_{x\to a^-} f(x) = \infty$ (b) $\lim\limits_{x\to a^+} f(x) = \infty$ (c) $\lim\limits_{x\to a^-} f(x) = -\infty$

Figures 1.11

Some curves have asymptotes that are *oblique,* that is neither horizontal nor vertical.

The line $y = kx + b$ is called a ***slant asymptote*** if $\lim\limits_{x\to\pm\infty}\left(f(x)-(kx+b)\right)=0$.

In this case, the vertical distance between the curve $y = f(x)$ and the line $y = kx + b$ approaches 0. For rational functions, slant asymptotes occur when the degree of the numerator is more than the degree of the denominator. In such case the equation of the slant asymptote can be found by a long division. The following formulas can be used to find coefficients of slant asymptote:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}; \qquad b=\lim_{x\to\pm\infty}\left(f(x)-kx\right).$$

Note (1) If at least one of the coefficients cannot be calculated or equals infinity, then the graph of $f(x)$ does not have slant asymptote. (2) A vertical asymptote of the graph of $f(x)$ is partial case of slant asymptote if $k = 0$ and $b = A$.

Example 2 Find the asymptotes of the graph of the function $y=\dfrac{2x^2}{x^2-1}$.

Solution The line $y = 2$ is a horizontal asymptote of the graph of $y=\dfrac{2x^2}{x^2-1}$ because

$$\lim_{x\to\pm\infty}\frac{2x^2}{x^2-1}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^2}=2.$$

The similar result can be obtained by using a slant asymptote:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\left(\frac{2x^2}{x^2-1}:x\right)=\lim_{x\to\pm\infty}\left(\frac{2x^2}{x^3-x}\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^3}=\lim_{x\to\pm\infty}\frac{2}{x}=0,$$

$$b=\lim_{x\to\pm\infty}\left(f(x)-kx\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^2-1}=\lim_{x\to\pm\infty}\frac{2x^2}{x^2}=2.$$

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x\to-1-0}\frac{2x^2}{x^2-1}=\left(\frac{2}{+0}\right)=+\infty,\quad \lim_{x\to-1+0}\frac{2x^2}{x^2-1}=\left(\frac{2}{-0}\right)=-\infty,$$

$$\lim_{x\to1-0}\frac{2x^2}{x^2-1}=\left(\frac{2}{-0}\right)=-\infty,\quad \lim_{x\to1+0}\frac{2x^2}{x^2-1}=\left(\frac{2}{+0}\right)=+\infty.$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes.

Exercise Set 1.4

In exercises 1 to 12 find the numbers at which $f(x)$ is discontinuous. Sketch the graph of $f(x)$.

1. $f(x)=\begin{cases}2^{-1/x^2}, & if\ x\neq 0;\\ 2, & if\ x=0.\end{cases}$ ***2.*** $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & if\ x\neq 3;\\ A, & if\ x=3.\end{cases}$ ***3.*** $f(x)=\begin{cases}x+4, & x<-1\\ x^2+2, & -1\le x<1\\ 2x, & x\ge 1.\end{cases}$

4. $f(x)=\begin{cases}-x^2, & x\le 0;\\ tgx, & 0<x\le \pi/4;\\ 4x-3, & x>\pi/4.\end{cases}$ **5.** $f(x)=\begin{cases}-2x, & x\le 0;\\ \sqrt{x}, & 0<x<4;\\ e^{x/4}, & x\ge 4.\end{cases}$ **6.** $f(x)=\begin{cases}-x, & x\le 0;\\ x^3, & 0<x\le 2;\\ x+4, & x>2.\end{cases}$

7. $f(x)=\dfrac{3x+3}{2x+4}$ **8.** $f(x)=\dfrac{2x+4}{3x+9}$ **9.** $f(x)=\dfrac{3x-2}{x^2-4}$

10. $f(x)=3^{1/(x-1)}+1$ **11.** $f(x)=2^{\frac{1}{x^2-1}}$ **12.** $f(x)=\dfrac{x^3+x}{|x|}$

In exercises 13 to 27 find the asymptotes of the graph of the functions.

13. $y=\dfrac{3x^2-x+4}{x+2}$ **14.** $y=\dfrac{x^2}{x-1}$ **15.** $y=\dfrac{(x-1)^2}{x}$

16. $y=\dfrac{x^2-6x+10}{x-3}$ **17.** $y=\dfrac{x^3}{(x-3)^2}$ **18.** $y=\dfrac{3x}{x+2}$

19. $y=2x+arctg\,x$ **20.** $y=x^2e^x$ **21.** $y=\dfrac{\ln x}{\sqrt{x}}$

22. $y=\sqrt{1+x^2}$ **23.** $y=x\cdot\ln\left(e+\dfrac{1}{x}\right)$ **24.** $y=e^{-\frac{1}{x}}$

25. $y=2x^2+\dfrac{1}{x}$ **26.** $y=\dfrac{3x-2}{5x^2}$ **27.** $y=\dfrac{x^3}{2(x+1)^2}$

Individual Tasks 1.4

1-3. Find the numbers at which $f(x)$ is discontinuous. Sketch the graph of $f(x)$.

4-5. Find the asymptotes of the graph of the functions.

I.	II.
1. $f(x)=\begin{cases}x-1, & x\le 0;\\ x^2, & 0<x<2;\\ 2x, & x\ge 2.\end{cases}$	1. $f(x)=\begin{cases}x-1, & x\le 0;\\ \sin x, & 0\le x<\pi;\\ 3, & x\ge\pi.\end{cases}$
2. $f(x)=\dfrac{3x+5}{2x-8}$	2. $f(x)=\dfrac{x+3}{3x-6}$
3. $f(x)=e^{\frac{1}{x^2-4}}$	3. $f(x)=5^{\frac{x-3}{x^2-9}}$
4. $y=\dfrac{2x^2-3x+4}{x-2}$	4. $y=\dfrac{x^2-x-7}{x+2}$
5. $y=x^2-\dfrac{1}{x}$	5. $y=\dfrac{x^3}{2(x-1)^2}$

1.5 Derivative

Let $f(x)$ is a function determined at points x_1 and x_2, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ are corresponding values of the function. Then $\Delta x = x_2 - x_1$ is called the ***increment of the argument*** and $\Delta y = f(x_2) - f(x_1)$ is called the ***increment of the function*** in the line segment $[x_1; x_2]$.

Definition ***(Derivative of a function at the*** x***)*** If

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

exists, it is called the ***derivative*** of $f(x)$ at x and the function is said to be ***differentiable*** at x.

Definition ***(Slope of a curve)*** The slope of the graph of the function $f(x)$ at $(x; f(x))$ is the derivative of $f(x)$ at x.

Definition ***(Tangent line to a curve)*** The tangent line to the graph of the function $f(x)$ at the point $P(x; y)$ is the line through P that has a slope equal to the derivative of $f(x)$ at x.

Definition (*Velocity and speed of a particle moving on a line*) The velocity at time t of an object whose position on a line at time t is given by $f(t)$ is the derivative of $f(t)$ at time t. The speed of the particle is the absolute value of the velocity.

Definition ***(Magnification of a linear projector)*** The magnification at x of a lens that projects the point x of one line onto the point $f(x)$ of another line is the derivative of $f(x)$ at x.

Some common alternative notations for the derivative are as follows:

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

The symbol $\frac{d}{dx}$ is called ***differentiation operator*** because it indicate the operation of ***differentiation***, which is the process of calculating a derivative.

Differentiation Rules

Let $u = u(x)$, $v = v(x)$ are two differentiable functions at point x_0 and $C = const$. The following rules can be proved:

1. $C' = 0$.

2. $(Cu(x))' = C \cdot u'(x)$, $\left(\frac{u(x)}{C}\right)' = \left(\frac{1}{C} \cdot u(x)\right)' = \frac{u'(x)}{C}$.

3. $(u(x) \pm v(x))' = u'(x) \pm v'(x)$.

4. $(u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$.

5. $\left(\dfrac{u(x)}{v(x)}\right)' = \dfrac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$.

6. (Chain Rule) If $y = f(u(x))$ is a differentiable function of u and u is a differentiable function of x, then $y = f(u(x))$ is a differentiable function of x and $y'(x) = f'(u) \cdot u'(x)$, where x is called the ***basic argument***, u is called the ***temporary argument***.

Using the definition of the derivative, the limit and the rules of differentiation, the table of derivatives of elementary functions can be obtained.

Table of derivatives of elementary functions

1. $\left(x^{\alpha}\right)' = \alpha \cdot x^{\alpha-1}, \alpha \in R$	2. $\left(x\right)' = 1$	3. $\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$
4. $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$	5. $\left(a^x\right)' = a^x \cdot \ln a$	6. $\left(e^x\right)' = e^x$
7. $\left(\log_a x\right)' = \dfrac{1}{x \ln a}$	8. $\left(\ln x\right)' = \dfrac{1}{x}$	9. $\left(\sin x\right)' = \cos x$
10. $\left(\cos x\right)' = -\sin x$	11. $\left(tgx\right)' = \dfrac{1}{\cos^2 x}$	12. $\left(ctgx\right)' = -\dfrac{1}{\sin^2 x}$
13. $\left(\arcsin x\right)' = \dfrac{1}{\sqrt{1-x^2}}$	14. $\left(\arccos x\right)' = -\dfrac{1}{\sqrt{1-x^2}}$	15. $\left(arctgx\right)' = \dfrac{1}{1+x^2}$
16. $\left(arcctgx\right)' = -\dfrac{1}{1+x^2}$	17. $\left(shx\right)' = chx$	18. $\left(chx\right)' = shx$
19. $\left(thx\right)' = \dfrac{1}{ch^2 x}$	20. $\left(cthx\right)' = -\dfrac{1}{sh^2 x}$	

Note If x is the temporary argument, then each of derivatives must be multiplied by the derivative of the corresponding temporary argument.

Example 1 Differentiate

(a) $y = 10^{3x-5}$ *(b)* $y = \cos^3(8 - 5x^2)$ *(c)* $y = e^{3x} \cdot \sqrt{7x^2 + 3}$ *(d)* $y = \dfrac{x + \ln(3x)}{tg2x}$

Solution

(a) The given function can be represented as follows $y = 10^u$, $u = 3x - 5$. Then the derivative of the function with respect variable x equals:

$$y' = (10^u)'_u \cdot u' = (10^u)'_u \cdot (3x-5)'_x = 10^u \ln 10 \cdot 3 = 10^{3x-5} \ln 10 \cdot 3 = 3 \ln 10 \cdot 10^{3x-5}.$$

(b) Like in the previous example, the given function can be represented as follows: $y = u^3$, $u = \cos v$, $v = 8 - 5x^2$. Using the rules of differentiation and the table of derivatives of elementary functions, the following result can be obtained:

$$y' = \left(\cos^3(8-5x^2)\right)' = \left(u^3\right)'_u \cdot \left(\cos v\right)'_v \cdot \left(8-5x^2\right)'_x = 3u^2 \cdot (-\sin v)\cdot(-10x) =$$

$$= 3\cos^2(8-5x^2)\cdot\left(-\sin(8-5x^2)\right)\cdot\left(-10x\right) = 30x\cdot\cos^2(8-5x^2)\cdot\sin(8-5x^2).$$

(c) According to the Differentiation Rules and the table of derivatives of elementary functions, we have

$$y' = \left(e^{3x}\right)' \cdot \sqrt{7x^2+3} + e^{3x}\cdot\left(\sqrt{7x^2+3}\right)' =$$

$$= e^{3x}\cdot\left(3x\right)'\cdot\sqrt{7x^2+3} + e^{3x}\cdot\frac{1}{2\sqrt{7x^2+3}}\cdot\left(7x^2+3\right)' =$$

$$= e^{3x}\cdot 3\cdot\sqrt{7x^2+3} + e^{3x}\cdot\frac{1}{2\sqrt{7x^2+3}}\cdot 14x = e^{3x}\left(3\cdot\sqrt{7x^2+3}+\frac{7x}{\sqrt{7x^2+3}}\right).$$

(d) According to the Differentiation Rules and the table of derivatives of elementary functions, we have

$$y' = \frac{\left(x+\ln(3x)\right)'\cdot tg2x - \left(x+\ln(3x)\right)\cdot\left(tg2x\right)'}{\left(tg2x\right)^2} =$$

$$= \frac{\left(1+\frac{1}{3x}\cdot\left(3x\right)'\right)\cdot tg2x - \left(x+\ln(3x)\right)\cdot\frac{1}{\cos^2(2x)}\cdot\left(2x\right)'}{tg^2 2x} =$$

$$= \frac{\left(1+\frac{1}{3x}\cdot 3\right)\cdot tg2x - \left(x+\ln(3x)\right)\cdot\frac{1}{\cos^2(2x)}\cdot 2}{tg^2 2x} = \frac{\left(1+\frac{1}{x}\right)\cdot tg2x - \frac{2\left(x+\ln(3x)\right)}{\cos^2(2x)}}{tg^2 2x}.$$

Exercise Set 1.5

In exercises 1 to 12 differentiate using the definition.

1. $y = 3x-5$ ***2.*** $y = x^2-3$ ***3.*** $y = x^2-3x$ ***4.*** $y = x^3+2x-3$

5. $y = \frac{1}{x}$ ***6.*** $y = \frac{2x}{3x+1}$ ***7.*** $y = \ln(5x+6)$ ***8.*** $y = \ln(2x-3)$

9. $y = e^{3x-5}$ ***10.*** $y = 2^{5x}$ ***11.*** $y = \sin 3x$ ***12.*** $y = \cos 5x$

In exercises 13 to 72 differentiate using the Differentiation Rules and the table of derivatives of elementary functions.

