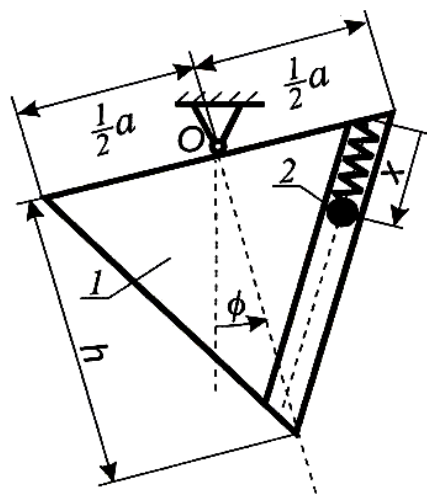# DYNAMICS OF RELATIVE MOTION OF THE MATERIAL POINT

TASKS AND METHODICAL INSTRUCTIONS
to performance calculated graphic work on a course
on theoretical mechanics
for students of specialty
1 – 70 02 01 «Industrial and civil engineering»

The theoretical mechanics is one of fundamental engineering disciplines. For fixing of theoretical material and acquiring of skills of engineering calculations students perform calculated graphic work on the main sections of a course.

The present methodical instructions contain the individual tasks for performance of calculated graphic work on a subject: «Dynamics of the relative motion of a material point», basic data and example of the solution. The requirements how to write of calculated graphic work are provided in instructions as well.

The main purpose of methodical instructions assistance to help the students in studying theoretical mechanics, to activate the individual work.

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CONTENT

INTRODUCTION

In the work the qualified engineer has to possess elements of fundamental knowledge of all-technical disciplines. The theoretical mechanics belongs to such disciplines. Tasks and methodical instructions are one of important sections of the course devoted to studying of the motion of a material point in noninertial reference systems. Methodical instructions allow the students to study and apply theoretical material to the solution of the practical tasks of dynamics of the relative motion of a material point.

REQUIREMENTS TO WRITING OF CALCULATED GRAPHIC WORK

1. CGW is carried out on single sheets of the A4 format.
2. The order of writing the CGW: title page; a task with the indication of initial data and schemes; the text of calculations with necessary explanations and calculated schemes; conclusion; bibliography.
3. Drawings and schemes are carried out with according to the rules of graphics and scales according to the standard of «BrSTU».
4. A text part is carried out according to requirements to fulfillment of text documents. Pages are numbered. Calculations are carried out in a general view, values of parameters are substituted. Numerical results with the indication of dimensions of the received values written down. All calculations are made to within a 100-th (hundredths) of a unit.

SHORT THEORETICAL DATA

The relative motion of a material point is called the motion of a point in the mobile system of coordinates. Let $O_1x_1y_1z_1$ fixed (motionless) system of coordinates; $Oxyz$ moving system of coordinates (figure 1).

The second law of dynamics in the $O_1x_1y_1z_1$ system:

$$m\overline{a} = \sum_{k=1}^{n} \overline{F_k}, \qquad (1)$$

where $\overline{a}$ is absolute acceleration of a point, which is equal to the geometrical sum of transport $\overline{a}_e$, relative $\overline{a}_r$ and Coriolis $\overline{a}_C$ accelerations, i.e.:

$$\overline{a} = \overline{a}_e + \overline{a}_r + \overline{a}_C. \qquad (2)$$
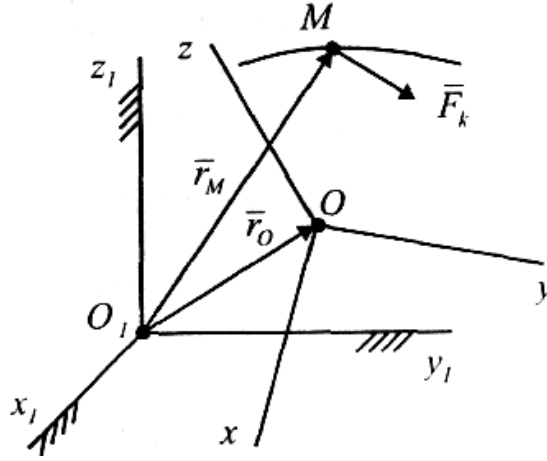
Figure 1 – System of coordinates

Having substituted (1) in (2), we will receive the second law of dynamics in the *Oxyz* system, i.e. in a noninertial reference system:

$$m\overline{a}_r = \sum_{k=1}^{n}\overline{F}_k + \overline{\Phi}_e + \overline{\Phi}_C, \qquad (3)$$

where $\overline{\Phi}_e = -m\overline{a}_e$ and $\overline{\Phi}_C = -m\overline{a}_C$ have dimension of force and are called transport and Coriolis forces of inertia.

