помощи известных табличных функций для уравнения Абеля (1), записанного в нормальной форме. Приведена программная реализация такого метода. На примере уравнения Абеля, которое интегрируется при помощи классического метода, построено также аналитическое решение, определенное через специальное кубическое алгебраическое уравнение. Показано, что для нового решения уравнения Абеля (26) и классического решения (25) задача Коши разрешима.

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SHVYCHKINA A.N. Computer realization of the analytical method of integrating the equations of Abel

In this paper we represent a mathematical technique leading to the construction of exact analytic solutions the Abel equation. The examined nonlinear ODEs admit exact analytic solutions in terms of known tabulated functions. The computer method of building a general solution the Abel differential equation and example are considered.

УДК 517.91, 004.9

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COMPUTER CONSTRUCTION OF THE GENERAL SOLUTION OF THE SPECIAL FORM OF THE ABEL DIFFERENTIAL EQUATION

1. Introduction and statement of the problem

In the papers [1, 2] the method of construction of the nonlinear differential equation of the second order of the form

$$a(x)\ y''+b_0(x)y'^3+b_1(x)y'^2+b_2(x)y'+b_3(x)=0\,, \tag{1}$$

the general solution of which has a special form

$$\varphi_3(x)=C_1\varphi_1(x)\cdot exp\left(\lambda_1\ y(x)\right)+C_2\varphi_2(x)\cdot exp\left(\lambda_2\ y(x)\right), \tag{2}$$

where $C_i\left(i=1,2\right)$ are arbitrary constants, $\varphi_j(x)$ $\left(j=1,2,3\right)$ are given twice continuously differentiable functions of variable x; λ_1, λ_2 are given constants was considered. Such problems are classical problems of the theory differential equations. For example, in the paper [3] the following formulation of the problem is given: "setting the form of a differential equation, it is necessary to seek different forms of a general solution of this equation and existence conditions of these forms". This task is interesting and because the equation (1) by substitution

$$y'=z\,, \tag{3}$$

reduces to the Abel equation of the first kind [4]

$$a(x)\ z'+b_0(x)z^3+b_1(x)z^2+b_2(x)z+b_3(x)=0 \tag{4}$$

which plays an important role in the theory of differential equations and its numerous applications [5, 6].

In this article the program listing by which the analytical method is implemented for the two differential equations (1) and (4) is given. We also give a visualization of the obtained partial solutions.

Considered analytical method based on the following two theorems, which have been proven in [1, 2].

Theorem 1. *Equation (1) has a general solution of the form*

$$C_1\cdot exp\left(\lambda_1\ y-\ \int\eta\ dx\right)+C_2\cdot exp\left(\lambda_2\ y+\ \int\xi\ dx\right)=1, \tag{5}$$

if the conditions

$$\frac{1}{b_0}\left[(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)a^2+3(b_0b_2+b_0a'-a\ b_0')-b_1^2\right]=0\,, \tag{6}$$

$$\begin{gathered}(2\ \lambda_1^3-3\lambda_1^2\cdot\lambda_2-3\ \lambda_1\cdot\lambda_2^2+2\ \lambda_2^3)a^3-3\left(\lambda_1^2-\lambda_1\ \lambda_2+\lambda_2^2\right)a^2\cdot b_1+\\+b_1^3+9b_0\ \left(b_1'a-a'\ b_1-3b_0\ b_3\right)=0\end{gathered} \tag{7}$$

fulfilled and the relations

$$\xi=\frac{\lambda_2}{3\ b_0}\left(\ b_1-(\lambda_2-2\lambda_1)\ a\right),\quad \eta=\frac{\lambda_1}{3\ b_0}\left(-\ b_1+(\lambda_1-2\lambda_2)\ a\right) \tag{8}$$

are held.

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Украина, 4300, г. Луцк, пр. Воли, 13.

Theorem 2. *Abel differential equation of the first kind (4) whose coefficients satisfy relations (6) and (7) has a general solution of the form*

$$z = \frac{e^{\lambda_1 y}\eta - C\xi e^{\int \eta\, dx + \int \xi\, dx + \lambda_2 y}}{\lambda_1 e^{\lambda_1 y} + C\lambda_2 e^{\int \eta\, dx + \int \xi\, dx + \lambda_2 y}}, \tag{9}$$

where C is an arbitrary constant, variables y *and* x *connected by the relation (5); functions* ξ, η *determined by the formulas (8).*

2. Description of the solution algorithm for the problem

We introduce the reductions: **ft=Flatten**; **sp=Simplify**; **fs=FullSimplify**;

Let us define the equation (5) as *eq1*

```
eq1=φ3[x] == e^(λ1 y[x]) c1 φ1[x] + e^(λ2 y[x]) c2 φ2[x];
```