13. $y = 5x^4 - 3\sqrt[7]{x^3} + \frac{7}{x^5}$ ***14.*** $y = 5\cdot 2^x - 4tgx$ ***15.*** $y = x^3\sin x$

16. $y = 2x^5 - \frac{4}{x^3} + \frac{1}{\sqrt{x}}$ ***17.*** $y = \frac{x^4+1}{x^4-1}$ ***18.*** $y = \frac{2x^2-4x+5}{3x}$

19. $y = x^2 + \frac{1}{x^2} - 2^x + 2x$ ***20.*** $y = \frac{1-\cos x}{x^2}$ ***21.*** $y = x\cdot chx + \frac{1}{x}$

22. $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}-tg\sqrt{2}$ **23.** $y=\left(\sqrt{x}+1\right)\cdot\arcsin x$ **24.** $y=\dfrac{x\cdot ctgx}{\arccos x}$

25. $y=\log_3 x-\dfrac{e^x}{arctgx}$ **26.** $y=\left(x+1\right)^{100}$ **27.** $y=\sqrt{tgx}$

28. $y=\arcsin\sqrt{x}$ **29.** $y=\ln\cos x$ **30.** $y=e^{ctgx}$

31. $y=\sin 3x+th^3x$ **32.** $y=x^3\sin 3x$ **33.** $y=\dfrac{e^x}{ctg4x}$

34. $y=2^{-\cos^4 5x}$ **35.** $y=x\cdot cth^2 7x$ **36.** $y=2^{-\cos^4 5x}+e^{arctg\sqrt{x}}$

37. $y=(x^5+3x-1)^4$ **38.** $y=\sqrt[3]{x^4+\sin^4 x}$ **39.** $y=\cos^2(2x+2^x)$

40. $y=\arcsin^5 x\cdot\sqrt[3]{x+9}$ **41.** $y=\left(\dfrac{x^2-5x+1}{x^2-4x+10}\right)^3$ **42.** $y=\dfrac{e^{arctg\sqrt{x}}}{x^2+1}$

43. $y=\dfrac{x+e^{3x}}{x-e^{3x}}$ **44.** $y=\sqrt[3]{\left(\dfrac{x^3+1}{x^3-1}\right)^2}$ **45.** $y=\left(2^{x^4}-tg^4x\right)^3$

46. $y=\ln^5\left(x-2^{-x}\right)$ **47.** $y=\sin(tg\sqrt{x})$ **48.** $y=\sin^2 x\cdot 2^{x^2}$

49. $y=2^{\frac{x}{\ln x}}$ **50.** $y=arctg\sqrt{1+x^2}$ **51.** $y=e^{-\sqrt{x^2+2x+2}}$

52. $y=\left(2^{tg\,3x}+tg\,3x\right)^2$ **53.** $y=3^{tg^3 5x}$ **54.** $y=\sin^3 2x\cdot\cos 8x^5$

55. $y=arctg^2 5x\cdot\ln(x-4)$ **56.** $y=tg^4 3x\cdot\arcsin 2x^3$ **57.** $y=(x-3)^4\cdot\arccos 5x^3$

58. $y=\dfrac{e^{\arccos^3 x}}{\sqrt{x+5}}$ **59.** $y=sh^3x^2$ **60.** $y=\dfrac{arcctg^4 5x}{sh\sqrt{x}}$

61. $y=\dfrac{\ln(x^3-1)}{\ln(2x-3)}$ **62.** $y=1+\sqrt{x+\sqrt{x+\sqrt{x}}}$ **63.** $y=\dfrac{\sqrt{\cos 3x^2}}{x^3+4x+1}$

64. $y=\ln^5(ctg6x+\sin^3 x)$ **65.** $y=\dfrac{arctg^3 2x}{ch\left(1/x\right)}$ **66.** $y=\ln^2(x+\sqrt[4]{x-3})$

67. $y=(1-x-x^2)e^{\frac{x-1}{2}}$ **68.** $y=e^{2x}+e^{-x^2}$ **69.** $y=ctg\sqrt{\dfrac{x}{1+x^3}}$

70. $y=\ln\dfrac{\sqrt[3]{x^2-1}}{x^4}$ **71.** $y=arctg\sqrt{\dfrac{1-x}{1+x}}$ **72.** $y=\dfrac{10^{\sqrt{x}}}{\arcsin 2x}$

In exercises 73 to 76 find the slope of the given curve $y=f\left(x\right)$ at the point $x=x_0$.

73. $f(x)=x\cdot arctgx,\ x_0=0$ **74.** $f(x)=x^4+x^3-17^5,\ x_0=1$

75. $f(x)=\dfrac{lnx}{x},\ x_0=e$ **76.** $f\left(x\right)=\sqrt{3x^3-x^2-5},\ x_0=2$

In exercises 77 to 80 find the angle between given curves at the intersection point.

77. $f_1(x)=\dfrac{1}{x},\ \ f_2(x)=x^2$

78. $f_1(x)=\dfrac{1}{x},\ \ f_2(x)=x^3$

79. $f_1(x)=3x^2,\ \ f_2(x)=1-x^2$

80. $f_1(x)=\dfrac{2}{x},\ \ f_2(x)=1+x^2$

Individual Tasks 1.5

1. Differentiate using the definition.

2-7. Differentiate using the Differentiation Rules and the table of derivatives of elementary functions.

8. Find the slope of the given curve $y=f\left(x\right)$ at the point $x=x_0$.

I.	**II.**
1. $y=\sin(4x-1)$	*1.* $y=\ln(5x+3)$
2. $y=\sqrt{x}-\dfrac{3}{\sqrt{x}}-tg\sqrt{3}$	*2.* $y=3\sqrt{x}-\dfrac{1}{\sqrt{x}}-\pi$
3. $y=\dfrac{2+3\cos x}{4x^2}$	*3.* $y=x^4\cos 3x$
4. $y=5^{-\cos^3 2x}$	*4.* $y=\cos^2\left(2/x+2^x\right)$
5. $y=ctg^3 4x\cdot\arcsin 3x^2$	*5.* $y=(2x-3)^5\cdot\arccos 3x^4$
6. $y=\dfrac{e^{\arcsin 2x}}{\sqrt{3x^2+5}}$	*6.* $y=\dfrac{arctg^3(5x+3)}{ch\sqrt{x}}$
7. $y=\ln\dfrac{\sqrt[4]{5x^2+3}}{(x^4-1)}$	*7.* $y=arcctg\sqrt{\dfrac{1-3x}{1+3x}}$
8. $f\left(x\right)=\sqrt[5]{(2x^2-4x^3)^4},\ \ x_0=1$	*8.* $f(x)=\dfrac{\sqrt{5-x^2}}{5+x},\ \ x_0=1$

1.6 Implicit Differentiation. Logarithmic Differentiation. Calculus with Parametric Curves

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable. Some functions, however, are defined implicitly by a relation between x and y. We do not need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of ***implicit differentiation***. It consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Example 1 Find y'_x, if $x^3+\ln y-x^2\cdot e^y=0$.

Solution Differentiate both sides of the equation $x^3+\ln y-x^2\cdot e^y=0$. Remember that y is a function of x and using the Chain Rule, we have

$$3x^2 + \frac{1}{y} \cdot y' - \left(2x \cdot e^y + x^2 e^y \cdot y'\right) = 0.$$

Now we solve this equation for y':

$$y'\left(\frac{1}{y} - x^2 e^y\right) = 2xe^y - 3x^2 \Rightarrow y' = \frac{(2xe^y - 3x^2) \cdot y}{1 - x^2 y e^y}.$$

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called ***logarithmic differentiation***. This method includes the implicit differentiation and the Laws of Logarithms.

Example 2 Differentiate $y = (tgx)^{\cos x}$.

Solution We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \ln(tgx)^{\cos x} \Rightarrow \ln y = \cos x \cdot \ln tgx.$$

Differentiating implicitly with respect to x gives

$$\frac{y'}{y} = (\cos x)' \cdot \ln tgx + \cos x \cdot (\ln tgx)' = (-\sin x) \cdot \ln tgx + \cos x \cdot \frac{1}{tgx} \cdot \frac{1}{\cos^2 x}.$$

Solving for y', we get $y' = y\left(-\sin x \cdot \ln tgx + \dfrac{1}{\sin x}\right)$.

Because we have an explicit expression for y, we can substitute and write

$$y' = (tgx)^{\cos x}\left(-\sin x \cdot \ln tgx + \frac{1}{\sin x}\right).$$

Some curves defined by parametric equations $x = x(t)$ and $y = y(t)$ can also be expressed, by eliminating the parameter, in the form of $y = F(x)$.

If we substitute $x = x(t)$ and $y = y(t)$ in the equation $y = F(x)$, we get $y(t) = F(x(t))$ and so, if $x = x(t)$, $y = y(t)$ and $y = F(x)$ are differentiable, the Chain Rule gives

$$y_t'(t) = F_x'(x) \cdot x_t'(t) \Rightarrow y_x' = \frac{y_t'(t)}{x_t'(t)} = \frac{dy}{dx}.$$

Example 3 Differentiate $\begin{cases} x = \dfrac{3t}{t+1}; \\ y = t^2 + 2t. \end{cases}$.

Solution Using the last formula, we get

$$x'(t) = \left(\frac{3t}{t+1}\right)' = \frac{(3t)' \cdot (t+1) - 3t \cdot (t+1)'}{(t+1)^2} = \frac{3(t+1) - 3t}{(t+1)^2} = \frac{3}{(t+1)^2};$$

$$y'(t) = \left(t^2 + 2t\right)' = 2t + 2 = 2(t+1);$$

$$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{2(t+1)\cdot(t+1)^2}{3}=\frac{2}{3}(t+1)^3.$$

Exercise Set 1.6

In exercises 1 to 45 differentiate.

1. $y=\dfrac{(x-3)^2(2x-1)}{(x+1)^3}$

2. $y=(\cos x-1)^{x^2}$

3. $y=\left(\dfrac{x}{x+1}\right)^x$

4. $y=\dfrac{(x-3)^5(x+2)^3}{\sqrt{(x-1)^3}}$

5. $y=(\sin 3x)^{\cos 5x}$

6. $y=(x^3+1)^{tg\,2x}$

7. $y=\dfrac{(3x-2)^3\sqrt{(7x+1)^5}}{(6x-4)^2}$

8. $y=(\cos(x+2))^{\ln x}$

9. $y=(th\,5x)^{\arcsin(x+1)}$

10. $y=\dfrac{\sqrt{x+7}(x-3)^4}{(x+2)^5}$

11. $y=\left(\log_2(x+4)\right)^{ctg\,7x}$

12. $y=(sh\,3x)^{arctg(x+2)}$

13. $y=\dfrac{(x-3)^2\sqrt{x+4}}{(x+2)^7}$

14. $y=(\cos(2x-5))^{arctg\,5x}$

15. $y=(\sin(7x+4))^{arcctg\,x}$

16. $y=\dfrac{(2x-7)^{10}\sqrt{3x-1}}{(x^2+2x+3)^5}$

17. $y=(tg\,3x)^{x^4}$

18. $y=(\sin x)^{x^3}$

19. $y^2=x+\ln\left(y/x\right)$

20. $xy^2-y^3=4x-5$

21. $x^2y^2+x=5y$

22. $x\sin y+y\sin x=4$

23. $x^3+y^3=5x$

24. $\sqrt{x}+\sqrt{y}=\sqrt{7}$

25. $y^2=\dfrac{x-y}{x+y}$

26. $\sin^2(3x+y^2)=5$

27. $ctg^2(x+y)=5x$

28. $y^2+x^2-\sin(x^2y^2)=5$

29. $2^x+2^y=2^{x+y}$

30. $e^{x^2y^2}-x^4+y^4=5$

31. $\ln y+\dfrac{x}{y}=x+y;$

32. $(x+y)^3=27(x-y)$

33. $e^y=e-xy$

34. $\begin{cases}x=t^3-t;\\ y=t^2+t.\end{cases}$

35. $\begin{cases}x=\sqrt{1-t^2};\\ y=t^{-2}.\end{cases}$

36. $\begin{cases}x=3\cos^2 t;\\ y=4\sin^2 t.\end{cases}$

37. $\begin{cases}x=\dfrac{\ln t}{t};\\ y=t^2\ln t.\end{cases}$

38. $\begin{cases}x=\arccos t;\\ y=\sqrt{1-t^2}.\end{cases}$

39. $\begin{cases}x=\dfrac{1}{t+1};\\ y=\dfrac{t}{t+1}.\end{cases}$

40. $\begin{cases}x=e^t\cos t;\\ y=e^t\sin t.\end{cases}$

41. $\begin{cases}x=2(t-\sin t);\\ y=2(1-\cos t).\end{cases}$

42. $\begin{cases}x=t^3+3t+1;\\ y=3t^5+5t^3.\end{cases}$

43. $\begin{cases}x=5\sin^3 t;\\ y=3\cos^3 t.\end{cases}$

44. $\begin{cases}x=e^{-3t};\\ y=e^{8t}.\end{cases}$

45. $\begin{cases}x=t-\sin t;\\ y=1-\cos t.\end{cases}$

In exercises 46 to 49 find the angle between the given curves at the intersection point.

46. $y=\dfrac{8}{x}$, $x^2-y^2=12$

47. $y^2=2x$, $x^2+y^2=8$

48. $y=x^3+3x^2+2x$, $y=-5x-5$

49. $y=\sin x$, $y=\cos x$, $0\le x\le \pi$.

Individual Tasks 1.6

1-6. Differentiate.

I.	II.
1. $y=\dfrac{(2x-3)^4(3x+2)^3}{\sqrt{(x-1)^5}}$	***1.*** $y=\dfrac{(x-2)^5\sqrt{(4x+1)^3}}{(3x-4)^6}$
2. $y=(\sin(2x-3))^{\ln 2x}$	***2.*** $y=\left(\ln(3x+4)\right)^{ctg\,2x}$
3. $xy^2-y^4=x^3-5y$	***3.*** $2xy^3+y^2=x-2$
4. $tg^2(x+2y^3)=2$	***4.*** $\cos^4(x-y^2)=\pi^{-1}$
5. $\begin{cases} x=2t^3-t+5; \\ y=4t^5+t^2. \end{cases}$	***5.*** $\begin{cases} x=3e^t\cos 2t; \\ y=5e^t\sin 2t. \end{cases}$
6. $\begin{cases} x=e^{-2t}\cos 3t; \\ y=e^{-2t}\sin 3t. \end{cases}$	***6.*** $\begin{cases} x=4t^3+1; \\ y=3t^3-5t^2-2. \end{cases}$

1.7 Differentials and Linear Approximations

Definition The ***differential*** of a function $y=f(x)$ is the principal part of its increment, linear with respect to the increment of the argument x. The differential of the argument is the increment of this argument $dx=\Delta x$.

The differential dy is defined in terms of dx by the equation $dy=f'(x)dx=y'dx$.

Basic properties of the differential

1. $dC=0,\ \ C=const$	*2.* $d(Cu(x))=Cdu(x)$
3. $d\left(u(x)\pm v(x)\right)=du(x)\pm dv(x)$	*4.* $d\left(u(x)\cdot v(x)\right)=v(x)du(x)+u(x)dv(x)$
5. $d\left(\dfrac{u}{v}\right)=\dfrac{vdu-udv}{v^2},\ \ v=v(x)\neq 0$	*6.* $d\left(f(u)\right)=f'(u)du$, where $u=u(x)$

The geometric meaning of differentials is shown in Figure 1.12. Let $P\left(x;f(x)\right)$ and $Q\left(x+\Delta x;f(x+\Delta x)\right)$ be the points on the graph of $y=f(x)$ and let $dx=\Delta x$. The corresponding change in y is $\Delta y=f(x+\Delta x)-f(x)$.

The slop RP of the tangent line is the derivative $f'(x)$. Thus the directed distance from S to R be $dy=f'(x)dx$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y=f(x)$ rises or falls when x changes by an amount dx.

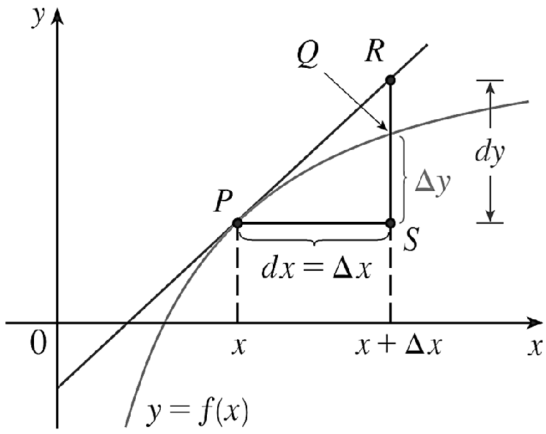


Figure 1.12

Notice that the approximation $\Delta y \approx dy = f'(x)\Delta x$ becomes better as Δx becomes smaller. Notice also, that dy was easier to compute than Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation at the point x_0 can be written as

$$y(x_0+\Delta x)-y(x_0)\approx f'(x_0)\cdot\Delta x \quad \text{or} \quad y(x_0+\Delta x)\approx y(x_0)+y'(x_0)\Delta x.$$

Example 1 Compare the values of Δy and dy at the point $M_0(1;5)$ if $y=2x^3+5x^2-3x+1$.