Projecting the equation (3) on the moving axes of *Oxyz*, we will receive the differential equations of the relative movement of a material point in the Cartesian axes:

$$m\ddot{x} = \sum F_{kx} + \Phi_{ex} + \Phi_{C_x};$$
$$m\ddot{y} = \sum F_{ky} + \Phi_{ey} + \Phi_{C_y};$$
$$m\ddot{z} = \sum F_{kz} + \Phi_{ez} + \Phi_{C_z}, k = 1,...,n. \qquad (4)$$

The equation (4) can also be written down in projections to natural axes.

If the point in moving system does not move, then $\overline{a}_r = 0$, $\overline{V}_r = 0$ and $\overline{\Phi}_C = -2m(\overline{\omega}_e \times \overline{V}_r) = 0$.

where ω_e – the angular velocity of transport rotation, V_r – the relative velocity of a point. Then (3) will take a form of the equation of relative rest of a point:

$$\sum_{k=1}^{n}\overline{F}_k + \overline{\Phi}_e = 0. \qquad (5)$$

For example, we will consider the body which is does not move upon the land surface (figure 2). The body is affected by forces: $\overline{P}$ – a body attractive force to the center of earth, $\overline{\Phi}_e^u$ – centrifugal force of inertia and $\overline{N}$ – normal reaction. Forces $\overline{P}$ also $\overline{\Phi}_e^u$ causes the body pressure upon a land surface of earth and their equally effective $\overline{G}$ represents the body weight, i.e. $\overline{G} = \overline{P} + \overline{\Phi}_e^u$.
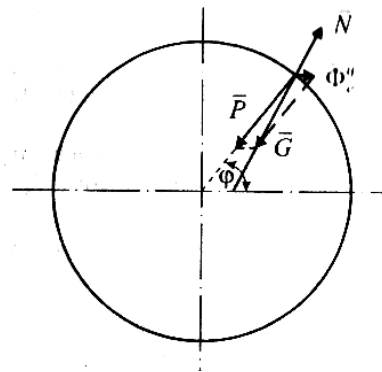
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Figure 2

Therefore at the solution of problems of a statics, accepting the system of coordinates connected with earth for fixed, any corrections because of rotation of earth it isn't entered. At motion of bodies on the land surface or near it with some relative velocity $\overline{V}_r$, there will be (will appear) a Coriolis acceleration and force of inertia $\Phi_C = 2m\omega_e V_r \sin\left(\overline{\omega}_e, \overline{V}_r\right)$ (figure 3) corresponding to it.
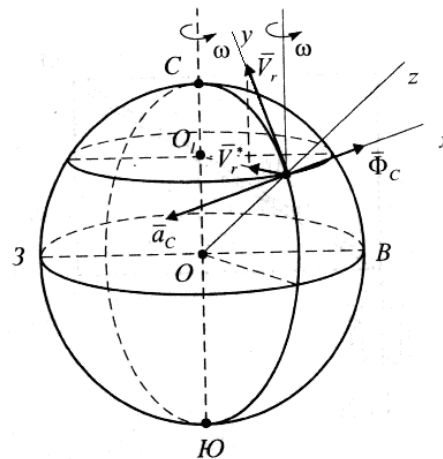

Figure 3

In the northern hemisphere at the motion of a body force $\overline{\Phi}_C$ seek to reject a body from motion direction to the right. Lateral pressure of the train on rails, undermining of the right coast of the rivers, a deviation from a vertical to the East of freely falling body is explained by it.

Concerning the reference system connected with Earth (fixed) we eliminate forces acting on a moving body (Coriolis force of inertia) that isn't always justified.

THE TASK FOR PERFORMANCE
IS CALCULATED GRAPHIC WORKS

The point (ball) of M attached to the spring fixed in a point O, moves on the cylindrical channel of a body of A. Body of A evenly rotates $\left(\omega = const\right)$ around a fixed horizontal axle x_1 or a vertical axis of z_1. In timepoint $t = 0$ (moment of time) the point which was in rest begins the motion under the set conditions x_0, V_0.

At a preset value of angular velocity to receive:

1. Law of the relative motion of a point *x=x(t)*. To determine amplitude *a* and the period of T_1 of own fluctuations, and also the beats period of T_2.

2. To determine coordinate *x* and the velocity of the point *V* and also point pressure *N* upon a wall of channel in the set timepoint $t = t_1$.

3. To plot diagrams of *x=x (t)* for intervals of *[0,T₁]* and *[0,T₂]*.

4. To plot the diagram *V=V(t)* and *N=N(t)* for an interval *[0,T₁]*.

5. To define at what values of angular velocity $\omega_{\kappa p}$ the motion of a point stops being oscillatory and the point makes the non-periodic motion.

6. To accept value of angular velocity $\omega \geq \omega_{\kappa p}$ for the non-periodic motion and to find the equation of the motion of a point *x*=x*(t)*. To determine passing time by a point of all channel (graphically).

In a task the following designations are accepted: *m* – mass of a point; ω – constant angular velocity of a body of *A*; *c* – a spring rigidity coefficient; l_0 – length of not deformed spring; x_0 – the initial coordinate of a point; V_0 – the initial velocity of a point; t_1 – time for which necessary calculations are made.