Differentiating the relation *eq1* we obtain

```
eq2 = D[eq1,x]//sp
```

$$e^{\lambda_1 y[x]} c_1(\lambda_1\varphi_1[x]y'[x] + \varphi_1'[x]) + e^{\lambda_2 y[x]} c_2(\lambda_2\varphi_2[x]y'[x] + \varphi_2'[x]) == \varphi_3'[x]$$

We solve the linear system of equations *eq1*, *eq2* with respect to unknown c_1 and c_2

```
sol1 = Solve[{eq1,eq2},{c1,c2}]//fl;
```

Differentiating the both sides of equation *eq2* and substitute values c_1, c_2 from *sol1*. As a result we obtain equation (1), which is denoted as *eq3*

```
eq3 = D[eq2,x]/.sol1//fl;
```

Let's find the relation between the coefficients of equations (1) and (2)

```
exp1 = Collect[Numerator[Together[eq3[[1]] − eq3[[2]]]],{y''[x],y'[x],y[x]}];
eq4 = a[x] == Coefficient[exp1,y''[x]]//sp
eq5 = b0[x] == Coefficient[exp1,y'[x],3]//sp
eq6 = b1[x] == Coefficient[exp1,y'[x],2]//sp
eq7 = b2[x] == Coefficient[exp1,y'[x],1]//sp
eq8 = b3[x] == Coefficient[Coefficient[exp1,y'[x],0],y''[x],0]//sp
```

$$a[x] == \lambda_2\varphi_2[x](\varphi_3[x]\varphi_1'[x] - \varphi_1[x]\varphi_3'[x]) + \lambda_1\varphi_1[x](-\varphi_3[x]\varphi_2'[x] + \varphi_2[x]\varphi_3'[x])$$

$$b_0[x] == \lambda_1\lambda_2(-\lambda_1 + \lambda_2)\varphi_1[x]\varphi_2[x]\varphi_3[x]$$

$$b_1[x] == 2\lambda_1\lambda_2\varphi_3[x](-\varphi_2[x]\varphi_1'[x] + \varphi_1[x]\varphi_2'[x]) + \lambda_2^2\varphi_2[x](\varphi_3[x]\varphi_1'[x] - \varphi_1[x]\varphi_3'[x]) + \lambda_1^2\varphi_1[x](-\varphi_3[x]\varphi_2'[x] + \varphi_2[x]\varphi_3'[x])$$

$$b_2[x] == \lambda_1(\varphi_3[x](-2\varphi_1'[x]\varphi_2'[x] + \varphi_1[x]\varphi_2''[x]) + \varphi_2[x](2\varphi_1'[x]\varphi_3'[x] - \varphi_1[x]\varphi_3''[x])) + \lambda_2(\varphi_3[x](2\varphi_1'[x]\varphi_2'[x] - \varphi_2[x]\varphi_1''[x]) + \varphi_1[x](-2\varphi_2'[x]\varphi_3'[x] + \varphi_2[x]\varphi_3''[x]))$$

$$b_3[x] == \varphi_3[x](-\varphi_2'[x]\varphi_1''[x] + \varphi_1'[x]\varphi_2''[x]) + \varphi_2[x](\varphi_3'[x]\varphi_1''[x] - \varphi_1'[x]\varphi_3''[x]) + \varphi_1[x](-\varphi_3'[x]\varphi_2''[x] + \varphi_2'[x]\varphi_3''[x])$$

To simplify these relations we introduce the functions

$$\xi = \frac{\varphi_2'}{\varphi_2} - \frac{\varphi_3'}{\varphi_3},\quad \eta = \frac{\varphi_3'}{\varphi_3} - \frac{\varphi_1'}{\varphi_1},\quad \psi = \varphi_1\varphi_2\varphi_3. \tag{10}$$

```
sys1 = {ξ[x] == φ2'[x]/φ2[x] − φ3'[x]/φ3[x], η[x] == φ3'[x]/φ3[x] − φ1'[x]/φ1[x]};
sol2 = Solve[sys1,{φ1'[x],φ2'[x]}]//ft;
ρ1 = {φ1[x]φ2[x]φ3[x] → ψ[x]};
eq4n = (eq4/.D[sol2,x]/.sol2//sp)/.ρ1
```

$$a[x] + (\lambda_2\eta[x] + \lambda_1\xi[x])\psi[x] == 0$$

```
eq5n = (eq5/.D[sol2,x]/.sol2//sp)/.ρ1
```

$$b_0[x] == \lambda_1\lambda_2(-\lambda_1 + \lambda_2)\psi[x]$$

```
eq6n = (eq6/.D[sol2,x]/.sol2//sp)/.ρ1
```