Solution We have

$$\Delta y = f(x+\Delta x)-f(x)=$$

$$=2(x+\Delta x)^3+5(x+\Delta x)^2-3(x+\Delta x)+1-(2x^3+5x^2-3x+1).$$

According to the initial condition $x=1$, the following result can be obtained

$$\Delta y(1)=f(1+\Delta x)-f(1)=2(1+\Delta x)^3+5(1+\Delta x)^2-3(1+\Delta x)+1-(2+5-3+1)=$$

$$=2(1+3\Delta x+3\Delta x^2+\Delta x^3)+5(1+2\Delta x+\Delta x^2)-3-3\Delta x-4=$$

$$=(2+5-3-4)+\Delta x(6+10-3)+\Delta x^2(6+5)+2\Delta x^3=13\Delta x+11\Delta x^2+2\Delta x^3.$$

$dy(1)=y'(1)dx$; $y'=6x^2+10x-3$; $y'(1)=6+10-3=13$; $dy(1)=13\Delta x$.

If $\Delta x=1$, then $\Delta y=13+11+2=26$ and $dy=13$.

If $\Delta x=0.1$, then $\Delta y=1{,}3+0{,}11+0{,}002=1{,}412$ and $dy=1{,}3$.

The obtained result shows that if Δx becomes smaller, the approximation $\Delta y\approx dy$ becomes better.

Example 2 Find the differential of the function $y=\dfrac{x}{2}\sqrt{49-x^2}+\dfrac{49}{2}\arcsin\dfrac{x}{7}$.

Solution The derivative of $y=f(x)$ is

$$y' = \frac{1}{2}\sqrt{49 - x^2} + \frac{x}{2} \cdot \frac{-2x}{2\sqrt{49 - x^2}} + \frac{49}{2} \cdot \frac{1}{7\sqrt{1 - \frac{x^2}{49}}} = \sqrt{49 - x^2}.$$

In particular, we have $dy = \sqrt{49 - x^2}\,dx$.

Example 3 Use a differential to estimate $\sqrt{67}$.

Solution The object is to estimate the value of the square root function $f(x) = \sqrt{x}$ at the input $x = 67$. In this case, $f(64)$ is known. We have

$$f(64) = 8 \text{ and } f'(x) = \frac{1}{2\sqrt{x}},\ f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16}.$$

Since $67 = 64 + 3$, $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$, $\Delta x = 3$. Therefore,

$$\sqrt{67} \approx f(64) + dy = f(64) + f'(64) \cdot 3 = 8 + \frac{1}{16} \cdot 3 = 8{,}1875.$$

A calculator shows that to four decimal places $\sqrt{67} \approx 8{,}1854$. So the estimate obtained by the differential is not far off.

Exercise Set 1.7

In exercises 1 to 18 find the differential of the functions.

1. $y = x^4 + 4x^3 + 6x^2$ ***2.*** $y = \frac{x^2 - 1}{x^2}$ ***3.*** $y = \sqrt{x^3 + 6x^2}$

4. $y = x\,tg^3 x$ ***5.*** $y = \sqrt{arctg\,x} + \arcsin^2 x$ ***6.*** $y = \ln(x + \sqrt{4 + x^2})$

7. $y = \frac{x}{1 - x}$ ***8.*** $y = x\,arctg x - \ln\sqrt{1 + x^2}$ ***9.*** $y = \cos^3 \frac{x + 1}{x^2}$

10. $y = ctg(3x^2 + \ln 6x)$ ***11.*** $y = sh^3 4x \cdot \arccos\sqrt{x}$ ***12.*** $y = th^2\sqrt{x} \cdot arcctg 3x^2$

13. $y = 10^{tg\sqrt{x}}$ ***14.*** $y = cth^4 2x \cdot \arcsin 7x^2$ ***15.*** $(x + y)^2 \cdot (2x + y)^3 = 1$

16. $y = e^{-\frac{x}{y}}$ ***17.*** $x^2 + 2xy - y^2 = a^2$ ***18.*** $\ln\sqrt{x^2 + y^2} = arctg\frac{y}{x}$

In exercises 19 to 26 use a linear approximation (or differentials) to estimate the given number.

19. $\sqrt[4]{17}$ ***20.*** $4^{1,2}$ ***21.*** $\sqrt[3]{26{,}19}$ ***22.*** $\sin 29^\circ 30'$

23. $e^{0,2}$ ***24.*** $\ln(e^2 + 0{,}2)$ ***25.*** $\ln tg\,47^\circ 15'$ ***26.*** $\frac{2{,}9}{\sqrt{2{,}9^2 + 16}}$

Individual Tasks 1.7

1-3. Find the differential of the functions.

4-5. Use a linear approximation (or differentials) to estimate the given number.

I.	**II.**
1. $y = \arcsin^5 x \cdot \sqrt[3]{x + 9}$	*1.* $y = \left(2^{x^4} - tg^4 x\right)^3$

2. $y=\sqrt[3]{\left(\dfrac{x^3+1}{x^3-1}\right)^2}$ 3. $y^2=x+\ln(y/x)$ 4. $\arcsin 0,51$ 5. $\sqrt[4]{16,64}$	2. $y=\dfrac{e^{arctg\sqrt{x}}}{x^2+1}$ 3. $\sqrt{x}+\sqrt{y}=\sqrt{7}$ 4. $arctg\,0,98$ 5. $ctg\,60^\circ 30'$

1.8 Higher Derivative

Definition The derivative of the derivative of a function $y=f(x)$ is called the ***second derivative*** of the function. It is denoted by $y''=\left(f'(x)\right)'=\dfrac{d^2y}{dx^2}=f''(x)$.

Definition The derivative of the second derivative is called the ***third derivative*** and denoted by $y'''=\left(f''(x)\right)'$.

Definition The derivatives $y^{(n)}=\left(f^{(n-1)}(x)\right)'$ for $n\geq 2$ are called the ***higher derivatives*** of $y=f(x)$.

If some curves defined by the parametric equation $x=x(t)$, $y=y(t)$ and $y'_x=\dfrac{y'_t(t)}{x'_t(t)}$, the second and third derivatives of the function $y=f(x)$ are differentiated by the formula:

$$y'_x=\frac{y'(t)}{x'(t)},\quad y''_{xx}=\frac{(y'_x)'_t}{x'(t)},\quad y'''_{xxx}=\frac{(y''_{xx})'_t}{x'(t)},\quad \cdots .$$

The differential of the second order is defined as the differential of the differential of the first order $d^2y=d(dy)$. Differentials of higher orders are defined similarly

$$d^3y=d(d^2y),\cdots,d^ny=d(d^{n-1}y).$$

If x is an independent variable, then differentials of higher orders are evaluated by the following formulas

$$d^2y=y''(dx)^2;\quad d^3y=y'''(dx)^3;\cdots;\quad d^ny=y^{(n)}(dx)^n.$$

Example 1 Compute $y^{(n)}$ if $y=x^5-4x^3+7x^2-8$ and n is a positive integer.

Solution

$$y'=5x^4-12x^2+14x,\qquad y''=20x^3-24x+14,\qquad y'''=60x^2-24,$$

$$y^{(4)}=120x,\qquad y^{(5)}=120,\qquad y^{(6)}=y^{(7)}=\cdots=0.$$

Example 2 Compute $y^{(n)}$ if $y=\ln x$ and n is a positive integer.

Solution

$$y'=\frac{1}{x}=x^{-1},\quad y''=(-1)x^{-2},\quad y'''=(-1)(-2)x^{-3},\quad y^{(-4)}=(-1)(-2)(-3)x^{-4},\ \cdots,$$

$$y^{(n)}=(-1)(-2)(-3)\cdots(-n+1)x^{-n}=(-1)^{n-1}(n-1)!x^{-n}=\frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Example 3 Find y'' if $x^4 + y^4 = 16$.

Solution Differentiating the equation implicitly with respect to x, we get $4x^3 + 4y^3 \cdot y' = 0$. Solving for y' gives $y' = -\dfrac{x^3}{y^3}$.

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \left(-\frac{x^3}{y^3}\right)'_x = -\frac{\left(x^3\right)' \cdot y^3 - \left(y^3\right)' \cdot x^3}{\left(y^3\right)^2} = -\frac{3x^2 \cdot y^3 - 3y^2 \cdot y' \cdot x^3}{\left(y^3\right)^2}.$$

If we now substitute the last equation into this expression, we get

$$y'' = -\frac{3x^2 \cdot y^3 - 3y^2 \cdot \left(-\dfrac{x^3}{y^3}\right) \cdot x^3}{\left(y^3\right)^2} = -\frac{3x^2 \cdot y^4 + 3x^6}{y^7} = -\frac{3x^2 \cdot (y^4 + x^4)}{y^7} = -\frac{48x^2}{y^7}.$$

Example 4 Find y'' if $\begin{cases} x = \ln t, \\ y = 1/t. \end{cases}$

Solution Using the formula $y'_x = \dfrac{y'(t)}{x'(t)}$ we get

$$y'(t) = -\frac{1}{t^2}, \quad x'(t) = \frac{1}{t}, \quad y'_x = \frac{dy}{dx} = -\frac{1}{t^2} : \frac{1}{t} = -\frac{1}{t}.$$

Using formula $y''_{xx} = \dfrac{(y'_x)'_t}{x'(t)}$, we get

$$y''_{xx} = \frac{d^2y}{dx^2} = \frac{(y'_x)'_t}{x'_t} = \frac{1}{t^2} : \frac{1}{t} = \frac{1}{t}.$$

Exercise Set 1.8

In exercises 1 to 8 compute $y^{(n)}$.

1. $y = 1/x$ ***2.*** $y = 2^x$ ***3.*** $y = \cos x$ ***4.*** $y = \dfrac{1}{2x+5}$

5. $y = e^{-2x}$ ***6.*** $y = x^n \cdot \sqrt{x}$ ***7.*** $y = xe^{3x}$ ***8.*** $y = \ln(3+x)$

In exercises 9 to 20 find $y'''(x_0)$ at the given point x_0.

9. $y = \sin^2 x$, $x_0 = \dfrac{\pi}{2}$ ***10.*** $y = arctg\, x$, $x_0 = 1$ ***11.*** $y = \ln(2 + x^2)$, $x_0 = 0$

12. $y = e^x \cos x$, $x_0 = 0$ ***13.*** $y = e^x \sin 2x$, $x_0 = 0$ ***14.*** $y = e^{-x} \cos x$, $x_0 = 0$

15. $y = \sin 2x$, $x_0 = \pi$ ***16.*** $y = (2x+1)^5$, $x_0 = 1$ ***17.*** $y = \ln(1+x)$, $x_0 = 2$

18. $y = \dfrac{1}{2}x^2 e^x$, $x_0 = 0$ ***19.*** $y = \arcsin x$, $x_0 = 0$ ***20.*** $y = (5x-4)^5$, $x_0 = 2$

In exercises 21 to 32 find y''.

21. $\begin{cases} x = 2\cos^2 t; \\ y = 3\sin^2 t. \end{cases}$ **22.** $\begin{cases} x = \ln(1+t^2), \\ y = t - arctg\, t. \end{cases}$ **23.** $\begin{cases} x = 2t - t^2, \\ y = 4t - t^4. \end{cases}$

24. $\begin{cases} x = \cos(t^2+1), \\ y = \sin^2 t. \end{cases}$ **25.** $\begin{cases} x = \arccos\sqrt{t}, \\ y = \sqrt{t - t^2}. \end{cases}$ **26.** $\begin{cases} x = \sqrt{t}; \\ y = \sqrt[5]{t}. \end{cases}$

27. $\begin{cases} x = \dfrac{2t}{1+t^3}; \\ y = \dfrac{t^2}{1+t^2}. \end{cases}$ **28.** $\begin{cases} x = \sqrt{t^2-1}; \\ y = \dfrac{t+1}{\sqrt{t^2-1}}. \end{cases}$ **29.** $\begin{cases} x = 4t + 2t^2; \\ y = 5t^3 - 3t^2. \end{cases}$

30. $\begin{cases} x = \dfrac{\ln t}{t}; \\ y = t\ln t. \end{cases}$ **31.** $\begin{cases} x = e^t \cos t; \\ y = e^t \sin t. \end{cases}$ **32.** $\begin{cases} x = t^4; \\ y = \ln t. \end{cases}$

In exercises 33 to 47 find y''.

33. $y^2 = 8x$ **34.** $\dfrac{x^2}{5} + \dfrac{y^2}{7} = 1$ **35.** $y = x + arctg\, y$

36. $y^2 = 5x - 4$ **37.** $arctg\, y = 4x + 5y$ **38.** $y^2 - x = \cos y$

39. $3x + \sin y = 5y$ **40.** $tg\, y = 3x + 5y$ **41.** $xy = ctg\, y$

42. $y = e^y + 4x$ **43.** $\ln y - \dfrac{y}{x} = 7$ **44.** $y^2 + x^2 = \sin y$

45. $3y = 7 + xy^3$ **46.** $4\sin^2(x+y) = x$ **47.** $\sin y = 7x + 3y$

Individual Tasks 1.8

1-2. Compute $y^{(n)}$.

3. Find $y'''(x_0)$ at the given point x_0.

4-6. Find y''.

I.	**II.**
1. $y = \ln(5+2x)$	*1.* $y = (x-7)^{-1}$
2. $y = \cos 3x$	*2.* $y = e^{-5x}$
3. $y = x\sin x$, $x_0 = \pi/2$	*3.* $y = x^2 \ln x$, $x_0 = 1/3$
4. $\begin{cases} x = 5(t - \sin t), \\ y = 5(1 - \cos t). \end{cases}$	*4.* $\begin{cases} x = \sin 2t; \\ y = \cos^2 t. \end{cases}$
5. $2x + \cos y = 4y$	*5.* $ctg\, y = 2x - 5y$
6. $x^3 + y^2 = 2y$	*6.* $x^2 + y^2 = 3x$

1.9 L'Hospital's Rule

Let $f(x)$ and $\varphi(x)$ are two differentiable functions at the point x_0 such that $\varphi'(x) \neq 0$.

Theorem 1*(L'Hospital's Rule)* If $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} \varphi(x) = 0$ and the limit of a quotient of the derivatives $\left(\lim\limits_{x\to x_0} \dfrac{f'(x)}{\varphi'(x)}\right)$ exists, then $\lim\limits_{x\to x_0} \dfrac{f(x)}{\varphi(x)} = \lim\limits_{x\to x_0} \dfrac{f'(x)}{\varphi'(x)}$.