For schemes with numbers No. 1, 2, 4, 8, 9, 11, 12, 13, 14, the size *a* bodies *A*, radius *r* of a body *A* in the scheme No. 7, length *l* (rod *A*) in the scheme No. 15, or the size *h* of a body *A* in schemes No. 3, 5, 6, 10 to determine by a formula $a, R, l, h = 5l_0$

Numerical data for each variant choose according to table 1. At the same time (herewith) time of t_1 is set it (is specified) the teacher. Schemes choose according to index number of the student in group (number of the student according to the teacher's magazine) on fig. 4.

Table 1 – Numerical data

№ of items	m, kg	ω, rad/s	c, N/m	l_0, m	x_0, m	V_0, m/s	α, deg
1	2	3	4	5	6	7	8
1	0,01	9	1	0,05	0,01	0	-
2	0,02	8	1,5	0,06	0,02	0,01	-
3	0,03	7,5	2	0,07	0,03	0,02	-
4	0,04	7	2,5	0,08	0,04	0,03	30
5	0,05	6,5	3	0,09	0,05	0,04	-
6	0,06	6	3,5	0,1	0,15	0,05	45
7	0,07	5,5	4	0,11	0,16	0,06	-
8	0,08	5	4,5	0,12	0,17	0,07	30
9	0,09	4,5	5	0,13	0,18	0,08	-
10	0,1	7	6	0,15	0,19	0,09	45
11	0,11	7,2	7	0,16	0,2	0,1	-
12	0,12	7,4	8	0,17	0,22	0,15	60
13	0,13	7,6	9	0,18	0,23	0,2	-
14	0,14	7,8	10	0,19	0,24	0,25	-
15	0,15	8	11	0,2	0,25	0,3	45
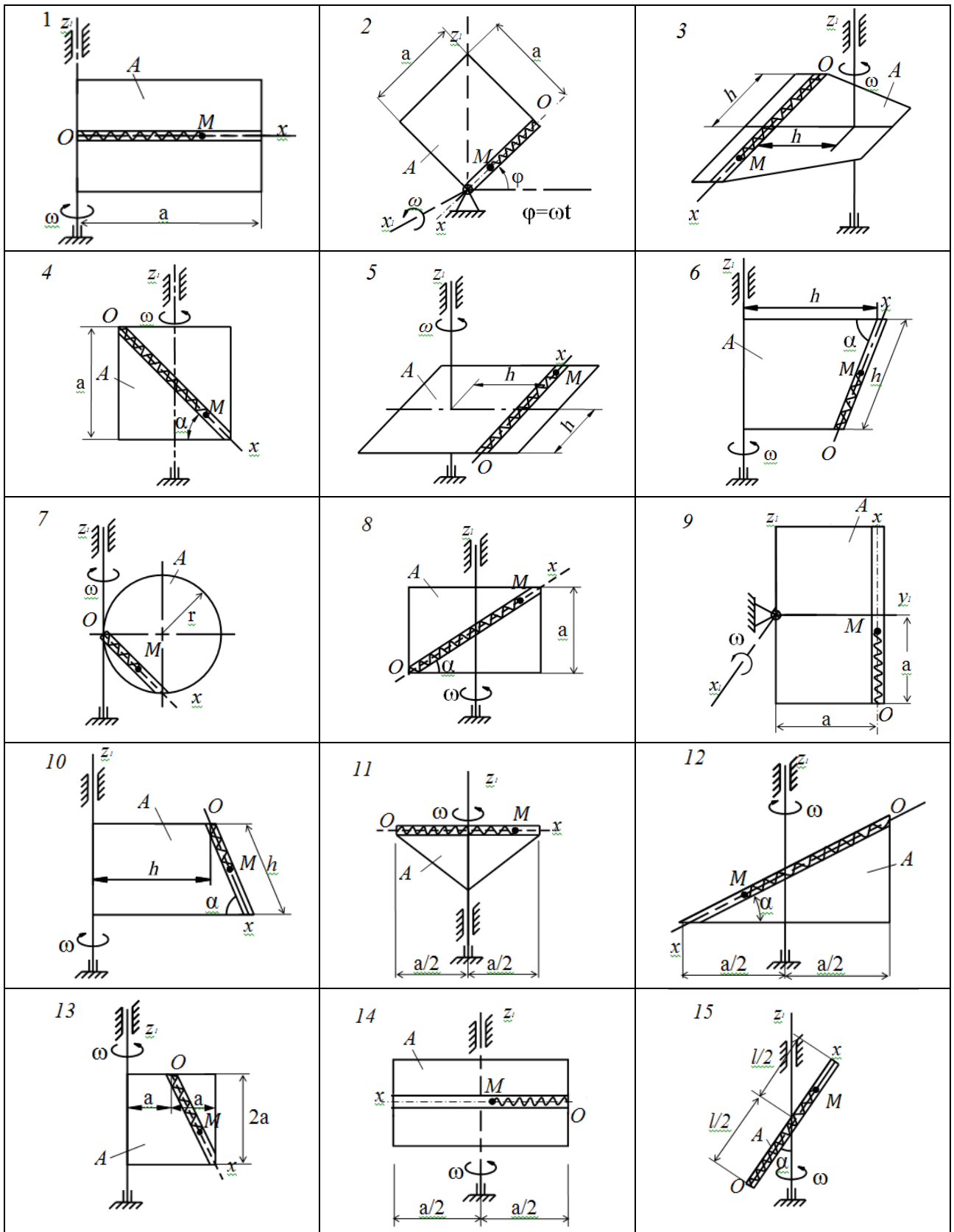
Figure 4 – Initial schemes