$$(\lambda_2^2\eta[x] + \lambda_1^2\xi[x] - 2\lambda_1\lambda_2(\eta[x] + \xi[x]))\psi[x] + b_1[x] == 0$$

```
eq7n = (eq7/.D[sol2,x]/.sol2//sp)/.ρ1
```

$$b_2[x] + \psi[x](\lambda_2(\eta[x]^2 + 2\eta[x]\xi[x] - \eta'[x]) - \lambda_1(2\eta[x]\xi[x] + \xi[x]^2 + \xi'[x])) == 0$$

```
eq8n = (eq8/.D[sol2,x]/.sol2//sp)/.ρ1
```

$$b_3[x] + \psi[x](\eta[x]^2\xi[x] - \xi[x]\eta'[x] + \eta[x](\xi[x]^2 + \xi'[x])) == 0$$

From equation $eq5n$ we find function ψ and substitute it into equations $eq4n$ and $eq6n$. Then from the resulting equations we find functions ξ, η.

```
sol3 = Solve[eq5n,ψ[x]]//ft;
sol4 = Solve[{eq4n,eq6n},{η[x],ξ[x]}]/.sol3//sp//ft
```

$$\left\{\eta[x]\to\frac{\lambda_1(a[x](\lambda_1-2\lambda_2)-b_1[x])}{3b_0[x]},\ \xi[x]\to\frac{\lambda_2(a[x](2\lambda_1-\lambda_2)+b_1[x])}{3b_0[x]}\right\}$$

We substitute these functions into equations $eq7n$, $eq8n$

```
eq9 = eq7n/.sol3/.D[sol4,x]/.sol4//sp
```

$$\frac{a[x]^2(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)-b_1[x]^2+3b_0[x](b_2[x]+a'[x])-3a[x]b_0'[x]}{b_0[x]}==0$$

```
eq10 = eq8n/.sol3/.D[sol4,x]/.sol4//sp
```

$$\frac{1}{b_0[x]}\left(a[x]^3(-2\lambda_1^3+3\lambda_1^2\lambda_2+3\lambda_1\lambda_2^2-2\lambda_2^3)+3a[x]^2(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)b_1[x]-b_1[x]^3+27b_0[x]^2b_3[x]+9b_0[x]b_1[x]a'[x]-9a[x]b_0[x]b_1'[x]\right)==0$$

Equations $eq9$, $eq10$ coincide with the equations (6), (7) and replace rules *sol4* have the form (8).

We check that the relations (5), (8) define the general solution of equation (1) in an implicit form with coefficients satisfying the conditions (6)-(7).

```
int1 = 1 == c1 e^(λ1 y[x] − ∫η[x]dx) + c2 e^(λ2 y[x] + ∫ξ[x]dx);
ρ2 = Solve[{eq9,eq10},{b2[x],b3[x]}]//ft;
ρ3 = Solve[D[int1,x],y'[x]]//ft//sp;
ρ4 = Solve[D[int1,{x,2}]/.ρ3,y''[x]]//ft//sp;
eq11 = a[x]y''[x] + b0[x]y'[x]^3 + b1[x]y'[x]^2 + b2[x]y'[x] + b3[x] == 0;
eq11/.D[ρ2,x]/.ρ2/.ρ3/.ρ4/.D[sol4,x]/.sol4//sp
True
```

We prove that the forms of the general solution of (1) and (5) are connected to each other by the first two equations of the system (10)

```
ρ5 = sys1/.Equal → Rule;
int1/.ρ5//sp
```

$$\frac{e^{\lambda_1 y[x]}c_1\varphi_1[x]+e^{\lambda_2 y[x]}c_2\varphi_2[x]}{\varphi_3[x]}==1$$

Let's turn to the study of the Abel equation (4), which is defined as *eq12*

```
eq12 = eq11/.{y''[x] → z'[x],y'[x] → z[x]}
```

$$z[x]^3b_0[x]+z[x]^2b_1[x]+z[x]b_2[x]+b_3[x]+a[x]z'[x]==0$$

We define the general solution (9) as the *int2*

```
int2 = Solve[D[int1,x]/.y'[x] → z[x],z[x]]//sp//ft
```

$$\left\{z[x]\to\frac{e^{\lambda_1 y[x]}c_1\eta[x]-e^{\int\eta[x]\,dx+\int\xi[x]\,dx+\lambda_2 y[x]}c_2\xi[x]}{e^{\lambda_1 y[x]}c_1\lambda_1+e^{\int\eta[x]\,dx+\int\xi[x]\,dx+\lambda_2 y[x]}c_2\lambda_2}\right\}$$

We check that the replacement rule *int2* determines the general solution of equation *eq12*

```
eq12/.D[int2,x]/.y'[x] → z[x]/.int2/.D[sol4,x]/.sol4/.ρ2//sp
True
```

Remark 1. If you enter a substitution $c_2 = c\cdot c_2$ in the ratio *int2*, there remains only one arbitrary constant.