L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of a quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of $f(x)$ and $\varphi(x)$ before using l'Hospital's Rule. *L'Hospital's Rule is also valid for one-sided limits and for the limits at infinity or negative infinity.*

L'Hospital's Rule can be used every time when a quotient of functions satisfies the conditions of Theorem 1. For example, if $\dfrac{f'(x)}{\varphi'(x)}$ is an indeterminate of type $(0/0)$ or (∞/∞) and it satisfies the conditions of Theorem 1 then $\lim\limits_{x\to x_0} \dfrac{f'(x)}{\varphi'(x)} = \lim\limits_{x\to x_0} \dfrac{f''(x)}{\varphi''(x)}$.

To uncover the indeterminate form of the types $(\infty - \infty)$, $(0\cdot\infty)$, (1^∞), (0^∞), (∞^0) additional algebraic transformations and properties $a^b = e^{\ln a^b} = e^{b\ln a}$, $\lim\limits_{x\to x_0} e^{f(x)} = e^{\lim\limits_{x\to x_0} f(x)}$ are required.

Example 1 Find the limit if it exists.

a) $\lim\limits_{x\to 1} \dfrac{x^2 - 1 + \ln x}{e^x - e}$ *b)* $\lim\limits_{x\to\infty} \dfrac{xe^{2x}}{x + e^{4x}}$ *c)* $\lim\limits_{x\to 0}\left(x^2 \cdot \ln x\right)$

d) $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$ *e)* $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{\ln x}}$

Solution

a) Since $\lim\limits_{x\to 1}(x^2 - 1 + \ln x) = 0$, $\lim\limits_{x\to 1}(e^x - e) = 0$, we can apply l'Hospital's Rule:

$$\lim_{x\to 1} \frac{x^2 - 1 + \ln x}{e^x - e} = \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{(x^2 - 1 + \ln x)'}{(e^x - e)'} = \lim_{x\to 1} \frac{2x + \frac{1}{x}}{e^x} = \frac{3}{e}.$$

b) Since $\lim\limits_{x\to\infty} xe^{2x} = \infty$, $\lim\limits_{x\to\infty}\left(x + e^{4x}\right) = \infty$, we can apply l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{xe^{2x}}{x + e^{4x}} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{e^{2x} + 2xe^{2x}}{1 + 4e^{4x}}.$$

Since $\lim\limits_{x\to\infty}(e^{2x} + 2xe^{2x}) = \infty$, $\lim\limits_{x\to\infty}\left(1 + 4e^{4x}\right) = \infty$ the limit on the right side is also indeterminate and a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^{2x}+2xe^{2x}}{1+4e^{4x}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2e^{2x}+2e^{2x}+4xe^{2x}}{16e^{4x}}=\lim_{x\to\infty}\frac{1+x}{4e^{2x}}.$$

Since $\lim_{x\to\infty}(1+x)=\infty$, $\lim_{x\to\infty}4e^{2x}=\infty$, the limit on the right side is also indeterminate and a third application of l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{1+x}{4e^{2x}}=\left(\frac{\infty}{\infty}\right)=\frac{1}{4}\lim_{x\to\infty}\frac{1}{2e^{2x}}=0.$$

c) Substituting $x=0$ in the function, we get the indeterminate form of the type $(0\cdot\infty)$. We transform the expression under the limit sign and apply l'Hospital's Rule.

$$\lim_{x\to 0}x^2\cdot\ln x=(0\cdot\infty)=\lim_{x\to 0}\frac{\ln x}{x^{-2}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{1}{x}\cdot\frac{x^3}{-2}=\lim_{x\to 0}\frac{x^2}{-2}=0.$$

d) Substituting $x=0$ in the function, we get the indeterminate form of the type $(\infty-\infty)$. We transform the expression under the limit sign and apply l'Hospital's Rule:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=(\infty-\infty)=\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}=\left(\frac{0}{0}\right).$$

We simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x}{2e^x+xe^x}=\lim_{x\to 0}\frac{1}{2+x}=\frac{1}{2}.$$

e) First notice that as $x\to\infty$, the given limit is the indeterminate form of the type (∞^0). To uncover the indeterminate form of this type, additional algebraic transformations and properties $a^b=e^{\ln a^b}=e^{b\ln a}$, $\lim_{x\to x_0}e^{f(x)}=e^{\lim_{x\to x_0}f(x)}$ are required.

$$\lim_{x\to\infty}(1+x)^{\frac{1}{\ln x}}=(\infty^0)=\lim_{x\to\infty}e^{\frac{1}{\ln x}\ln(1+x)}=\lim_{x\to\infty}e^{\frac{\ln(1+x)}{\ln x}}=e^{\lim_{x\to\infty}\frac{\ln(1+x)}{\ln x}}.$$

Substituting $x\to\infty$ in the function, we get the indeterminate form of the type (∞/∞) and we can apply l'Hospital's Rule:

$$\lim_{x\to\infty}\frac{\ln(1+x)}{\ln x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{1/(x+1)}{1/x}=\lim_{x\to\infty}\frac{x}{1+x}=1.$$

Then

$$\lim_{x\to\infty}(1+x)^{\frac{1}{\ln x}}=e^{\lim_{x\to\infty}\frac{\ln(1+x)}{\ln x}}=e^1=e.$$

Exercise Set 1.9

In exercises 1 to 48 find the limit if it exists.

1. $\lim_{x\to 1}\frac{x^3-7x^2+4x+2}{x^3-5x+4}$ **2.** $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos 2x}$ **3.** $\lim_{x\to\infty}\frac{\pi-2arctgx}{e^{3/x}-1}$

4. $\lim_{x\to 1}x^{\frac{1}{1-x}}$ **5.** $\lim_{x\to 0}x^{\frac{3}{4+\ln x}}$ **6.** $\lim_{x\to\infty}(x+2^x)^{\frac{1}{x}}$

7. $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6 2x}$ **8.** $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-ctg^2 x\right)$ **9.** $\lim\limits_{x\to 0}\dfrac{e^{7x}-1}{tg\,3x}$

10. $\lim\limits_{x\to 0}\dfrac{tg\,x-x}{x-\sin x}$ **11.** $\lim\limits_{x\to 1}\dfrac{1-4\sin^2(\pi x/6)}{1-x^2}$ **12.** $\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^2-\sin x^2}$

13. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{tg\,x}{tg\,5x}$ **14.** $\lim\limits_{x\to\infty}\dfrac{e^x}{x^5}$ **15.** $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt[3]{x}}$

16. $\lim\limits_{x\to 0}\dfrac{\pi/x}{ctg(\pi x/2)}$ **17.** $\lim\limits_{x\to+\infty}\dfrac{\ln(x+7)}{\sqrt[7]{x-3}}$ **18.** $\lim\limits_{x\to 0}\dfrac{1-\cos 7x}{x\sin 7x}$

19. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{x}{e^x-1}\right)$ **20.** $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{x}{ctg\,x}-\dfrac{\pi}{2\cos x}\right)$ **21.** $\lim\limits_{x\to 0}\left(\dfrac{1}{x\sin x}-\dfrac{1}{x^2}\right)$

22. $\lim\limits_{x\to\infty}x\sin\dfrac{3}{x}$ **23.** $\lim\limits_{x\to 0}\dfrac{x\cos x-\sin x}{x^3}$ **24.** $\lim\limits_{x\to 0}(1-\cos x)\cdot ctg\,x$

25. $\lim\limits_{x\to 1}\ln x\cdot\ln(x-1)$ **26.** $\lim\limits_{x\to 0}(x\cdot\ln x)$ **27.** $\lim\limits_{x\to\infty}x^4e^{-x}$

28. $\lim\limits_{x\to 0}\dfrac{tg\,x-x}{2\sin x+x}$ **29.** $\lim\limits_{x\to 0}\dfrac{tg\,x-\sin x}{4x-\sin x}$ **30.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{tg\,3x}{tg\,5x}$

31. $\lim\limits_{x\to\infty}\dfrac{\ln(x+5)}{\sqrt[4]{x+3}}$ **32.** $\lim\limits_{x\to 0}\dfrac{\frac{\pi}{x}}{ctg\frac{\pi x}{2}}$ **33.** $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

34. $\lim\limits_{x\to 0}(\arcsin x\cdot ctg\,x)$ **35.** $\lim\limits_{x\to 1}(x-1)^{x-1}$ **36.** $\lim\limits_{x\to\infty}x\sin\dfrac{3}{x}$

37. $\lim\limits_{x\to 0}\dfrac{e^{tgx}-e^x}{tgx-x}$ **38.** $\lim\limits_{x\to\infty}\dfrac{\ln e^x}{1-xe^x}$ **39.** $\lim\limits_{x\to\infty}\dfrac{e^{1/x^2}-1}{2arctgx^2-\pi}$

40. $\lim\limits_{x\to 0}\dfrac{2-(e^x+e^{-x})\cos x}{x^4}$ **41.** $\lim\limits_{x\to 2}\dfrac{ctg(\pi x/4)}{x-2}$ **42.** $\lim\limits_{x\to\infty}\dfrac{\pi-2arctgx}{\ln(1+1/x)}$

43. $\lim\limits_{x\to\infty}(\pi-2arctgx)\cdot\ln x$ **44.** $\lim\limits_{x\to 1}(1-x)^{\ln x}$ **45.** $\lim\limits_{x\to\infty}(\ln x)^{\frac{1}{x}}$

46. $\lim\limits_{x\to 0}\left(\ln\dfrac{1}{x}\right)^x$ **47.** $\lim\limits_{x\to\infty}\left(\dfrac{x-4}{x+3}\right)^{3x}$ **48.** $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+3}\right)^{2x-1}$

Individual Tasks 1.9

1-6. Find the limit if it exists.

I.	II.
1. $\lim\limits_{x\to 4}\dfrac{\ln(x-4)}{\ln(e^x-e^4)}$ *2.* $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-ctgx\right)$	*1.* $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln\sin x}{(\pi-2x)^2}$

3. $\lim\limits_{x\to 0}(1-e^{2x})\cdot ctg\,x$ 4. $\lim\limits_{x\to 0}x\cdot ctg\pi x$ 5. $\lim\limits_{x\to 0}(\sin x)^{tg\,x}$ 6. $\lim\limits_{x\to 4}\left(2-\frac{x}{4}\right)^{tg\frac{\pi x}{8}}$	2. $\lim\limits_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$ 3. $\lim\limits_{x\to 1/2}\sin(2x-1)\cdot tg\pi x$ 4. $\lim\limits_{x\to 0}(x^2\ln x)$ 5. $\lim\limits_{x\to 0}\left(\cos 2x\right)^{\frac{3}{x^2}}$ 6. $\lim\limits_{x\to 0}\left(\frac{tg\,x}{x}\right)^{\frac{1}{x^2}}$

1.10 Taylor and Maclaurin Polynomials

If a function $y=f(x)$ has derivatives of the n order inclusively in some interval containing the point $x=a$, then it can be represented as a sum of a polynomial of degree n and a remainder term $R_n(x)$:

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x) \qquad (1)$$

where $f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$ is called ***Taylor polynomial*** and $R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$, $\xi\in(a;\,x)$.

Formula 1 is called ***Taylor formula*** with the remainder term in the form of Lagrange. If $a=0$ then ***Maclaurin formula*** can be obtained

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},\ \xi\in(0;\,x) \qquad (2)$$

The following representations of some elementary functions are widely used for the calculations of limits and an approximation of the functions at the given point.

1. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\frac{e^{\xi}}{(n+1)!}x^{n+1},\ \xi\in(0;\,x)$

2. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}+\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cos\xi,\ \xi\in(0;\,x)$

3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cos\xi,\ \xi\in(0;\,x)$

4. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n-1}x^n}{n}+\frac{(-1)^n}{n+1}\cdot\frac{x^{n+1}}{(1+\xi)^{n+1}},\ \xi\in(0;\,x)$

5. $\frac{1}{1-x}=1+x+x^2+\ldots+x^{n-1}+\frac{x^n}{(1-\xi)^n},\ \xi\in(0;x)$

6. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}x^m +$

$+\frac{n(n-1)(n-2)\cdots(n-m)(1+\xi)^{n-m-1}}{(m+1)!}x^{m+1},\ \xi \in (0;\ x)$

An approximation of the given functions can be represented as follows:

$$e^x \approx 1+x; \quad e^x \approx 1+x+\frac{x^2}{2}; \quad \sin x \approx x; \quad \sin x \approx x - \frac{x^3}{6}; \quad \cos x \approx 1-\frac{x^2}{2};$$

$$\cos x \approx 1-\frac{x^2}{2}+\frac{x^4}{24}; \quad \ln(1+x)\approx x; \quad \ln(1+x)\approx x-\frac{x^2}{2}; \quad \sqrt{1+x}\approx 1+\frac{x}{2}-\frac{x^2}{8},\ |x|<1.$$

Example 1 Find the limit $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-ctg^2 x\right)$, if it exists.

Solution

$$\lim_{x\to 0}\left(\frac{1}{x^2}-ctg^2x\right)=\lim_{x\to 0}\left(\frac{1}{x}-ctgx\right)\left(\frac{1}{x}+ctgx\right)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\left(\frac{1}{x}+\frac{\cos x}{\sin x}\right)=$$

$$=\lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x}\cdot\frac{\sin x + x\cos x}{x\sin x}=\lim_{x\to 0}\frac{1}{x^4}\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-R_5\right)-x\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-R_4\right)\right)\times$$

$$\times\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-R_5\right)+x\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-R_4\right)\right)=$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left(x^2\left(\frac{1}{2}-\frac{1}{6}\right)+x^4\left(\frac{1}{5!}-\frac{1}{4!}\right)+R_5\right)\times\left(2x-x^3\left(\frac{1}{3!}+\frac{1}{2!}\right)+x^5\left(\frac{1}{5!}+\frac{1}{4!}\right)-R_5\right)=$$

$$=\lim_{x\to 0}\left(\frac{2}{6}-x^2\frac{4}{120}+R_2\right)\left(2-\frac{4}{6}x^2+R_2\right)=\frac{1}{3}\cdot 2=\frac{2}{3}.$$

Exercise Set 1.10

In exercises 1 to 4 expand the polynomial in powers $x-x_0$, using Taylor formula.

1. $P(x)=x^4-x^3+5x^2-4x+1$, $x_0=1$

2. $P(x)=x^3+4x^2-6x-8$, $x_0=-1$

3. $P(x)=x^5-3x^4+7x+2$, $x_0=2$

4. $P(x)=x^3-4x^2+7x-11$, $x_0=2$

In exercises 5 to 8 find the first three terms in the expansion of a given function in powers $x-2$. Find approximate values of the function at the given points.

5. $f(x)=x^5-5x^3+x$, $f(2{,}1)$

6. $f(x)=5x^5+3x^3+x^2$, $f(1{,}99)$

7. $f(x)=2x^5-4x^3+3x^2+x$, $f(1{,}96)$

8. $f(x)=3x^6-4x^3+x^2+1$, $f(2{,}2)$

In exercises 9 to 12 expand the given function in powers x using Taylor formula.