EXAMPLE OF PERFORMANCE OF THE TASK
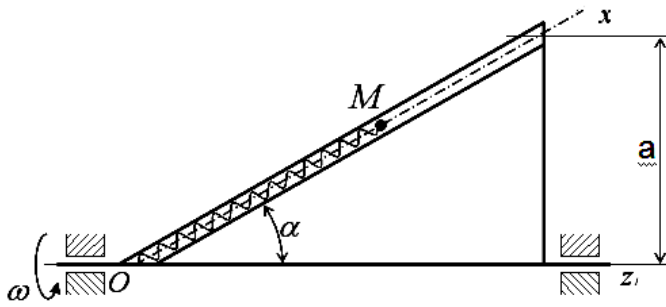
$m=0,009\ kg$
$\omega=9\ rad/s$
$C=0,1\ N/m$
$l_0=0,2\ m$
$\alpha=30^o$
$x_0=0,9\ m$
$V_0=0,7\ m/s$
$t_1=0,16\ s$

Figure 5. – Initial scheme
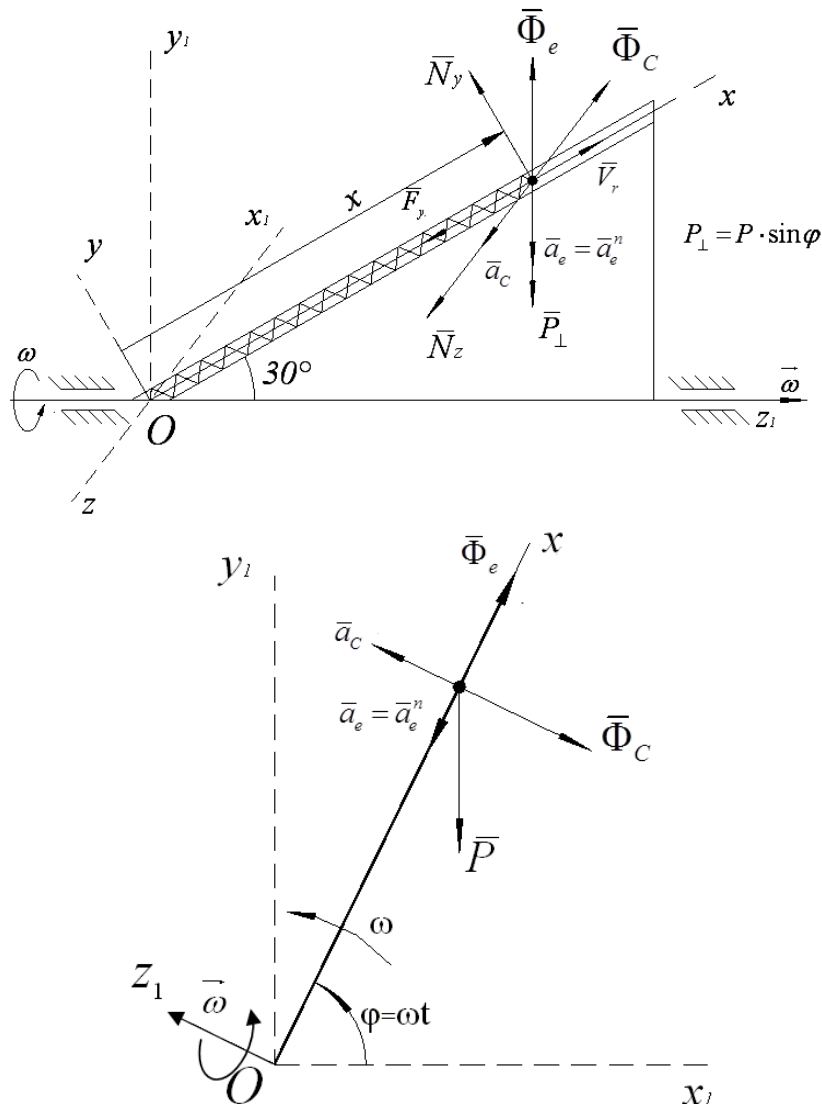
Solution:

Figure 6. – Calculated scheme

Rotation of a plate (lamella) **A** around an axis of z_1 impart (convey) to the **M** point the transport motion. The motion of a point on the channel of a lamella - the relative motion. We connect a moving system of coordinates **Oxyz** with a lamella (the

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origin of coordinates in a point **O** fastenings of a spring). We will direct an axis **x** along an axis of the channel. An axis **y** – in the drawing plane.