The other simulations of the general solution for the special Abel differential equation were constructed in [7].

3. Visualization of partial solutions of the Abel equation

Example 1. Let

$$\xi=\frac{2}{x},\ \eta=-\frac{3}{x}. \tag{11}$$

Then equation (9) takes the form

$$z=-\frac{3c_1xe^{\lambda_1 y}+2c_2e^{\lambda_2 y}}{c_1\lambda_1x^2e^{\lambda_1 y}+c_2\lambda_2xe^{\lambda_2 y}}, \tag{12}$$

and the corresponding Abel equation (4) is written as

$$z'=\frac{3\lambda_2^2x^2z^2-\lambda_1^2x^2z^2(\lambda_2xz+2)+\lambda_1xz(\lambda_2^2x^2z^2-2\lambda_2xz-10)+6\lambda_2xz-6}{(2\lambda_1-3\lambda_2)x^2}. \tag{13}$$

In this case equation (5) takes the form

$$x^2\left(c_1 x e^{\lambda_1 y} + c_2 e^{\lambda_2 y}\right) = 1. \qquad (14)$$

Thus, the general solution of the Abel equation (13) is defined by equations (12), (14). Eliminating from equations (12), (14) the variable y we obtain the function relating to the variables x and z

$$\left(\frac{\lambda_1 x z + 3}{c_2 x^2\left((\lambda_1 - \lambda_2) x z + 1\right)}\right)^{\lambda_1} = \left(\frac{\lambda_2 x z + 2}{c_1 x^3\left((\lambda_2 - \lambda_1) x z - 1\right)}\right)^{\lambda_2}. \qquad (15)$$

By using command Manipulate [8], we generate a version of the partial solution (15) with the control added to allow interactive manipulation of the values of $\lambda_1, \lambda_2, c_1, c_2$

```
ap = Appearance → "Labeled";
Manipulate[ContourPlot[
Evaluate[((3 + λ1 x z)/(c1 x^2(1 + x(λ1 − λ2)z)))^λ1 − ((2 + λ2 x z)/(c2 x^3((λ2 − λ1)xz − 1)))^λ2 == 0],
{x, −5, 5}, {z, −5, 5}, Exclusions → {x == 0},
ContourStyle → {Thickness[0.008]},
FrameLabel → Automatic, PlotPoints → 500],
  {{λ1, 2, λ1}, −50, 50, ap}, {{λ2, 3, λ2}, −10, 10, ap},
  {{c1, −1, c1}, −10, 10, ap}, {{c2, 1, c2}, −10, 10, ap}]
```