9. $y=\sqrt{1+x}$

10. $y=xe^x$

11. $y=tg\,x$

12. $f(x)=(x^2-3x+1)^3$

13. $y=\arcsin x$

In exercises 14 to 16 find the limit using Maclaurin formula.

14. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{e^x - 1 - x - 0,5x^2}$ **15.** $\lim\limits_{x\to 0}\dfrac{\ln(1+x) - x + 0,5x^2}{x(1-\cos 2x)}$

16. $\lim\limits_{x\to 0}\dfrac{e^x - 1 - x - 0,5x^2}{\ln(x+1) - x + 0,5x^2}$

Individual Tasks 1.10

1-2. Expand the polynomial in powers $x - x_0$ using Taylor formula.

3-4. Expand the given function in powers x using Taylor formula.

5. Find the limit using Maclaurin formula.

I.	**II.**
1. $P(x) = 3x^4 - 2x^3 + x^2 - 11x + 4,$ $x_0 = -1$	*1.* $P(x) = 5x^4 - 2x^3 - 3x^2 + 6x - 9,$ $x_0 = 1$
2. $P(x) = 7x^3 - 4x^2 + 6x + 5,$ $x_0 = -1$	*2.* $P(x) = 2x^3 - 3x^2 + 5x + 1,$ $x_0 = -1$
3. $f(x) = \sqrt[3]{1+x}$	*3.* $f(x) = \arccos x$
4. $f(x) = arctgx$	*4.* $f(x) = \cos 3x$
5. $\lim\limits_{x\to 0}\dfrac{x(\ln(1+x) - x)}{\sin 2x - 2x}$	*5.* $\lim\limits_{x\to 0}\dfrac{(\cos x - 1 + 0,5x^2)x^2}{e^{x^2} - 1 - x^2 - 0,5x^4}$

1.11 Using the Derivative and Limits when Graphing a Function

Definition *(monotonic function)* If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is an ***increasing function***. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is a ***decreasing function***. These two types of functions are also called ***monotonic***.

Theorem 1 *(increasing/decreasing test)*

(a) If $f'(x) > 0$ on an interval $(a;b)$, then $y = f(x)$ is increasing on that interval.

(b) If $f'(x) < 0$ on an interval $(a;b)$, then $y = f(x)$ is decreasing on that interval.

Definition *(Critical number and critical point)* A number $x = x_0$, at which $f'(x_0) = 0$ or does not exist is called a ***critical number*** for the function $y = f(x)$. The corresponding point $(x_0; f(x_0))$ on the graph of $y = f(x)$ is a ***critical point*** on that graph.

Definition *(Relative maximum (local maximum))* The function $y = f(x)$ has a ***relative maximum (or local maximum)*** at the number $x = x_0$ if there is an open interval $(a;b)$ around $x = x_0$ such that $f(x) \le f(x_0)$ for all x in $(a;b)$ that lie in the domain of $y = f(x)$. A local or relative minimum is defined analogously.

Definition *(Global maximum)* The function $y = f(x)$ has a ***global maximum*** (or ***absolute maximum***) at the number $x = x_0$ if $f(x) \le f(x_0)$ for all x in the domain of $y = f(x)$. A global minimum is defined analogously.

Theorem 2 *(first-derivative test for local maximum (minimum))* Let $y = f(x)$ be a function and let $x = x_0$ be a number in its domain. Assume that such numbers a and b exist that $x_0 \in (a;\ b)$ and

1. $y = f(x)$ is continuous on the open interval $(a; b)$.
2. $y = f(x)$ is differentiable on the open interval $(a; b)$, except possibly at $x = x_0$.
3. $f'(x) > 0$ for all $x < x_0$ in the interval and is negative for all $x > x_0$ in the interval.

Then $y = f(x)$ has a local maximum at $x = x_0$.

A similar test, with ‘'positive" and "negative" interchanged, is used for a local minimum.

Theorem 3 ***(the second derivative test)*** Suppose $f''(x)$ is continuous near $x = x_0$.

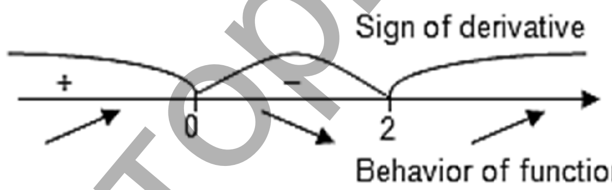
(a) If $f'(x) = 0$ and $f''(x) > 0$, then $y = f(x)$ has a local minimum at $x = x_0$.

(b) If $f'(x) = 0$ and $f''(x) < 0$, then $y = f(x)$ has a local maximum at $x = x_0$.

Example 1 Find the intervals on which the function $y = x^3 - 3x^2$ is increasing or decreasing. Find the local maximum and minimum values of $y = f(x)$.

Solution If $y = x^3 - 3x^2$, then $y' = 3x^2 - 6x = 3x(x - 2)$.

To use the first-derivative test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the two factors of $f'(x)$, namely x and $x - 2$.



We divide the real line into intervals whose endpoints are the critical numbers $x = 0$ and $x = 2$. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. It means that $y = f(x)$ is increasing on interval $x \in (-\infty; 0) \cup (2; +\infty)$ and decreasing on interval $x \in (0, 2)$. Consequently, $y = f(x)$ has a local maximum at $x = 0$ and local minimum at $x = 2$.

$y_{\max}(0) = 0$ is a local maximum value of function and

$$y_{\min}(2) = 2^3 - 3 \cdot 2^2 = 8 - 12 = -4$$

is a local minimum value of function.

Definition ***(Concave upward)*** A function $y = f(x)$ whose first derivative is increasing throughout the open interval $(a; b)$ is called ***concave upward*** on that interval.

Definition ***(Concave downward)*** A function $y = f(x)$ whose first derivative is decreasing throughout an open interval $(a; b)$ is called ***concave downward*** on that interval.

Definition ***(Inflection point and inflection number)*** Let $y = f(x)$ be a function and let $x = x_0$ be a number. Assume that there are numbers a and b such that $a < x_0 < b$ and

1. $y = f(x)$ is continuous on the open interval $(a; b)$;

2. $y = f(x)$ is concave upward on the interval $(a; x_0)$ and concave downward on the interval $(x_0; b)$ or vice versa.

Then the point $(x_0; f(x_0))$ is called ***an inflection point*** or ***point inflection***. The number $x = x_0$ is called an ***inflection number***.

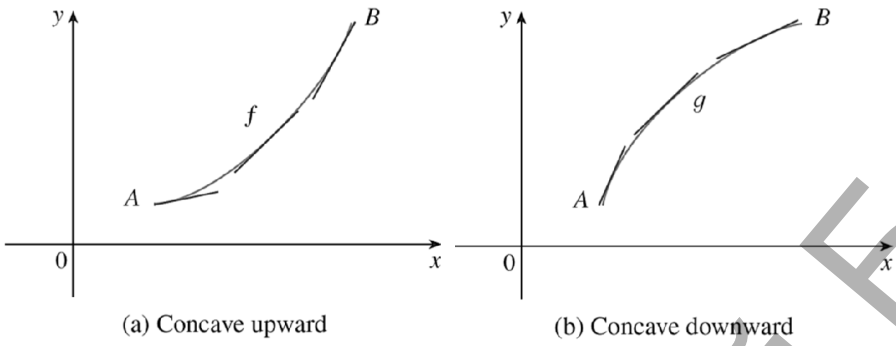


Figure 1.13

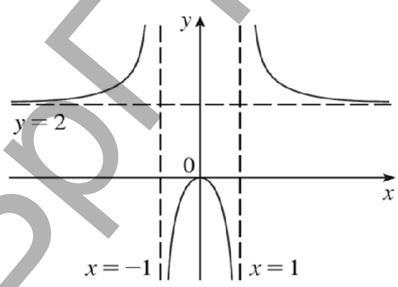


Figure 1.14

Note that when a function is concave upward, it is shaped like a part of a cup. It can be proved that where a curve is concave upward, it lies above its tangent lines and below its chords, as shown in Figure 1.13(a). If a curve is concave downward, it lies below its tangent lines and above its chords, as shown in Figure 1.13(b).

Theorem 4 ***(concavity test)*** (a) If $f''(x) > 0$ for all x in $(a; b)$, then the graph of $y = f(x)$ is concave upward on $(a; b)$.

(b) If $f''(x) < 0$ for all x in $(a; b)$, then the graph of $y = f(x)$ is concave downward on $(a; b)$.

Theorem 5 Let $y = f(x)$ be a function and let $x = x_0$ be a number in its domain. Assume that numbers a and b exist such that $x_0 \in (a;\ b)$ and

1. $y = f(x)$ is continuous and differentiable on the open interval $(a; b)$.

2. $f''(x) = 0$ or $f''(x)$ does not exit.

3. $f''(x) > 0$ ($f''(x) < 0$) for all $x < x_0$ on the interval and $f''(x) < 0$ ($f''(x) > 0$) for all $x > x_0$ in the interval.

Then the point $(x_0; f(x_0))$ is an inflection point.

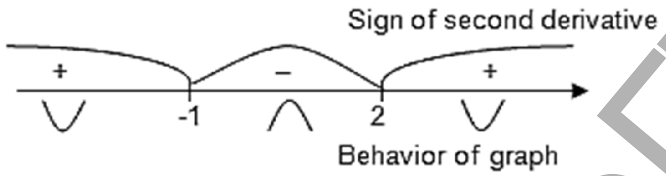
Example 2 Find the intervals of concavity and the inflection points of function $y = x^4 - 2x^3 - 12x^2 - 6x + 5$.

Solution If $y = x^4 - 2x^3 - 12x^2 - 6x + 5$, then

$$y' = 4x^3 - 6x^2 - 24x - 6;$$

$$y'' = 12x^2 - 12x - 24 = 12(x^2 - x - 2) = 12(x-2)(x+1).$$

To use the concavity test we have to know where $f''(x) > 0$ and where $f''(x) < 0$. This depends on the signs of the two factors of $f''(x)$, namely $x+1$ and $x-2$. We divide the real line into intervals whose endpoints are the numbers $x=-1$ and $x=2$. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative.



It means that the graph of $y = f(x)$ is concave upward on the interval $x \in (-\infty; -1) \cup (2; +\infty)$ and concave downward on the interval $x \in (-1, 2)$. We find the values of the function at the points $x=-1$ and $x=2$.

$$y(-1) = 1 + 2 - 12 + 6 + 5 = 2;\ y(2) = 16 - 16 - 48 - 12 + 5 = -55.$$

The point $(2; -55)$ is an inflection point since the curve changes from concave upward to concave downward there. Also, $(-1; 2)$ is an inflection point since the curve changes from concave downward to concave upward there.

Guidelines for Sketching a Curve

The following checklist is intended as a guide to the sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess any symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain
B. Intercepts
C. Symmetry
- *even function*
- *odd function*
- *periodic function*

D. Asymptotes
- *vertical asymptotes*
- *slant asymptotes*

E. Intervals of Increase or Decrease
F. Local Maximum and Minimum Values
G. Concavity and Points of Inflection
H. Sketch the Curve

Example 3 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

Solution

A. The domain is

$$D=(-\infty;-1)\cup(-1;1)\cup(1;+\infty).$$

B. The x- and y- intercepts are both 0.

C. Since $f(-x)=f(x)$, the function is even. The curve is symmetric about the y axis.

D.

$$\lim_{x\to\pm\infty}\frac{2x^2}{x^2-1}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^2}=2.$$

Therefore the line $y=2$ is a horizontal asymptote.

Since the denominator is 0 when $x=\pm1$, we compute the following limits:

$$\lim_{x\to-1-0}\frac{2x^2}{x^2-1}=\left(\frac{2}{+0}\right)=+\infty;\quad \lim_{x\to-1+0}\frac{2x^2}{x^2-1}=\left(\frac{2}{-0}\right)=-\infty;\quad \lim_{x\to1-0}\frac{2x^2}{x^2-1}=\left(\frac{2}{-0}\right)=-\infty;$$

$$\lim_{x\to1+0}\frac{2x^2}{x^2-1}=\left(\frac{2}{+0}\right)=+\infty.$$

Therefore the lines $x=\pm1$ are vertical asymptotes.

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\left(\frac{2x^2}{x^2-1}:x\right)=\lim_{x\to\pm\infty}\left(\frac{2x^2}{x^3-x}\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^3}=\lim_{x\to\pm\infty}\frac{2}{x}=0,$$

$$b=\lim_{x\to\pm\infty}\left(f(x)-kx\right)=\lim_{x\to\pm\infty}\frac{2x^2}{x^2-1}=\lim_{x\to\pm\infty}\frac{2x^2}{x^2}=2.$$

It means that $y=f(x)$ has not a slant asymptote.

E.

$$y'=\left(\frac{2x^2}{x^2-1}\right)'=\frac{4x\cdot\left(x^2-1\right)-2x\cdot2x^2}{\left(x^2-1\right)^2}=-\frac{4x}{\left(x^2-1\right)^2}.$$

Since $f'(x)>0$ when $x<0$ and $f'(x)<0$ when $x>0$, $y=f(x)$ is increasing on $(-\infty;-1)\cup(-1;0)$ and decreasing on $(0;1)\cup(1;+\infty)$.

F. The only critical number is $x=0$. Since $f'(x)$ changes from positive to negative at 0, $f(0)=0$ is a local maximum by the First Derivative Test.

G.

$$y''=\left(-\frac{4x}{\left(x^2-1\right)^2}\right)'=-4\cdot\frac{\left(x^2-1\right)^2-4x^2\cdot\left(x^2-1\right)}{\left(x^2-1\right)^4}=4\cdot\frac{3x^2+1}{\left(x^2-1\right)^3}.$$

Since $3x^2+1>0$ for all x, we have

$$f''(x)>0\Leftrightarrow x^2-1>0\Leftrightarrow |x|>1$$

and

$$f''(x)<0\Leftrightarrow x^2-1<0\Leftrightarrow |x|<1.$$

Thus the curve is concave upward on the intervals $(-\infty;-1)\cup(1;+\infty)$ and concave downward on $(-1;1)$. It has no point of inflection since 1 and -1 are not in the domain of $y=f(x)$.

H. Using the information in E–G, we finish the sketch in Figure 1.14.

Exercise Set 1.11

In exercises 1 to 12 find the intervals on which the function is increasing or decreasing. Find the local maximum and minimum values of the given function.

1. $y=2x^3-6x^2-18x$ ***2.*** $y=(x-2)^5(2x+1)^4$ ***3.*** $y=\dfrac{x^2-2x+2}{x-1}$

4. $y=xe^{-x}$ ***5.*** $y=x\ln x$ ***6.*** $y=x-e^x$

7. $y=(2-x)(x+1)^2$ ***8.*** $y=\dfrac{2x+3}{3x-5}$ ***9.*** $y=\sqrt[3]{(x^2-6x+5)^2}$

10. $y=\sqrt{3x-7}$ ***11.*** $y=x\ln^2 x$ ***12.*** $y=e^{3-6x-x^2}$

In exercises 13 to 18 find the absolute maximum and absolute minimum values of the function on the given interval.