We apply the weight $\overline{P} = m\overline{g}$, elastic force of a spring $\overline{F}_y$ and reaction $\overline{N}$ of a wall of the channel to a point **M**. As a walls smooth (there is no friction), then $\overline{N}$ lies in the plane a perpendicular axis **x** (**Oyz** plane). The direction of it is unknown therefore we resolution $\overline{N}$ to components $\overline{N}y, \overline{N}z$ in the direction of axes **y, z**.

As **Oxyz** noninertial frame of reference (the lamella rotates), for a research of the relative motion of a point to forces acting on a point, we add transport $\overline{\Phi}_e$ and Coriolis $\overline{\Phi}_C$ forces of inertia.

$$\overline{\Phi}_e = -m\overline{a}_e; \ \ \overline{\Phi}_C = -m\overline{a}_C; \ \ \overline{F}_y = c(x - l_0).$$

The vector $\overline{a}_C$ is perpendicular the planes of vectors $\overline{\omega}_e$ and $\overline{V}_r$, that is, is parallel to an axis **z**.

$$a_C = 2\omega_e \cdot V_r \cdot \sin 30°; \ \ \omega_e = \omega; \ \ V_r = \dot{x}$$

$$a_e^n = \omega^2 x \sin \alpha$$

We work out the differential equations of the relative motion of a point in projections to **Oxyz** axes:

$$\begin{cases} m\ddot{x} = -P\sin\varphi\sin\alpha - F_y + \Phi_e\sin\alpha; \\ 0 = N_y - P\sin\varphi\cos\alpha + \Phi_e\cos\alpha; \\ 0 = N_z - P\cos\varphi - \Phi_C; \end{cases} \quad (*)$$

At the same time it is considered that $a_{r.y} = 0, a_{r.z} = 0$ (the relative motion takes place only along an axis **x**). From the second and third equations reactions $\overline{N}y, \overline{N}z$ of a wall of the channel (to a point) are defined.

For definition $\overline{\Phi}_e$ and $\overline{\Phi}_C$ it is also necessary to find the solution of the first equation of system. It takes a form:

$$m\ddot{x} = -mg\sin\alpha\sin\omega t - c(x - l_0) + m\omega^2 x \sin^2\alpha$$

or

$$\ddot{x} + \left(\frac{c}{m} - \omega^2 \sin^2\alpha\right)x = -g\sin\alpha\sin\omega t + \frac{cl_0}{m}.$$

At the set numerical values

$$\left(\frac{c}{m} - \omega^2 \sin^2\alpha\right) = \frac{0,1 \cdot 100}{0,09} - 9^2 \cdot 0,5^2 = 90,56 \ c^{-2}$$

As $\left(\dfrac{c}{m} - \omega^2 \sin^2\alpha\right) > 0$, designate

$$\left(\frac{c}{m} - \omega^2 \sin^2\alpha\right) = k^2; \ -g\sin\alpha = A^*; \ \frac{cl_0}{m} = D^*.$$

At the same time

$$k = \sqrt{90{,}561} = 9{,}53 \; s^{-1};$$

$$A^* = -9{,}81 \cdot 0{,}5 = -4{,}91 \; m/s^2;$$

$$D^* = \frac{10 \cdot 0{,}2}{0{,}09} = 22{,}22 \; m/s^2.$$

Then the equation takes a form:

$$\ddot{x} + k^2 x = A^* \sin(\omega t) + D^*.$$

It is the ordinary heterogeneous linear differential equation of the 2nd order with constant coefficients.

Its common solution has an appearance:

$$x = x_1 + x_2,$$

where x_1 – the common solution of the uniform equation, x_2 – the particular solution of the heterogeneous equation.

The characteristic equation for this uniform differential has an appearance:

$$\lambda^2 + k^2 = 0.$$

Its roots

$$\lambda_{1,2} = \pm ki.$$

As roots imaginary, the common solution x_1 has an appearance:

$$x_1 = C_1 \cos(kt) + C_2 \sin(kt).$$

We look for the particular solution in a view:

$$x_2 = A\sin(\omega t) + D.$$

Substituting the particular solution in the differential equation, we will receive:

$$-A\omega^2 \sin(\omega t) + k^2 A \sin(\omega t) + k^2 D = A^* \sin(\omega t) + D^*$$

or

$$(-\omega^2 + k^2) A \sin(\omega t) + k^2 D = A^* \sin(\omega t) + D^*.$$

The particular solution will satisfy to the differential equation if:

$$(-\omega^2 + k^2) A = A^* \quad \text{and} \quad k^2 D = D^*.$$

From here

$$A = \frac{A^*}{k^2 - \omega^2} = \frac{-4{,}91}{9{,}53^2 - 9^2} = -0{,}5 \; m,$$

$$D = \frac{D^*}{k^2} = \frac{22{,}22}{9{,}53^2} = 0{,}245 \; m.$$

The equation of the relative motion of a point receives a form:

$$x = C_1 \cos kt + C_2 \sin(kt) + A\sin(\omega t) + D..$$

Point velocity

$$V = \frac{dx}{dt} = -kC_1 \sin(kt) + kC_2 \cos(kt) + A\omega\cos(\omega t).$$

We define constants of integration of C_1 and C_2 from initial conditions:
at $t = 0$: $\quad x_0 = 0,9\,m, \quad \dot{x}_0 = 0,7\,m/s.$

$$\begin{cases} 0,9 = C_1 + 0,245, \\ 0,7 = 9,53 \cdot C_2 - 0,5 \cdot 9. \end{cases}$$

From here

$$C_1 = 0,9 - 0,245 = 0,655\,m;$$

$$C_2 = \frac{0,7 + 0,5 \cdot 9}{9,53} = 0,546\,m/s.$$

Then

$$x = 0,655\cos(9,53t) + 0,546\sin(9,53t) - 0,497\sin(9t) + 0,245;$$

$$\dot{x} = -6,242\sin(9,53t) + 5,203\cos(9,53t) - 4,736\cos(9t).$$

We substitute the common solution of the uniform equation in an amplitude form for what we enter (introduce) the following substitution:

$$C_1 = a\sin\beta; \quad C_2 = a\cos\beta.$$

Then the solution take a form:

$$x = a\sin\beta\cos(kt) + a\cos\beta\sin(kt) = a\sin(kt + \beta).$$

At the same time a – amplitude of fluctuations, $(kt + \beta)$ – a phase of fluctuations, β – an initial phase of fluctuations.
We substitute numerical values:

$$a = \sqrt{C_1^2 + C_2^2} = \sqrt{0,655^2 + 0,546^2} = 0,853\,m;$$

$$\beta = arctg\frac{C_1}{C_2} = arctg\frac{0,655}{0,546} = 0,846\,rad = 50,2°.$$

Law of the relative motion of a point:

$$x = 0,853\sin(9,53t + 0,846) - 0,497\sin(9t) + 0,245;$$

$$\dot{x} = 8,114\cos(9,53t + 0,846) - 4,736\cos(9t).$$

We determine coordinate x and velocity $\dot{x}$ in timepoint $t = t_1 = 0,16\,s.$

$$x = 0,34\,m; \qquad \dot{x} = -6,58\,m/s.$$

We build diagrams of a point in time $0 \le t \le T_1$. Where the period of fluctuations of T_1 is determined by a formula: $T_1 = 2 \cdot \dfrac{\pi}{k}\,s.$

a)

$x = 0,34\ m$

0.5

$x(t)$

$t_1 = 0,16\ s$

0 0.1 0.2 0.3 0.4 0.5 0.6

-0.5

t

b)