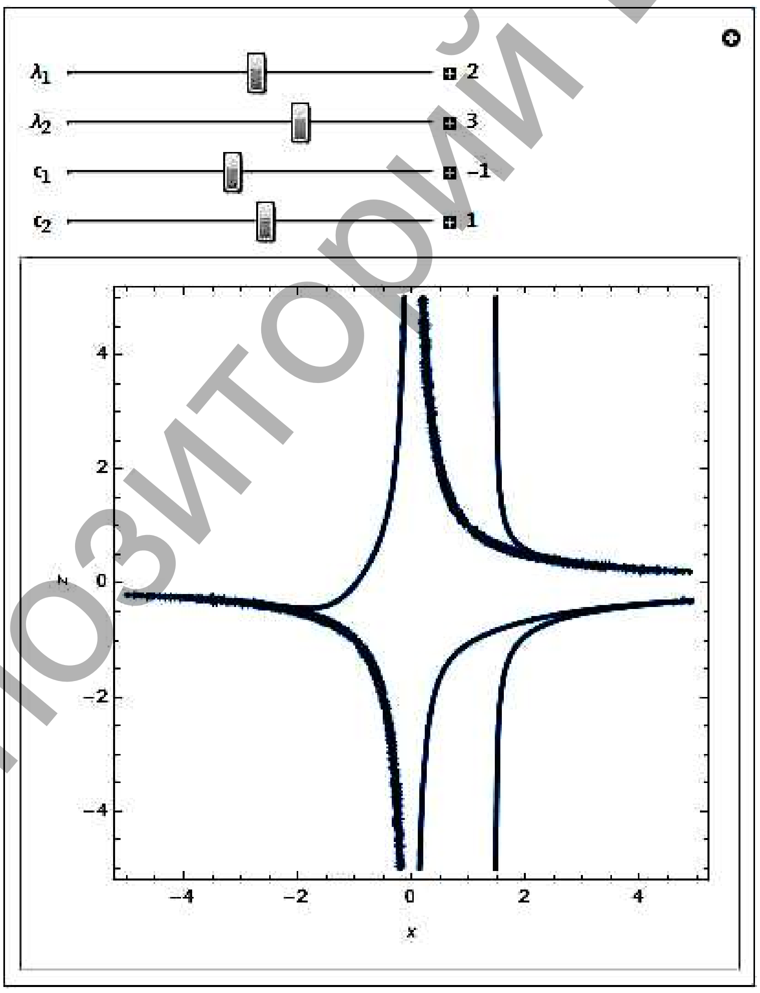

Figure 1. The graph of the partial solution of the equation (13) for $\lambda_1 = 2$, $\lambda_2 = 3$, $c_1 = -1$, $c_2 = 1$

4. Solution of the Cauchy problem for the Abel equation

Consider the solution algorithm [9] of the Cauchy problem for the equation (4) with given initial condition

$$z(x_0) = z_0. \qquad (16)$$

The stages of the algorithm as follows:

a) Suppose that the first three coefficients a, b_0, b_1, of the equation (4) and values λ_1, λ_2 are known;

b) Then, according to equations (6) and (7) we can find the coefficients b_2, b_3;

c) Then from equation (8) we find functions ξ, η;

d) Substituting these functions and values x_0, z_0 from condition (16) into equations (5) and (9), we obtain the linear system of two equations with unknowns values c_1, c_2. We solve this linear system.

e) Let's substitute the values c_1, c_2 into equations (5), (9) with known values λ_1, λ_2 (from equation (18)). From the first obtained relation we find the function $y(x)$ and substitute it in the second equation, which will determine the solution of the Cauchy problem. If such solutions are several, it is necessary to choose the one that satisfies the initial conditions (16).

We demonstrate action of this algorithm on the example.

Example 2. Suppose that coefficients of the equation (4) have form

$$a = \mathrm{tg}(x) - 2\,\mathrm{ctg}(x), \quad b_0 = 2, \quad b_1 = -3\mathrm{tg}(x), \tag{17}$$

and values λ_1, λ_2 from the equation (5) are the following

$$\lambda_1 = 1,. \quad \lambda_2 = 2. \tag{18}$$

Set the initial condition

$$x_0 = \frac{\pi}{4}, z_0 = 2. \tag{19}$$

According to item b) we find the other coefficients of equation (4)

$$b_2 = -\frac{1}{2}\left(3\cos(2x) + 5\right)\csc^2(x), b_3 = \left(\cos(2x) + 2\right)\csc(x)\sec(x). \tag{20}$$

According to item c) we find functions

$$\xi = -\mathrm{tg}(x), \eta = \mathrm{ctg}(x). \tag{21}$$

Substituting functions (21), values (18) and the values x_0, z_0 from condition (19) into equations (5), (9) we get a system of two equations. Solving this system we obtain (according to item d))

$$c_1 = \frac{3e^{-y(\frac{\pi}{4})}}{2\sqrt{2}}, c_2 = -\frac{e^{-2y(\frac{\pi}{4})}}{\sqrt{2}}. \tag{22}$$

Substituting values λ_1, λ_2 from (18) and c_1, c_2 from (22) into equations (5), (9) we obtain

$$e^{y(x)-2y\left(\frac{\pi}{4}\right)}\left(3e^{y\left(\frac{\pi}{4}\right)}\csc(x) - 2e^{y(x)}\cos(x)\right) = 2\sqrt{2}, \tag{23}$$

$$z(x) = \frac{3e^{y(\frac{\pi}{4})}\mathrm{ctg}(x) - 2e^{y(x)}\sin^2(x)}{3e^{y(\frac{\pi}{4})} - 2e^{y(x)}\sin(2x)}. \tag{24}$$

We solve equation (23) with respect to $y(x)$ and substitute this function into equation (24). From the two branches of the solution

$$z(x) = \frac{\sin^2(x)\csc(2x)\left(\sqrt{9\csc^2(x) - 16\sqrt{2}\cos(x)} + \left(6\cot^2(x) - 3\right)\csc(x)\right)}{\sqrt{9\csc^2(x) - 16\sqrt{2}\cos(x)}}, \tag{25}$$

$$z(x) = \frac{\csc(2x)\left(\sin(x)\left(\sin(x)\sqrt{9\csc^2(x) - 16\sqrt{2}\cos(x)} + 3\right) - 6\cos(x)\cot(x)\right)}{\sqrt{9\csc^2(x) - 16\sqrt{2}\cos(x)}} \tag{26}$$

we have chosen the solution (25), since it satisfies the initial condition (16), (19). Graphs of these solutions represents on the fig. 2.