13. $y=\dfrac{x-1}{x+1}$, $[0;4]$ ***14.*** $y=x-x\sqrt{-x}$, $[-4;0]$ ***15.*** $y=\sqrt{100-x^2}$, $[-6;8]$

16. $y=\dfrac{1-x+x^2}{1-x-x^2}$, $[0;1]$ ***17.*** $y=2x-\sqrt{x}$, $[0;4]$ ***18.*** $y=tg\,x-x$, $\left[-\dfrac{\pi}{4};\dfrac{\pi}{4}\right]$

In exercises 19 to 24 find the intervals of concavity and the inflection points.

19. $y=\ln(x^2+2x+5)$ ***20.*** $y=\dfrac{x^4}{x^3-1}$ ***21.*** $y=\dfrac{x^2}{x-1}$

22. $y=x^2e^x$ ***23.*** $y=e^{-x^2}$ ***24.*** $\begin{cases} y=3t+t^3; \\ x=t^2. \end{cases}$

In exercises 25 to 35 use the guidelines of this section to sketch the curve.

25. $y=\dfrac{x^2-6x+10}{x-3}$ ***26.*** $y=\dfrac{(x-1)^2}{x}$ ***27.*** $y=\dfrac{\ln x}{\sqrt{x}}$

28. $y=e^{\frac{1}{x+2}}$ ***29.*** $y=\sqrt[3]{(x+3)x^2}$ ***30.*** $y=\dfrac{3x+2}{5x^2}$

31. $y=x\cdot e^{-x}$ ***32.*** $y=\sqrt[3]{6x^2-x^3}$ ***33.*** $y=\dfrac{x^3}{2(x+1)^2}$

34. $y=\sqrt[3]{4x^3-12x}$ ***35.*** $y=\ln(x^2+2x+2)$

Individual Tasks 1.11

1. Find the intervals on which $y=f(x)$ is increasing or decreasing. Find the local maximum and minimum values of the given function.

2. Find the absolute maximum and absolute minimum values of the function on the given interval.

3. Find the intervals of concavity and the inflection points.

4-5. Use the guidelines of this section to sketch the curve.

I.	**II.**
1. $y = x^{\frac{2}{3}} - x$	*1.* $y = x^2(1 - x\sqrt{x})$
2. $y = x^4 - 8x^2, [-2;2]$	*2.* $y = x + 3\sqrt[3]{x}$, $[-1;1]$
3. $y = 2x^2 + \dfrac{1}{x}$	*3.* $y = \dfrac{3x-2}{5x^2}$
4. $y = x^2 \cdot e^{-x}$	*4.* $y = x \cdot e^{-x^2/2}$
5. $y = \dfrac{x^3 + 2x^2 + 7x - 3}{2x^2}$	*5.* $y = \dfrac{x^3}{3 - x^2}$

1.12 Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life (a businessperson wants to minimize costs and maximize profits; a traveler wants to minimize transportation time). In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized.

Steps in solving optimization problems

1. Understand the Problem

The first step is to read the problem carefully until it is clearly understood. Ask yourself: What are the given quantities? What are the given conditions?

2. Draw a Diagram

In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. Introduce Notation

Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, ..., x, y)$ for other unknown quantities and label the diagram with these symbols. It may help use initials as suggestive symbols, for example, S for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to *find relationships* (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be expressed as a function of single variable x. Write the domain of this function.

6. Use the methods of Section 1.11 to find the absolute maximum or minimum value of $f(x)$. In particular, if the domain of $f(x)$ is a closed interval, then the Closed Interval Method can be used.

Example 1 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution Draw the diagram as in Figure 1.15, where r is the radius and h is the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 1.16 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h.

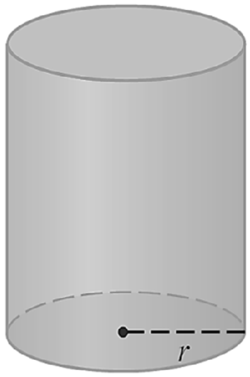


Figure 1.15

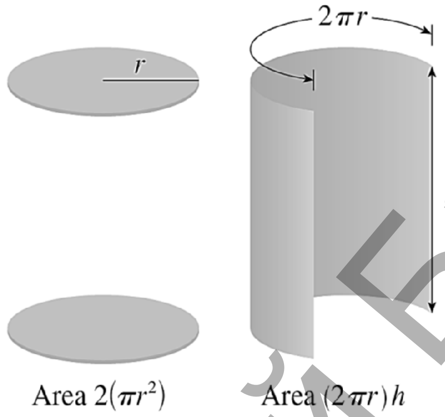


Figure 1.16

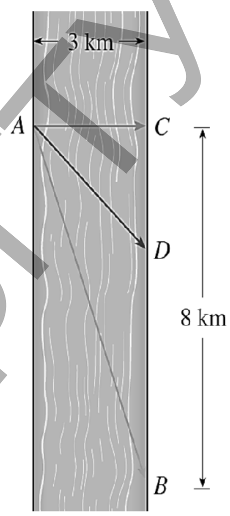


Figure 1.17

So the surface area is $S = 2\pi r^2 + 2\pi rh$.

To eliminate h we use the fact that the volume is given as 1 L, which we take to be $1000\,cm^3$. Thus $\pi r^2 h = 1000$, which gives $h = 1000/\left(\pi r^2\right)$. The substitution of this into the expression for S gives $S(r) = 2\pi r^2 + \dfrac{2000}{r}$.

Therefore the function that we want to minimize is

$$S(r) = 2\pi r^2 + \frac{2000}{r}\ ,\ r > 0.$$

To find the critical numbers, we differentiate:

$$S'(r) = \left(2\pi r^2 + \frac{2000}{r}\right)' = 4\pi r - \frac{2000}{r^2} = \frac{4\left(\pi r^3 - 500\right)}{r^2}.$$

Then $S'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$. Since the domain of $S(r)$ is $(0;+\infty)$, we can observe that $S'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $S'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so $S(r)$ is decreasing for all r to the left of the critical number and increasing for all r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

The value of corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(500/\pi\right)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter.

Example 2 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 1.17). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B, then run to B. If he can row $6\,km/h$ and run 8 km/h, where should he land to reach as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

Solution If we let x be the distance from C to D, then the running distance is $DB = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $AD = \sqrt{x^2+9}$. We use the equation $time = \frac{dist}{rate}$. Then the rowing time is $\sqrt{x^2+9}/6$ and the running time is $(8-x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}.$$

The domain of this function $T(x)$ is $[0;8]$. Notice that if $x=0$, he rows to C and if $x=8$, he rows directly to B. The derivative of $T(x)$ is

$$T'(x) = \left(\frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}\right)' = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}.$$

Thus, using the fact that $x \ge 0$, we have

$$T'(x) = 0 \Rightarrow \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} = 0 \Leftrightarrow 4x = 3\sqrt{x^2+9} \Leftrightarrow 16x^2 = 9x^2 + 81$$

$$7x^2 = 81 \Leftrightarrow x = \pm 9/\sqrt{7}.$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0;8]$, we evaluate at all three points:

$$T(0) = 1{,}5 \qquad T(9/\sqrt{7}) = 1 + \sqrt{7}/8 \approx 1{,}33 \qquad T(8) = \sqrt{73}/6 \approx 1{,}42$$

Since the smallest of these values of $T(x)$ occurs when $x = 9/\sqrt{7}$, the absolute minimum value of $T(x)$ must occur there. Thus the man should land the boat at a point $9/\sqrt{7}\ km\ (\approx 3{,}4\ km)$ downstream from his starting point.

Exercise Set 1.12

1. Find the area of the largest rectangle that can be inscribed in a semicircle of radius R .

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

4. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

5. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

6. A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of the material used.

7. If 1200 cm^2 of the material is available to make a box with a square base and an open top, find the largest possible volume of the box.

8. Find the dimensions of the rectangle of the largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

9. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

10. A right circular cylinder is inscribed in a sphere of radius R. Find the largest possible volume of such a cylinder.

11. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

12. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W .

Individual Tasks 1.12

I. *1.* Find two positive numbers whose product is 100 and whose sum is a minimum. *2.* Find the dimensions of the rectangle of the largest area that can be inscribed in a circle of radius R. *3.* A right circular cylinder is inscribed in a cone with height H and base radius R. Find the largest possible volume of such a cylinder.
II. *1.* Find a positive number such that the sum of the number and its reciprocal is as small as possible. *2.* Find the dimensions of the isosceles triangle of the largest area that can be inscribed in a circle of radius R. *3.* A right circular cylinder is inscribed in a sphere of radius R. Find the largest possible surface area of such a cylinder.

II FUNCTIONS OF SEVERAL VARIABLES

2.1 Functions of Two Variables. Partial Derivatives. Directional Derivatives and the Gradient Vector.

Definition A ***function of two variables*** is a rule that assigns to each ordered pair of real numbers $(x;y)$ in a set D a unique real number denoted by $f(x,y)$. The set D is the ***domain*** of f and its ***range*** is the set of the values that f takes on, that is $\{f(x,y):(x,y)\in D\}$.

Example 1 Find the domain and the range of $z=\sqrt{9-x^2-y^2}$.

Solution The domain of z is

$$D=\{(x,y):9-x^2-y^2\geq 0\}=\{(x,y):x^2+y^2\leq 9\}$$

which is the disk with center $(0;0)$ and radius 3.

The graph has equation $z=\sqrt{9-x^2-y^2}$. We square both sides of this equation to obtain $z^2=9-x^2-y^2$, or $x^2+y^2+z^2=9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z\geq 0$, the graph of z is just the top half of this sphere.

Functions of any number of variables can be considered. A ***function of n variables*** is a rule that assigns a number $z=f(x_1,x_2,...,x_n)$ to an n-tuple $(x_1,x_2,...,x_n)$ of real numbers. We denote R^n by the set of all such n-tuples.

Definition Let f be a function of two variables whose domain D includes the points arbitrarily close to $(a;b)$. Then we say that the ***limit*** of $f(x,y)$ as $(x;y)$ approaches $(a;b)$ is L and we write $\lim\limits_{(x;y)\to(a;b)} f(x,y)=L$ if for every number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that if $(x;y)\in D$ and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$, then $|f(x,y)-L|<\varepsilon$.

Other notations for the limit in Definition are

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y)=L \text{ and } f(x,y)\to L \text{ as } (x;y)\to(a;b).$$

If f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y=b$, where number b is a constant. Then we really consider a function of a single variable x, namely $g(x)=f(x,b)$.

Definition If g has a derivative at $x=a$, then we call it the ***partial derivative of*** f ***with respect to x at*** $(a;b)$ and denote it by $f'_x(a;b)$. By the definition of a derivative, we have

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x;y)-f(x;y)}{\Delta x}=\frac{\partial z}{\partial x}=z'_x=f'_x(x,y).$$

Similarly, the ***partial derivative of f with respect to y at*** $(a;b)$, denoted by $f_y'(a;b)$, is obtained by keeping x fixed $(x=a)$ and finding the ordinary derivative at b of the function $f(a,y)$:

$$\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x;y+\Delta y)-f(x;y)}{\Delta y}=\frac{\partial z}{\partial y}=z_y'=f_y'(x,y).$$

For a differentiable function of two variables $z=f(x,y)$, we define the ***differentials*** dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the ***total differential***, is defined by

$$dz=z_x'dx+z_y'dy=f_x'(x,y)dx+f_y'(x,y)dy,\qquad \Delta x=dx,\qquad \Delta y=dy.$$

The differential du is defined in terms of the differentials dx, dy and dz of the independent variables by

$$u=f(x,y,z):\qquad du=f_x'(x,y,z)dx+f_y'(x,y,z)dy+f_z'(x,y,z)dz$$

Figure 2.1 shows the geometric interpretation of the differential dz and the increment Δz: represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z=f(x,y)$ when $(x;y)$ changes from $(a;b)$ to $(a+\Delta x,b+\Delta y)$.

If we take $x=x_0+\Delta x$ and $y=y_0+\Delta y$ in the formula of total differential, then the differential of z is $dz=f_x'(x,y)(x-x_0)+f_y'(x,y)(y-y_0)$.

So, in the notation of differentials, the linear approximation can be written as

$$f(x;y)\approx f(x_0;y_0)+f_x'(x_0;y_0)\cdot\Delta x+f_y'(x_0;y_0)\cdot\Delta y.$$

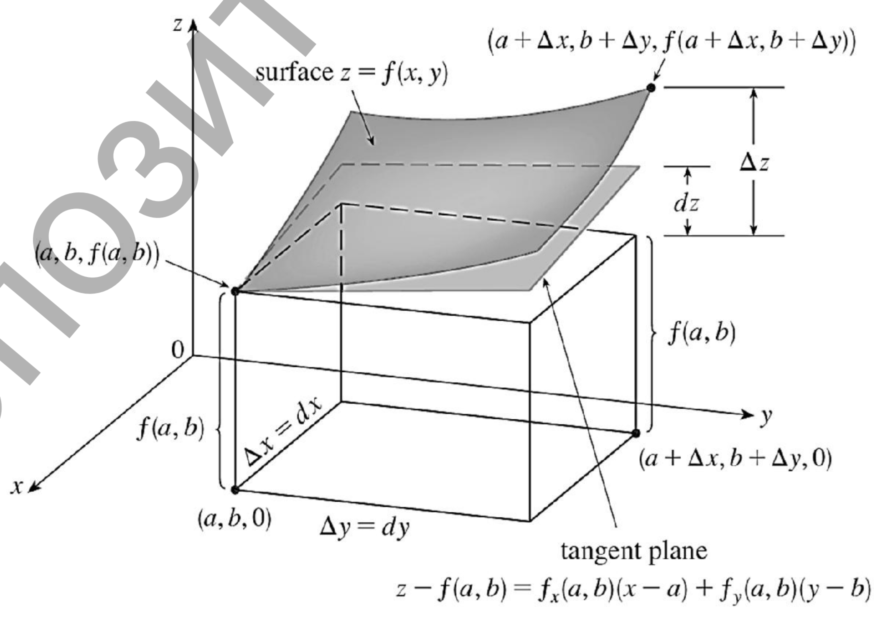


Figure 2.1

Definition The ***directional derivative*** of $u = f(x,y,z)$ at $M_0(x_0;y_0;z_0)$ in the direction of a vector $\vec{a} = (l,m,n)$ is $\lim\limits_{M\to M_0} \frac{\Delta u(M_0)}{|M_0M|} = \frac{\partial u(M_0)}{\partial \vec{a}}$, $\vec{a} = \overline{M_0M}$, if this limit exists.

This derivative is found by the formula

$$\frac{\partial u(M_0)}{\partial \vec{a}} = u'_x(M_0)\cdot\cos\alpha + u'_y(M_0)\cdot\cos\beta + u'_z(M_0)\cdot\cos\gamma,$$

$$\cos\alpha = \frac{l}{|\vec{a}|}, \qquad \cos\beta = \frac{m}{|\vec{a}|}, \qquad \cos\gamma = \frac{n}{|\vec{a}|}$$

The directional derivative shows the rate of change in the function at the particular point in this direction.

Definition If f is a function of two variables x and y, then the ***gradient*** of f is the vector function defined by $\nabla f(x,y) = grad\ f = \left(f'_x, f'_y\right)$.