5

$t_1 = 0,16\ s$

$\upsilon(t)$

0 0.1 0.2 0.3 0.4 0.5
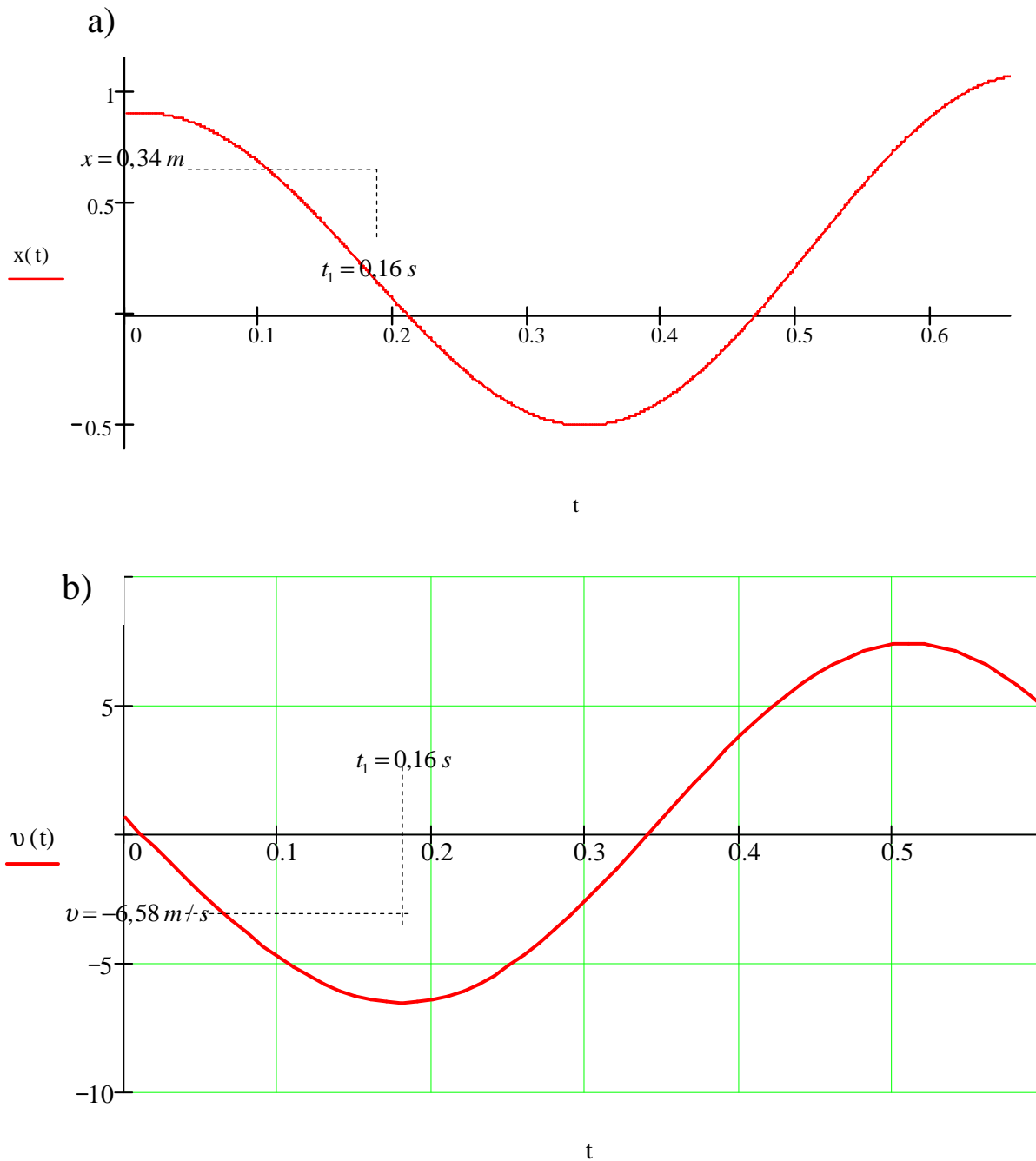
$\upsilon = -6,58\ m/\!\cdot\! s$

-5

-10

t

Figure 7. – Dependence of change of:
a) coordinate on an interval $[0,T_1]$; b) velocity on an interval $[0,T_1]$

We look for the beats period of T_2 on a formula:

$$T_2 = \frac{4\pi}{|\Delta|},$$

where $\Delta = \omega - k;\ k = \sqrt{\dfrac{c}{m} - \omega^2 \cdot \sin^2\alpha}.$

Thus, the beats period or the lower vibratory frequency will be:

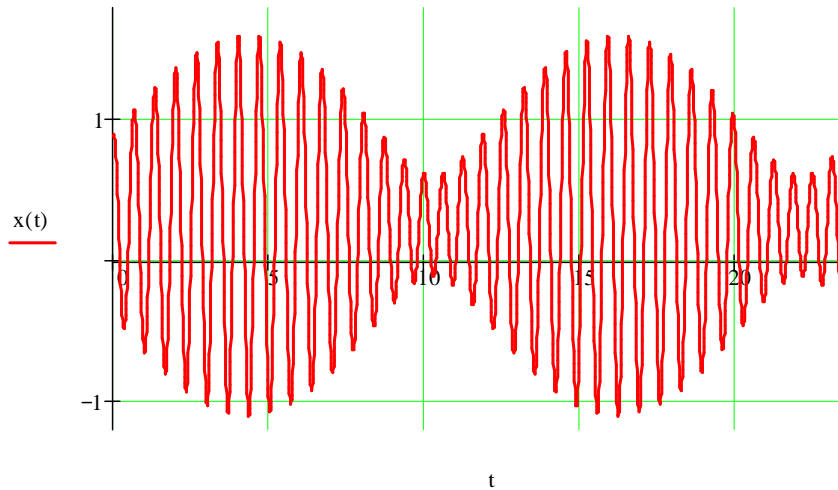$$T_2 = \frac{4\pi}{\left|9 - \sqrt{90,561}\right|} = 24,34\ s.$$

Figure 8. – Dependence of coordinate on time on an interval of time [0, T₂]

From the system equations (*) we define reactions and for timepoint $t_1 = 1.16$ sec.:

$$N_y = 0,09 \cdot 9,81 \cdot \sin(9 \cdot \frac{180}{\pi} \cdot 0,16) \cdot 0,866 - 0,09 \cdot 9^2 \cdot 0,34 \cdot 0,5 \cdot 0,866 = -0,136 \, N;$$

$$N_z = 0,09 \cdot 9,81 \cdot \cos(9 \cdot \frac{180}{\pi} \cdot 0,16) + 0,09 \cdot 2 \cdot 9 \cdot 6,58 \cdot \sin 30° = -4,01 \, N.$$

Reaction of a wall of a channel:

$$N = \sqrt{N_y^2 + N_z^2} = \sqrt{(-0,136)^2 + (-4,01)^2} = 4,011 \, N.$$

We investigate the non-periodic motion.