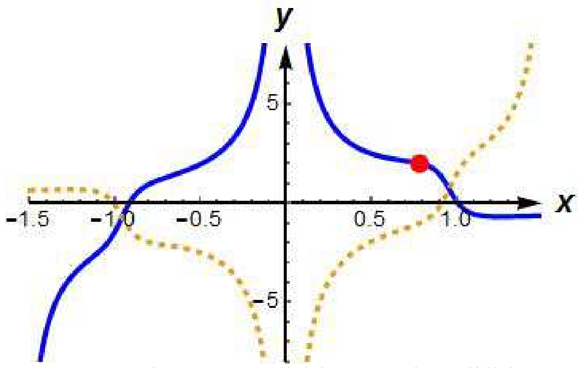

Figure 2. The graphs of the solutions (25) (blue curve) and (26) (dashed curve)

The red dot is determined by the initial condition (19).

5. Solution of the Cauchy problem for equation (1)

The solution of the Cauchy problem for equation (1) will be shown by the following example. At the same time we will use the algorithm from paper [2].

Example 3. Consider the equation (1) with the following coefficients

$$a=\frac{3(\beta+2\alpha x)}{\gamma+\alpha x^2+\beta x}-\frac{4(\mu+2\delta x)}{\nu+\delta x^2+\mu x},\quad b_0=84,\quad b_1=\frac{33(\beta+2\alpha x)}{\gamma+\alpha x^2+\beta x}+\frac{40(\mu+2\delta x)}{\nu+\delta x^2+\mu x}, \tag{27}$$

where $\alpha,\beta,\gamma,\delta,\mu,\nu$ are constants. Let, for example

$$\lambda_1=-4,\quad \lambda_2=3. \tag{28}$$

We substitute relations (27), (28) into system (6)–(7) and find functions b_2, b_3

$$b_2=\frac{2\left(3\left(\beta^2-\alpha\gamma+3\alpha^2x^2+3\alpha\beta x\right)+\frac{(\gamma+x(\beta+\alpha x))\left(2\delta(2\gamma+16\alpha x^2+9\beta x)+7\mu(\beta+2\alpha x)\right)}{\nu+x(\mu+\delta x)}\right)}{(\gamma+x(\beta+\alpha x))^2}, \tag{29}$$

$$b_3=\frac{2\left(\beta^2\mu-\alpha\gamma\mu+\beta\gamma\delta+8\alpha^2\delta x^3+3\alpha x^2(\alpha\mu+3\beta\delta)+3\beta x(\alpha\mu+\beta\delta)\right)}{(\gamma+x(\beta+\alpha x))^2(\nu+x(\mu+\delta x))}. \tag{30}$$

Then we substitute relations (27), (28) into equation (8) and find functions ξ, η.

$$\xi=\frac{\mu+2\delta x}{\nu+\delta x^2+\mu x},\quad \eta=\frac{\beta+2\alpha x}{\gamma+\alpha x^2+\beta x}. \tag{31}$$

The general solution of the equation (4), (27)–(30) is

$$\frac{c_1e^{-4y}}{\gamma+\alpha x^2+\beta x}+c_2e^{3y}\left(\nu+\delta x^2+\mu x\right)=1. \tag{32}$$

We define the following initial conditions

$$y(0)=1,\quad y'(0)=-1. \tag{33}$$

We differentiate the relation (32)

$$c_2e^{7y}\left(\mu+2\delta x+3y'\left(\nu+x(\mu+\delta x)\right)\right)-\frac{c_1\left(\beta+2\alpha x+4y'\left(\gamma+x(\beta+\alpha x)\right)\right)}{(\gamma+x(\beta+\alpha x))^2}=0. \tag{34}$$

We substitute the initial conditions (33) in equations (32), (34) and solve the obtained system as regards constants c_1, c_2. We substitute values c_1, c_2 into equation (32). As a result we find solution of the Cauchy problem

$$\frac{e^{-4y(x)-3}\left(\frac{e^7\gamma^2(\mu-3\nu)}{\gamma+x(\beta+\alpha x)}+(\beta-4\gamma)e^{7y(x)}\left(\nu+x(\mu+\delta x)\right)\right)}{\beta\nu+\gamma(\mu-7\nu)}=1. \tag{35}$$

The graph of the partial solution (35) for values of

$$\alpha=4,\beta=2,\gamma=1,\delta=-2,\mu=-3,\nu=-4 \tag{36}$$

represented on Fig. 3. Two closed curves which correspond to two branches of the solution showed on Fig. 3. Through the point, defined by the initial conditions (33), the right branch of the solution passes. Fig. 4 shows the positions of the branches of the solution (35)–(36) in the neighborhood of point with coordinates (0;1).