The derivative in the direction of its gradient takes the maximum value.

Example 2 Find the directional derivative of the function $u = x + y^2 - z^3$ at the given point $M_0(1;2;-1)$ in the direction of the vector $\vec{a} = (2;-6;3)$. Find the gradient of $u(x,y,z)$.

Solution We find particular derivatives at the point of M_0.

$$u'_x = 1, \qquad u'_x(1,2,-1) = 1.$$

$$u'_y = 2y, \qquad u'_y(1,2,-1) = 2\cdot 2 = 4.$$

$$u'_z = -3z^2, \qquad u'_z(1,\ 2,-1) = -3\cdot(-1)^2 = -3.$$

$$|\vec{a}| = \sqrt{2^2 + (-6)^2 + 3^2} = \sqrt{4+36+9} = \sqrt{49} = 7.$$

$$\cos\alpha = \frac{2}{|\vec{a}|} = \frac{2}{7}; \qquad \cos\beta = \frac{-6}{|\vec{a}|} = -\frac{6}{7}; \qquad \cos\gamma = \frac{3}{|\vec{a}|} = \frac{3}{7}.$$

Then the desired derivative is equal

$$\frac{\partial u(M_0)}{\partial \vec{a}} = 1\cdot\frac{2}{7} + 4\cdot\left(-\frac{6}{7}\right) - 3\cdot\frac{3}{7} = \frac{2-24-9}{7} = -\frac{31}{7}.$$

$$\nabla u(x,y,z) = gradu = (u'_x, u'_y, u'_z) = (1, 2y, -3z^2);$$

$$gradu(M_0) = \left(u'_x(M_0), u'_y(M_0), u'_z(M_0)\right) = (1,4,-3).$$

Exercise Set 2.1

In exercise 1 to 6 find and sketch the domain of the function.

1. $z = \sqrt{1-x^2} + \sqrt{y^2-1}$ **2.** $z = \arccos\frac{x}{x+y}$ ***3.*** $z = \arcsin(2x-y)$

4. $z = \sqrt{y^2-2x+4}$ **5.** $z = \ln x + \ln\cos y$ ***6.*** $z = \sqrt{x^2-4} + \sqrt{4-y^2}$

In exercise 7 to 17 find the particular derivatives and total differential.

7. $z = 2x^3 - 6x^2y + y^3$ ***8.*** $z = x^3y - y^3x$ ***9.*** $z = \ln(x^2 + y^2)$

10. $z = arctg\left(y/x\right)$ ***11.*** $z = x^y$ ***12.*** $z = e^{\sin(4x^2 - 3y)}$

13. $z = x^3y + \cos x - 3tgx \cdot \ln y + 5$ ***14.*** $z = \ln tg\dfrac{y}{6x}$ ***15.*** $u = \dfrac{z}{\sqrt{x^2 + y^2}}$

16. $z = \ln(x^2 + y^2)$ ***17.*** $z = \cos\dfrac{x - y}{x^2 + y^2}$

In exercise 18 to 20 find the directional derivative of the function at the given point M_0 in the direction of the vector $\vec{a}\,(\overrightarrow{M_0M_1})$. Find the gradient of the function.

18. $z = x^3 - 2x^2y + xy^2 + 1$, $M_0(1;\,2)$, $\vec{a} = (3;\,-4)$

19. $u = \dfrac{\sin(x - y)}{z}$, $M_0\left(\dfrac{\pi}{2};\,\dfrac{\pi}{3};\,\sqrt{3}\right)$, $M_1\left(\pi;\,\dfrac{\pi}{6};\,2\sqrt{3}\right)$

20. $u = 8 \cdot \sqrt[5]{x^3 + y^2 + z}$, $M_0(3;\,2;\,1)$, $M_1(5;\,8;\,4)$

In exercise 21 to 24 calculate.

21. $\sqrt{3{,}12^2 + 3{,}98^2}$ ***22.*** $\sqrt{5{,}86^2 + 8{,}11^2}$ ***23.*** $\sqrt{5{,}18^2 + 11{,}97^2}$ ***24.*** $\sqrt{3{,}03^2 + 3{,}87^2}$

Individual Tasks 2.1

1. Find and sketch the domain of the function.

2-3. Find the particular derivatives and the total differential.

4. Find the directional derivative of the function at the given point M_0 in the direction of the vector $\vec{a}\,(\overrightarrow{M_0M_1})$. Find the gradient of the function.

5. Calculate.

I.	**II.**
1. $z = \arcsin\left(y / x\right)$ *2.* $z = arcctg\left(xy^2\right)$ *3.* $u = \left(xy^2\right)^z$ *4.* $u = \ln\left(x + \dfrac{y}{2z}\right)$, $M_0(1;2;1)$, $M_1(-2;3;5)$ *5.* $\sqrt{5{,}13^2 + 11{,}91^2}$	*1.* $z = \arccos(x + y)$ *2.* $z = \ln\left(3x^2 - y^4\right)$ *3.* $u = (x - y)(y - z)(z - x)$ *4.* $u = \dfrac{y}{x} + \dfrac{z}{y} - \dfrac{x}{z}$, $M_0(1;\,1;\,2)$, $M_1(8;\,-1;\,-4)$ *5.* $\sqrt{6{,}12^2 + 7{,}98^2}$

2.2 Chain Rule. Implicit Differentiation

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version deals with the case where $z = f(u,v)$ and each of the variables u and v is, in turn, a function of a variable x. This means that z is indirectly a function of x, $z = f\big(u(x), v(x)\big)$ and the Chain Rule gives a formula for differentiating z as a function of x. We assume that it is differentiable.

Chain Rule (Case 1)

Suppose that $z = f(u,v)$ is a differentiable function of u and v, where $u = u(x)$, $v = v(x)$ and are both differentiable functions of x.

Then z is a differentiable function of x and $\dfrac{dz}{dx} = \dfrac{\partial z}{\partial u}\cdot\dfrac{du}{dx} + \dfrac{\partial z}{\partial v}\cdot\dfrac{dv}{dx}$.

Chain Rule (Case 2)

Suppose that $z = f(u,v)$ is a differentiable function of u and v, where $u = u(x,y)$ and $v = v(x,y)$ are differentiable functions of x and y. Then

$$\begin{cases} z'_x = z'_u \cdot u'_x + z'_v \cdot v'_x, \\ z'_y = z'_u \cdot u'_y + z'_v \cdot v'_y. \end{cases}$$

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of the implicit differentiation. We suppose that an equation of the form $F(x,y) = 0$ defines implicitly as a differentiable function of x, that is, $y = f(x)$, where $F\big(x, f(x)\big) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x,y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{dy}{dx} = -\frac{F'_x(x,y)}{F'_y(x,y)}.$$

Now we suppose that it is given implicitly as a function $z = f(x,y)$ by an equation of the form $F(x,y,z) = 0$. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x,y,z) = 0$ as follows:

$$\frac{\partial z}{\partial x} = -\frac{F'_x(x,y,z)}{F'_z(x,y,z)}, \qquad \frac{\partial z}{\partial y} = -\frac{F'_y(x,y,z)}{F'_z(x,y,z)}.$$

The equation $z = f(x,y)$ represents a surface S (the graph of f). Suppose f has continuous partial derivatives. An equation of the ***tangent plane*** to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ can be written as follows:

$$z - z_0 = f'_x(x_0, y_0)\cdot(x - x_0) + f'_y(x_0, y_0)\cdot(y - y_0).$$

The canonical equations of ***normal line*** to this surface, carried out through the point $P(x_0, y_0, z_0)$ will be written down thus

$$\frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$

An equation of the tangent plane to the surface $F(x,y,z)=0$ at the point $P(x_0,y_0,z_0)$ can be written as follows:

$$F'_x(x_0,y_0,z_0)\cdot(x-x_0)+F'_y(x_0,y_0,z_0)\cdot(y-y_0)+F'_z(x_0,y_0,z_0)\cdot(z-z_0)=0.$$

The canonical equations of normal line to this surface, carried out through the point $P(x_0,y_0,z_0)$ will be written down thus

$$\frac{x-x_0}{F'_x(P_0)}=\frac{y-y_0}{F'_y(P_0)}=\frac{z-z_0}{F'_z(P_0)}.$$

Exercise Set 2.2

In exercise 1 to 12 find the particular derivatives of the function.

1. $z=e^{x^2+y^2}$, $x=a\cos t$, $y=a\sin t$

2. $z=3^{x^2}\,arctg y$, $x=\frac{u}{v}$, $y=uv$

3. $z=tg^2(x^2+4y)$, $y=\sin\sqrt{x}$

4. $z=arctg\frac{y}{x}$, $y=x\cos^2 x$

5. $z=x^5+2xy-y^3$, $x=\cos 2t$, $y=arctg\,t$

6. $z=\cos(2t+4x^2-y)$, $x=\frac{1}{t}$, $y=\frac{\sqrt{t}}{\ln t}$

7. $z=\frac{x^2}{y}$, $x=u-2v$, $y=2u+v$

8. $z=\sqrt{x^2-y^2}$, $x=u^v$, $y=u\ln v$

9. $z=\arccos\frac{u}{v}$, $u=x+\ln y$, $v=-2e^{-x^2}$

10. $z=e^{u^2-3\sin v}$, $u=x\cos y$, $v=x/y$

11. $z=\frac{u^2}{v}$, $u=\ln(x^2-y^2)$, $v=xy^2$

12. $z=\frac{v^2}{u}$, $u=x^2-4\sqrt{y}$, $v=xe^y$

In exercise 13 to 16 check if the function satisfies the given equation.

13. $z=\frac{xy}{x+y}$, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

14. $z=\frac{2x+3y}{x^2+y^2}$, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+z=0$

15. $z=x\ln\frac{y}{x}$, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

16. $z=\frac{y}{\left(x^2+y^2\right)^5}$, $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{z}{y^2}$

In exercise 17 to 21 find an equation of the tangent plane and the normal line to the given surface at the specified point.

17. $S: z=x^2+2y^2+4xy-5y-10$, $M_1(-7;1;8)$

18. $S: z=4y^2+4xy-x$, $M_1(1;-2;7)$

19. $S: z=x^2-y^2-4x+2y$, $M_1(3;1;-2)$

20. $S: z=x^2+y^2-4xy+3x-15$, $M_1(-1;3;4)$

21. $z=\frac{1}{2}(x^2-y^2)$, $M_0(3;1;4)$

Individual Tasks 2.2

1-2. Find the particular derivatives of the function.

3. Check if the function satisfies the given equation.

4. Find an equation of the tangent plane and the normal line to the given surface at the specified point.

I.	**II.**
1. $z = e^{xy}\ln(x+y),\ x = t^3,\ y = 1 - t^3$	*1.* $z = \sqrt{x^2+y^2},\ x = u\sin v,\ y = v\sin u$
2. $2x^2 - 3y^2 + 5xy - y^3x + x^5 - 37 = 0$	*2.* $\sin(xy) - x^2 - y^2 - 5 = 0$
3. $z = \dfrac{xy}{x+y},\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$	*3.* $z = x\ln\dfrac{x}{y},\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$
4. $S:\ x^2 - y^2 + z^2 - 4x + 2y = 14$, $M_0(3;1;4)$	*4.* $S:\ xyz^2 + 2y^2 + 3yz + 4 = 0$, $M_0(0;2;-2)$

2.3 Higher Derivatives

If f is a function of two variables, then its partial derivatives $f'_x(x,y)$ and $f'_y(x,y)$ are also functions of two variables, so we can consider their partial derivatives $(f'_x)'_x$, $(f'_x)'_y$, $(f'_y)'_x$, and $(f'_y)'_y$, which are called the ***second partial derivatives*** of f. If $z = f(x,y)$, we use the following notation:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = (z'_x)'_x = z''_{xx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = (z'_x)'_y = z''_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = (z'_y)'_x = z''_{yx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = (z'_y)'_y = z''_{yy}.$$

Thus the notation $f''_{xy}(x,y)$ (or $\dfrac{\partial^2 z}{\partial x \partial y}$) means that we first differentiate with respect to x and then with respect to y, whereas in computing $f''_{yx}(x,y)$ the order is reversed.

Clairaut's Theorem Suppose z is defined on a disk D that contains the point $(a;b)$. If the functions z''_{xy} and z''_{yx} are both continuous on D, then

$$z''_{xy}(a;b) = z''_{yx}(a;b).$$

Partial derivatives of order 3 or higher can also be defined. For instance,

$$z'''_{xyy} = (z''_{xy})'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y \partial x}\right).$$

Then the differential d^2z of $z = f(x,y)$, also called the ***total differential order by two***, is defined by

$$d^2z = \frac{\partial^2 z}{\partial x^2}\cdot dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}\cdot dxdy + \frac{\partial^2 z}{\partial y^2}\cdot dy^2.$$

Exercise Set 2.3

In exercise 1 to 12 find the differential d^2z of the given function.

1. $z = arctg(x - 3y)$ ***2.*** $z = \ln(5x^2 - 3y^4)$

3. $z = ctg\dfrac{y}{x^3}$

4. $z = x^2 + xy + y^2 - 4\ln x - 10\ln y$

5. $z = \ln\left(x + \sqrt{x^2 + y^2}\right)$

6. $z = f(t),\ t = x^2 + y^2$

7. $z = x^2 + y^2 - 3xy - 4x + 6y - 7$

8. $z = x^3 + y^2 - 6xy - 39x + 18y + 20$

9. $z = e^{xy}$

10. $z = 2x^2 - 3y^2 + xy + 3x + 1$

11. $z = x\ln\dfrac{y}{x}$

12. $z = x^3 + 8y^3 - 6xy + 5$

In exercise 13 to 15 check if the function satisfies the given equation.

13. $u = e^{-(x+3y)}\sin(x+3y),\ 9u''_{xx} + u''_{yy} = 0$

14. $u = \sin^2(x - 2y),\ 4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$

15. $x^2 \cdot u''_{xx} + 2xy \cdot u''_{xy} + y^2 \cdot u''_{yy} = 0,\ u = \dfrac{y}{x}$

Individual Tasks 2.3

1-2. Find the differential $d^2 z$ of the given function.

3. Check if the function satisfies the given equation.

I.	**II.**
1. $z = 2x^2 + xy - 3y^2 + 3x + 1$	*1.* $z = x^2 + 2y^2 + 3z^2 - 2xy + 4x + 2yz$
2. $z = arctg\dfrac{x+y}{1-xy}$	*2.* $z = e^x(\sin y + \cos x)$
3. $z = f(y/x)$, $x^2 \cdot z''_{xx} + 2xy \cdot z''_{xy} + y^2 \cdot z''_{yy} = 0$	*3.* $z = \ln(x + e^{-y})$, $\dfrac{\partial z}{\partial x} \cdot \dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial^2 z}{\partial x^2} = 0$

2.4 Maximum and Minimum Values

Definition A function f of two variables has a ***local maximum*** at the point $(a;b)$ if $f(x,y) \le f(a;b)$, when $(x;y)$ is near $(a;b)$. The number $f(a;b)$ is called a ***local maximum value***. If $f(x,y) \ge f(a;b)$ and $(x;y)$ is near $(a;b)$, then f has a ***local minimum*** at the point $(a;b)$ and $f(a;b)$ is called a ***local minimum value***.