The motion will be non-periodic if roots of the characteristic equation are valid, that is:

$$\omega^2 \sin^2 \alpha - \frac{c}{m} > 0,$$

or $\omega \geq \sqrt{\dfrac{c}{m \cdot \sin^2 \alpha}}; \quad \omega_{cr} = \sqrt{\dfrac{10}{0,09 \cdot 0,5^2}} = 21,08 \, s^{-1}.$

We accept $\omega = 25 c^{-1} \rangle \omega_{cr}$.

Then roots of the characteristic equation:

$$\lambda_{1,2} = \pm\sqrt{\omega^2 \sin^2 \alpha - \frac{c}{m}} = \pm\sqrt{25^2 \cdot 0,5^2 - \frac{10}{0,09}} = \pm\sqrt{45,14} = \pm 6,72 \, s^{-2}..$$

Denote:

$$\left(\omega^2 \sin^2 \alpha - \frac{c}{m}\right) = k_1^2; \; -g \sin \alpha = A^*; \frac{cl_0}{m} = D^*.$$

$$k_1^2 = 45,14 \, s^{-1}; \quad A^* = -4,91 \, m/s^2; \quad D^* = 22,22 \, m/s^2.$$

The differential equation take a form:

$$\ddot{x} - k_1^2 x = A^* \sin(\omega t) + D^*.$$

We look for the solution of the differential equation in a form:
$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + A \sin(\omega t) + D.$$
At the same time:
$$A = \frac{-A^*}{k_1^2 + \omega^2} = \frac{4,91}{6,45^2 + 25^2} = 0,007\,m;$$
$$D = -\frac{D^*}{k_1^2} = -\frac{22.22}{6,72^2} = -0,492\,m.$$
We define constants C_1 and C_2.
$$\dot{x} = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} + \omega \cdot A \cos(\omega t).$$
We substitute initial conditions:
at $\quad t = 0:\quad x_0 = 0,9\,m, \quad \dot{x}_0 = 0,7\,m/s.$
$$\begin{cases} 0,9 = C_1 + C_2 - 0,492, \\ 0,7 = 6,72 \cdot C_1 - 6,72 \cdot C_2 + 25 \cdot 0,007. \end{cases}$$
From here $\quad C_1 = 0,734\,m, \quad C_2 = 0,658\,m/s.$
So
$$x = 0,734 \cdot e^{6,45t} + 0,658 \cdot e^{-6,45t} + 0,007 \sin(25t) - 0,492,\,m.$$

We determine time t_2 – passing time by a point of all channel. We make calculation one of the existing mathematical packages (MathCAD, Mathematica, etc.) or graphically.

We define time of t_2 graphically.

For this purpose we appoint length of the channel. On a statement of the problem $a=5l_0=1m$.

Length of the channel on which the point on a lamella moves:
s=a/sin(α)=1/0,5=2 m.

We plot the diagram of the non-periodic motion. From the diagram it is visible that the point will leave the channel through $t_2 \sim 0,18$ sec. It is important to note that the solution for $t > t_2$ loses meaning.
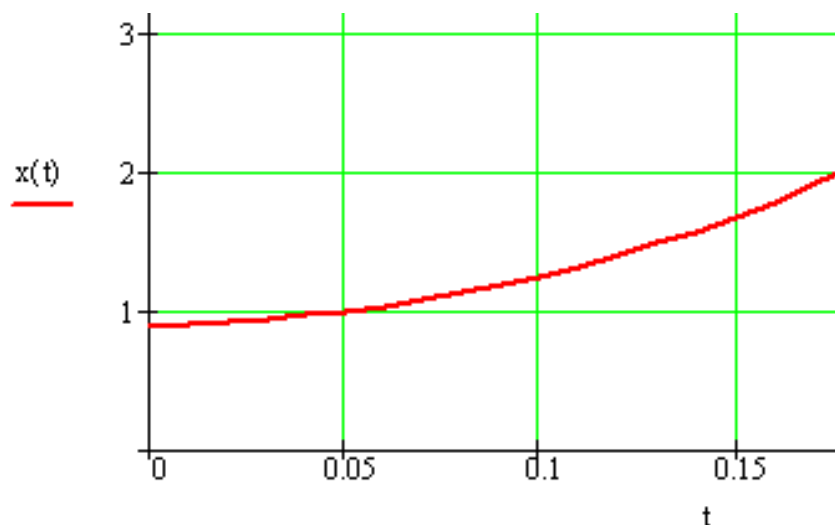

Figure 9. – Diagram of the non-periodic motion on time interval [0, t_2]

LITERATURE

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