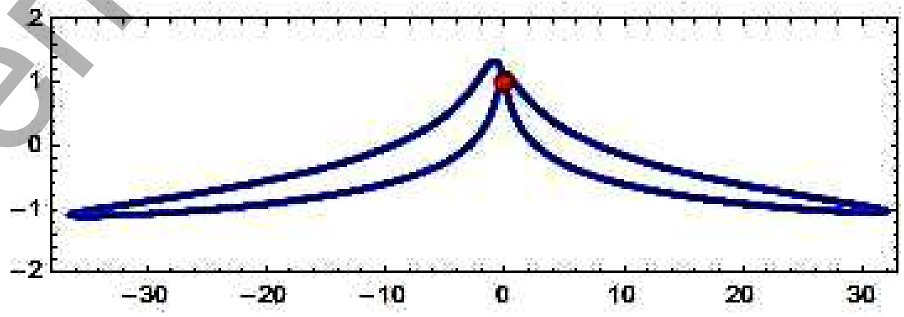

Figure 3. The graph of solution (35)–(36) is two closed curves. The coordinates of the red point are (0; 1)

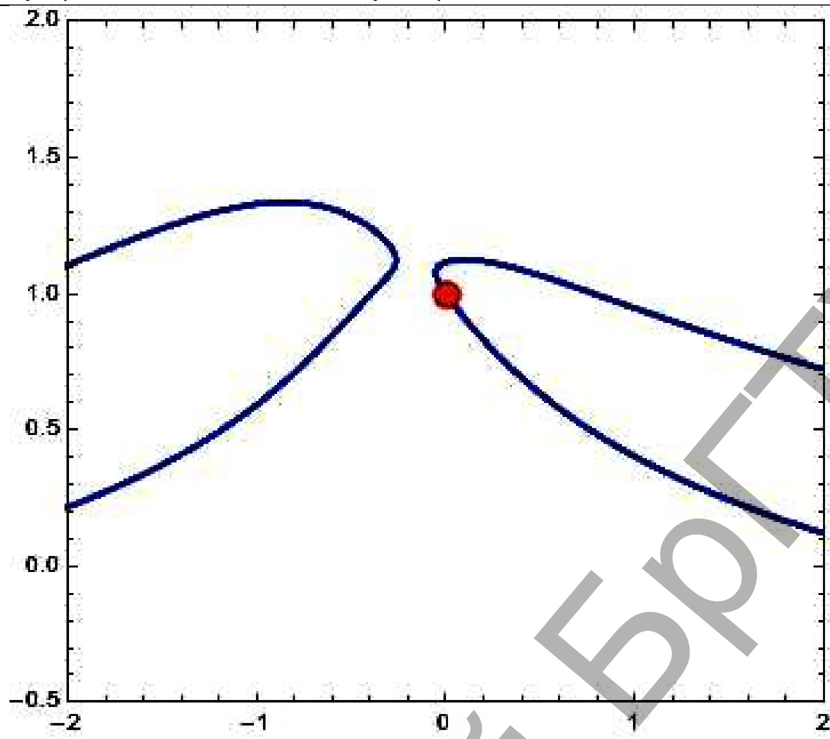

Figure 4. The graph of solution (35)–(36) in the neighborhood of the origin

Method considered in the article can be generalized for the equations of a higher order than the second. In paper [10] the solution of the corresponding problem for a differential equation of the third order was considered. For this purpose the function

$$\varphi_4(x) = F(\varphi_1, \varphi_2, \varphi_3, y, C_1, C_2, C_3), \quad (37)$$

where

$$F = C_1\ \varphi_1 \cdot exp(\lambda_1\ y) + C_2\ \varphi_2 \cdot exp(\lambda_2\ y) + C_3\ \varphi_3 \cdot exp(\lambda_3\ y) \quad (38)$$

and $C_i\ (i = 1, 2, 3)$ are arbitrary constants; $\varphi_j\ (j = \overline{1,4})$ three times continuously differentiable functions of variable x is considered. The following remark holds [10].

Remark 2. Applying the method used in the article, we can get a differential equation of the fourth or higher orders, for which the general solution has the form

$$\varphi_{n+1}(x) = \sum_{i=1}^{n} C_i \varphi_i(x) \cdot exp(\lambda_i\ y(x)),$$

where $\varphi_i(x)$ $(i = \overline{1,n})$ are arbitrary n times continuously differentiable functions of variable x, λ_i $(i = \overline{1,n})$ are certain constants, C_i $(i = \overline{1,n})$ are arbitrary constants. This yields to the differential equation of the n order.