If the inequalities in Definition are used for *all* points $(x;y)$ in the domain of f, then f has an ***absolute maximum*** (or ***absolute minimum***) at $(a;b)$.

Theorem If f has a local maximum or minimum at $(a;b)$ and the first-order partial derivatives of f exist there, then $f'_x(a;b) = 0$ and $f'_y(a;b) = 0$.

Thus the geometric interpretation of Theorem is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

Definition A point $(a;b)$ is called a ***critical point*** (or ***stationary point***) of f if $f'_x(a;b) = 0$ and $f'_y(a;b) = 0$, or if one of these partial derivatives does not exist.

The theorem says that if f has a local maximum or minimum at $(a;b)$, then $(a;b)$ is a critical point of f. However, as in the single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

The following test, which is proved at the end of this section, is analogous to the Second Derivative Test for functions of single variable.

Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center $(x_0;y_0)$, and suppose that $f'_x(x_0;y_0)=0$ and $f'_y(x_0;y_0)=0$ (that is, $(x_0;y_0)$ is a critical point of f). Let

$$\Delta=\begin{vmatrix} f''_{xx}(x_0;y_0) & f''_{xy}(x_0;y_0) \\ f''_{yx}(x_0;y_0) & f''_{yy}(x_0;y_0) \end{vmatrix}.$$

(a) If $\Delta>0$ and $f''_{xx}(x_0;y_0)>0$, then $f(x_0;y_0)$ is ***a local minimum***.

(b) If $\Delta>0$ and $f''_{xx}(x_0;y_0)<0$, then $f(x_0;y_0)$ is ***a local maximum***.

(c) If $\Delta<0$, then $f(x_0;y_0)$ is not a local maximum or minimum.

Note 1 In case (c) the point $(x_0;y_0)$ is called a ***saddle point*** of f and the graph of f crosses its tangent plane at $(x_0;y_0)$.

Note 2 If $\Delta=0$, the test gives no information: f could have a local maximum or local minimum at $(x_0;y_0)$, or $(x_0;y_0)$ could be a saddle point of f.

Note 3 If f is a function of three variables, Second Derivatives Test has the following form. Let M_0 be a critical point of f. We will compose the so-called ***Hesse matrix***

$$H(M_0)=\begin{pmatrix} f''_{xx}(M_0) & f''_{xy}(M_0) & f''_{xz}(M_0) \\ f''_{yx}(M_0) & f''_{yy}(M_0) & f''_{yz}(M_0) \\ f''_{zx}(M_0) & f''_{zy}(M_0) & f''_{zz}(M_0) \end{pmatrix}.$$

The basic minors of the matrix can be denoted by

$$\Delta_1=f''_{xx}(M_0),\qquad \Delta_2=\begin{vmatrix} f''_{xx}(M_0) & f''_{xy}(M_0) \\ f''_{yx}(M_0) & f''_{yy}(M_0) \end{vmatrix},\qquad \Delta_3=\det H(M_0).$$

(a) If $\Delta_1>0,\ \Delta_2>0,\ \Delta_3>0$, then $f(M_0)$ is ***a local minimum***.

(b) If $\Delta_1<0,\ \Delta_2>0,\ \Delta_3<0$, then $f(M_0)$ is ***a local maximum***.

Example 1 Find the local maximum and minimum values and saddle points of

$$z(x,y)=2x^3-12xy+3y^2-18x-6y+3.$$

Solution We first locate the critical points:

$$z'_x=\left(2x^3-12xy+3y^2-18x-6y+3\right)'_x=6x^2-12y-18;$$

$$z'_y=\left(2x^3-12xy+3y^2-18x-6y+3\right)'_y=-12x+6y-6.$$

Setting these partial derivatives equal to 0, we obtain the equations $\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases}$.

$$\begin{cases} 6x^2 - 12y - 18 = 0 \\ -12x + 6y - 6 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - 2y - 3 = 0 \\ -2x + y - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - 2(2x+1) - 3 = 0 \\ y = 2x + 1 \end{cases} \Leftrightarrow \begin{cases} x^2 - 4x - 5 = 0, \\ y = 2x + 1. \end{cases}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4\cdot(-5)}}{2} = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm \sqrt{36}}{2} = \frac{4 \pm 6}{2};$$

$$\begin{cases} x = \dfrac{4-6}{2} = -1 \\ y = 2\cdot(-1) + 1 = -1 \end{cases} \quad \text{or} \quad \begin{cases} x = \dfrac{4+6}{2} = 5, \\ y = 2\cdot 5 + 1 = 11. \end{cases}$$

The two critical points are $M_1(-1;-1)$ and $M_2(5;11)$.

Next we calculate the second partial derivatives and Δ:

$$z''_{xx} = (z'_x)'_x = \left(6x^2 - 12y - 18\right)'_x = 6\left(x^2\right)'_x - 0 - 0 = 12x;$$

$$z''_{xy} = (z'_x)'_y = \left(6x^2 - 12y - 18\right)'_y = 0 - 12(y)'_y - 0 = -12;$$

$$z''_{yx} = (z'_y)'_x = (-12x + 6y - 6)'_x = -12(x)'_x + 0 - 0 = -12;$$

$$z''_{yy} = (z'_y)'_y = (-12x + 6y - 6)'_y = 0 + 6(y)'_y - 0 = 6.$$

$$\Delta = \begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{vmatrix} = z''_{xx}\cdot z''_{yy} - z''_{xy}\cdot z''_{yx} = 12x\cdot 6 - (-12)\cdot(-12) = 72x - 144.$$

Since $\Delta\big(M_1(-1;-1)\big) = 72\cdot(-1) - 144 = -216 < 0$, it follows from case (c) of the Second Derivatives Test that the point $M_1(-1;-1)$ is a saddle point; that is, $z(x,y)$ has no local maximum or minimum at $M_1(-1;-1)$.

Since $\Delta\big(M_2(5,11)\big) = 72\cdot 5 - 144 = 216 > 0$ and $z''_{xx}(M_2) = z''_{xx}(5;11) = 12\cdot 5 = 60 > 0$, we see from case (a) of the test that $z_{\min}(x;y) = z(5;11) = -200$ is a local minimum.

Definition The extremum of a function $z = f(x,y)$ found under a condition $\varphi(x,y) = 0$ is called a ***conditional extremum***. An equation $\varphi(x,y) = 0$ is called ***constraint (side) equation.***

If the constraint equation $\varphi(x,y) = 0$ is solvable with respect to x or y, then the problem of finding the conditional extremum is reduced to finding the extremum of a function of single variable.

If the constraint equation is not solvable with respect to its variables, then they form the so-called ***Lagrange function***, which is investigated for an extremum.

Theorem Let f and φ be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $\varphi(x,y)=0$. Suppose that f, when restricted to points on the curve $\varphi(x,y)=0$, has a local extremum at the point $(x_0;y_0)$ and that $grad\,\varphi(x_0,y_0)\neq 0$. Then there is a number λ called a ***Lagrange multiplier***, for which

$$grad\,f(x_0,y_0)=\lambda\cdot grad\,\varphi(x_0,y_0).$$

If we introduce a ***Lagrange function*** $F(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$ then the vector equation $grad\,f(x,y)=\lambda\cdot grad\,\varphi(x,y)$ in terms of its components can be rewritten by a system of three equations in the three unknowns:

$$\begin{cases} F'_x(x,y,\lambda)=0,\\ F'_y(x,y,\lambda)=0,\\ F'_\lambda(x,y,\lambda)=0. \end{cases} \tag{1}$$

Theorem Let $M_0(x_0,y_0,\lambda_0)$ be a solution of the system (1) and

$$d^2F(x,y)=F''_{xx}(x,y,\lambda)dx^2+2F''_{xy}(x,y,\lambda)dxdy+F''_{yy}(x,y,\lambda)dy^2.$$

(a) If $d^2F(M_0)<0$, then the function $z=f(x,y)$ has a local maximum at the point $(x_0;y_0)$.

(b) If $d^2F(M_0)>0$, then the function $z=f(x,y)$ has a local minimum at the point $(x_0;y_0)$.

(c) If $d^2F(M_0)=0$, then the test gives no information.

Example 2 Find the extreme values of the function $z=16-10x-24y$ on the circle $x^2+y^2=169$.

Solution We form the Lagrange function

$$F(x,y,\lambda)=16-10x-24y+\lambda(x^2+y^2-169).$$

Setting partial derivatives equal to 0, we obtain the equations

$$\begin{cases} F'_x(x,y,\lambda)=0,\\ F'_y(x,y,\lambda)=0,\\ F'_\lambda(x,y,\lambda)=0. \end{cases} \Leftrightarrow \begin{cases} -10+2x\lambda=0,\\ -24+2y\lambda=0,\\ x^2+y^2-169=0. \end{cases} \Leftrightarrow \begin{cases} x=5/\lambda,\\ y=12/\lambda,\\ \dfrac{25}{\lambda^2}+\dfrac{144}{\lambda^2}=169. \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda^2=1,\\ x=5/\lambda,\\ y=12/\lambda. \end{cases} \Leftrightarrow \left[\begin{array}{l} \lambda_1=-1,\ x_1=-5,\ y_1=-12.\\ \lambda_2=1,\ x_2=5,\ y_2=12. \end{array} \right.$$

We find a differential of the second order:

$$d^2F(x,y)=F''_{xx}(x,y,\lambda)dx^2+2F''_{xy}(x,y,\lambda)dxdy+F''_{yy}(x,y,\lambda)dy^2;$$

$$F''_{xx}=2\lambda,\ \ F''_{xy}=0,\ \ F''_{yy}=2\lambda\ \ \Rightarrow\ d^2F(x,y)=2\lambda(dx^2+dy^2).$$

We determine the sign of the second differential at the stationary points $M_1(-5;-12)$ and $M_2(5;12)$.

$$d^2F(M_1) = -2(dx^2 + dy^2) < 0 \;\Rightarrow\; z_{\max} = z(M_1) = 354.$$

$$d^2F(M_2) = 2(dx^2 + dy^2) > 0 \;\Rightarrow\; z_{\min} = z(M_2) = -322.$$

If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum value $f(x_1;y_1)$ and an absolute minimum value $f(x_2;y_2)$ at some points $(x_1;y_1)$ and $(x_2;y_2)$ in D.

We have the following extension of the Closed Interval Method.

To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D :

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 3 Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y)\colon 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so the last theorem tells us there are both an absolute maximum and an absolute minimum. According to step 1, we first find the critical points. These occur when

$$\begin{cases} f'_x = 2x - 2y, \\ f'_y = -2x + 2, \end{cases}$$

so the only critical point is $(1;1)$, and the value of it is $f(1;1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of four line segments L_1, L_2, L_3, L_4, shown in Figure 2.2.

On L_1 we have $y = 0$ and $f(x,0) = x^2$, $0 \le x \le 3$.

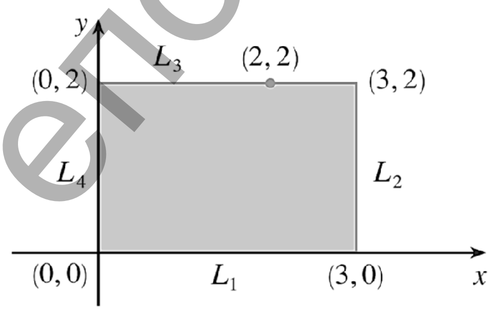


Figure 2.2

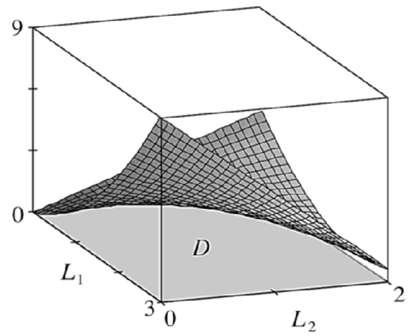


Figure 2.3

This is an increasing function of x, so its minimum value is $f(0;0) = 0$ and its maximum value is $f(3;0) = 9$. On L_2 we have $x = 3$ and $f(3,y) = 9 - 4y$, $0 \le y \le 2$.

This is a decreasing function of y, so its maximum value is $f(3;0)=9$ and its minimum value is $f(3;2)=1$. On L_3 we have $y=2$ and $f(x,0)=x^2-4x+4, 0\le x\le 3$.

By observing that $f(x,0)=(x-2)^2$, we see that the minimum value of this function is $f(2;2)=0$ and the maximum value is $f(0;2)=4$. Finally, on L_4 we have $x=0$ and $f(0,y)=2y$, $0\le y\le 2$ with maximum value $f(0;2)=4$ and minimum value $f(0;0)=0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1;1)=1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3;0)=9$ and the absolute minimum value is $f(0;0)=f(2;2)=0$. Figure 2.3 shows the graph of f.

Exercise Set 2.4

In exercise 1 to 8 Find the local maximum and minimum values and saddle points of the given function.

1. $z=x^3+y^2-6xy-39x+18y+20$

2. $z=x^3+3xy^2-15x-12y+3$

3. $u=x^2+y^2+z^2-4x+6y-2z$

4. $2x^2+2y^2+z^2+8yz-z+8=0$

5. $z=y\sqrt{x}-2y^2-x+14y-2$

6. $z=x\sqrt{y}-x^2-y+6x-3$

7. $x^2+y^2+z^2-4x-2y-4z-7=0$

8. $z=x^2+2xy-y^2-4x+2$

In exercise 9 to 12 use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

9. $z=2x^3+y^2\cdot(1-x)$, $x+y=2$

10. $z=16-10x-24y$, $x^2+y^2=169$

11. $z=\dfrac{1}{x}+\dfrac{1}{y}$, $x+y=2$

12. $z=x^2+xy+y^2-5x-4y+10$, $x+y=4$

In exercise 13 to 15 find the absolute maximum and minimum values of f on the set $\bar{D}$.

13. $z=x^2-2y^2+4xy-6x+5;\ \bar{D}: x=0,\ y=0,\ x+y=3$

14. $z=4(x-y)-x^2-y^2;\ \bar{D}: x+2y=4,\ x-2y=4,\ x=0$

15. $z=x^2-y^2+2xy-4x;\ \bar{D}: x-y+1=0,\ y=0,\ x=3$

Individual Tasks 2.4

1. Find the local maximum and minimum values and saddle points of the given function.

2. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

3. Find the absolute maximum and minimum values of f on the set $\bar{D}$.

I.	**II.**
1. $z=3x^3+3y^3-9xy+6$	*1.* $z=x^3+8y^3-6xy+5$

2. $z = x^{-1} - y^{-1}$, $4x - y = 1$	2. $z = x + 2y$, $x^2 + y^2 = 5$
3. $z = x^2 + 2xy - y^2 - 4x + 2$;	3. $z = x^2 + 2xy - 4x + 8y$;
$\overline{D}: y = x + 2,\ x = 4,\ y = 0$	$\overline{D}: x = 0,\ y = 0,\ x = 1,\ y = 2$

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Functions of a Single and Several Variables

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по дисциплине **«Математика»**

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