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CHICHURIN A.V., STEPANYK G.P. Computer construction of the general solution of the special form of the Abel differential equation

The computer method of building a general solution of the special form for the nonlinear differential equation of the second order and the Abel differential equation of the first kind is considered. Three examples which contain the solutions of the initial problem are presented.

The module allowing to visualize partial solutions of the Abel differential equation for the given values of parameters has been built. For the obtained solutions the visualization in the real numbers domain is represented. All the calculations and visualizations are realized in the *Mathematica* 10 system.

УДК 519.2:004.6

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О МЕДИАНЕ ЗАКОНА РАСПРЕДЕЛЕНИЯ ПУАССОНА И НЕКОТОРЫХ ЧИСЛОВЫХ ПОСЛЕДОВАТЕЛЬНОСТЯХ

Введение. Вначале приведем некоторые теоретические сведения, обозначения, используемые в работе.

Пуассона распределение – распределение вероятностей случайной величины X, принимающей целые неотрицательные значения $k=0,1,2,\ldots$ с вероятностями $P(X=k)=p_k=e^{-\lambda}\frac{\lambda^k}{k!}$, где $\lambda>0$ – параметр.

Функция распределения закона Пуассона:

$F(x)=P(X<x)=0$, если $x\le 0$,

$F(x)=P(X<x)=\sum\limits_{k=0}^{x-1}p_k=e^{-\lambda}\sum\limits_{k=0}^{x-1}\frac{\lambda^k}{k!}$, если $x\in N$ и

$F(x)=P(X<x)=\sum\limits_{k=0}^{[x]}p_k=e^{-\lambda}\sum\limits_{k=0}^{[x]}\frac{\lambda^k}{k!}$, если $x>0$, $x\notin N$, где $[x]$ – целая часть числа x.

Или $F(x)=P(X<x)=e^{-\lambda}\sum\limits_{k=0}^{\lceil x\rceil-1}\frac{\lambda^k}{k!}$, если $x>0$, где $\lceil x\rceil$ – наименьшее целое, большее или равное x: $\lceil x\rceil=\min\{n\in Z\mid n\ge x\}$).

Рассмотрим функцию

$$F(m+1,\lambda)=1-\frac{\gamma(m+1,\lambda)}{m!}=1-\frac{1}{m!}\int_0^\lambda t^m e^{-t}dt,$$

где $\gamma(m,\lambda)=\int_0^\lambda t^{m-1}e^{-t}dt$ – неполная нижняя гамма-функция (например, в [1]).

Заметим, что

$$F(1,\lambda)=1-\frac{1}{0!}\int_0^\lambda t^0e^{-t}dt=$$

$$=1-\int_0^\lambda e^{-t}dt=1+e^{-t}\Big|_0^\lambda=e^{-\lambda}=p_0.$$

Используя метод интегрирования по частям в определенном интеграле, получим:

$$F(m+1,\lambda)=1-\frac{1}{m!}\int_0^\lambda t^m e^{-t}dt=$$

$$=1+\frac{1}{m!}\int_0^\lambda t^m de^{-t}=1+\frac{1}{m!}\left(t^m e^{-t}\Big|_0^\lambda-\int_0^\lambda e^{-t}dt^m\right)=$$

$$=1+\frac{1}{m!}\left(\lambda^m e^{-\lambda}-m\int_0^\lambda e^{-t}t^{m-1}dt\right)=$$

$$=1+\frac{\lambda^m e^{-\lambda}}{m!}-\frac{1}{(m-1)!}\int_0^\lambda e^{-t}t^{m-1}dt=$$

$$=e^{-\lambda}\frac{\lambda^m}{m!}+F(m,\lambda)=p_m+F(m,\lambda)=\ldots=$$

$$=\sum_{k=1}^m p_k+F(1,\lambda)=\sum_{k=0}^m p_k.$$

Следовательно, функцию распределения $F(x)$ можно определить следующим образом:

$F(x)=P(X<x)=0$, если $x\le 0$,

$$F(x)=P(X<x)=\sum_{k=0}^{x-1}p_k=F(x,\lambda)=$$

$$=1-\frac{1}{(x-1)!}\int_0^\lambda t^{x-1}e^{-t}dt,$$

если $x\in N$ и

$$F(x)=P(X<x)=\sum_{k=0}^{[x]}p_k=F([x]+1,\lambda)=$$

$$=1-\frac{1}{[x]!}\int_0^\lambda t^{[x]}e^{-t}dt,$$

если $x>0$, $x\notin N$.

Или

$$F(x)=P(X<x)=\sum_{k=0}^{\lceil x\rceil-1}p_k=F(\lceil x\rceil,\lambda)=$$

$$=1-\frac{1}{(\lceil x\rceil-1)!}\int_0^\lambda t^{\lceil x\rceil-1}e^{-t}dt,$$

если $x>0$.

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