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Кафедра высшей математики

LINEAR ALGEBRA AND ANALYTIC GEOMETRY

for foreign first-year students

учебно-методическая разработка на английском языке
по дисциплине **«Математика»**
для студентов 1-го курса

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Настоящая методическая разработка предназначена для иностранных студентов первого курса факультета электронно-информационных систем. Настоящая разработка содержит необходимый материал по разделам «Линейная алгебра» и «Аналитическая геометрия» курса математики, который изучается студентами на первом курсе. Изложение теоретического материала по всем темам сопровождается рассмотрением большого количества примеров и задач, ведется на доступном, по возможности строгом языке.

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I LINEAR ALGEBRA

1.1 The Algebra of Matrices

A **matrix** is a rectangular array of numbers. Each number in the matrix is called an **element** of the matrix. The matrix below, with three rows and four columns, is called a 3×4 (read 3 by 4) matrix.

$$\begin{bmatrix} 2 & 5 & -2 & 5 \\ -3 & 6 & 4 & 0 \\ 1 & 3 & 7 & 2 \end{bmatrix}$$

Definition. A matrix of m rows and n columns is said to be of **order** $m\times n$ or **dimension** $m\times n$**.** The matrix above has order 3×4. We will use the notation a_{ij} to refer to the element of a matrix in the i-th row and j-th column. For the matrix given above, $a_{23}=4, a_{31}=1$ and $a_{13}=-2$.

The elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ form the **main diagonal** of a matrix. The Elements 2, 6, and 7 form the main diagonal of the matrix shown above.

Matrix is denoted by either a capital letter or by surrounding the corresponding lower-case letter with brackets. For example, a matrix could be denoted as A or $\left[a_{ij}\right]$.

Caution. Remember that $\left[a_{ij}\right]$ is a matrix and a_{ij} is the element in the *i*-th row and *j*-th column of the matrix.

Definition of the Zero Matrix. The $m\times n$ zero matrix, denoted 0 is the matrix whose elements are all zeros. Three example of zero matrices are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Definition of the Row Matrix. The matrix of order $1\times n$ is called the row matrix. For example, $A=\begin{bmatrix} 1 & 7 & -3 & 0 \end{bmatrix}$ is a row matrix of order 1×4.

Definition of the Column Matrix. The matrix of order $n\times1$ is called the column matrix. For example, $A=\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ is a column matrix of order 3×1.

Definition of the Identity Matrix. The square matrix that has a 1 for each element on the main diagonal and zeros elsewhere is called the identity matrix. The identity matrix of order n, denoted I_n is the $n\times n$ matrix $I_n=\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix}$.

The identity matrix has properties similar to the real number 1. For example, the product of a matrix A and I is A.

$$\begin{bmatrix} 2 & -3 & 0 \\ 4 & 7 & -5 \\ 9 & 8 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 4 & 7 & -5 \\ 9 & 8 & -6 \end{bmatrix}$$

If A is a square matrix of order $n \times n$ and I_n is the identity matrix of order n, then $AI_n = I_n A = A$.

Definition of Equality of Two Matrices. Two matrices $A = \left[a_{ij} \right]$ and $B = \left[b_{ij} \right]$ are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

For example, the 2×3 matrices $\begin{bmatrix} a & -2 & b \\ 3 & c & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & x & -4 \\ 3 & -1 & y \end{bmatrix}$ are equal if and only if $a = 3, x = -2, b = -1$, and $y = 1$.

Remark. The definition of equality implies that the two matrices have the same order.

Operations on matrices

Definition of Matrix Addition. If A and B are matrices of order $m \times n$ then the sum of the matrices is the $m \times n$ matrix given by $A + B = \left[a_{ij} + b_{ij} \right]$.

This definition states that the sum of two matrices is found by adding their corresponding elements. Note from the definition that both matrices have the same order. The sum of two matrices of different order is not defined.

Given $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 6 \\ -2 & 3 & 5 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 2+5 & -2+(-2) & 3+6 \\ 1+(-2) & 3+3 & -4+5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 9 \\ -1 & 6 & 1 \end{bmatrix}.$$

Now let $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -5 & 1 \end{bmatrix}$. Then $A + B$ is not defined because the matrices do not have the same order.

Properties of Matrix Addition

Given matrices A, B, C and the zero matrix 0 each of order $m \times n$ then the following properties hold:

1. Commutative law $A + B = B + A$.
2. Associative law $A + (B + C) = (A + B) + C$.
3. Additive inverse $A + (-A) = 0$.
4. Additive identity $A + 0 = 0 + A = A$.

Definition of Additive Inverse of a Matrix. Given the matrix $A = \left[a_{ij} \right]$ the additive inverse of A is $-A = \left[-a_{ij} \right]$. For example, $-A = -\begin{bmatrix} 3 & 2 & 0 \\ 1 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 0 \\ -1 & 5 & -1 \end{bmatrix}$.

Definition of Subtraction on Matrices. Given two matrices A and B of order $m \times n$, the subtraction of the two matrices $A - B$ is $A - B = A + (-B)$. For example,

$$A - B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -4 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 3 & 1 \\ -1 & 6 \end{bmatrix}.$$

Definition of the Product of a Real Number and a Matrix. Given the $m \times n$ matrix $A = [a_{ij}]$ and a real number c then $cA = [ca_{ij}]$.

Remark. The product of a real number and a matrix is referred to as **scalar multiplication.**

As an example of this definition, consider the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$ and the constant $c = -2$, then

$$-2A = -2\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -2\cdot 2 & -2\cdot(-3) & -2\cdot 1 \\ -2\cdot 3 & -2\cdot 1 & -2\cdot(-2) \\ -2\cdot 1 & -2\cdot(-1) & -2\cdot 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 & -2 \\ -6 & -2 & 4 \\ -2 & 2 & -8 \end{bmatrix}.$$

This definition is also used to factor a constant from a matrix.

Properties of Scalar Multiplication

Given real numbers a, b and c and matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ each of order $m \times n$, then:

1. $(b + c)A = bA + cA$.
2. $c(A + B) = cA + cB$.
3. $a(bA) = (ab)A$.

Example 1. Given $A = \begin{bmatrix} -2 & 3 \\ 4 & -2 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -2 \\ -3 & 2 \\ -4 & 7 \end{bmatrix}$ find $2A + 5B$.

Solution. $2A + 5B = 2\begin{bmatrix} -2 & 3 \\ 4 & -2 \\ 0 & 4 \end{bmatrix} + 5\begin{bmatrix} 8 & -2 \\ -3 & 2 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & -4 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 40 & -10 \\ -15 & 10 \\ -20 & 35 \end{bmatrix} = \begin{bmatrix} 36 & -4 \\ -7 & 6 \\ -20 & 43 \end{bmatrix}.$

Definition of the Product of Two Matrices. Let $A = [a_{ij}]$ be a matrix of order $m \times n$ and $B = [b_{ij}]$ be a matrix of order $n \times p$. Then the product AB is the matrix of order $m \times p$ given by $AB = [c_{ij}]$ where each element c_{ij} is

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + a_{i3}\cdot b_{3j} + \ldots + a_{in}\cdot b_{nj}.$$

Remark. This definition may appear complicated, but basically, to multiply two matrices multiply each row vector of the first matrix by each column vector of the second matrix.

For the product of two matrices to be possible, the number of columns of the first matrix must equal the number of rows of the second matrix: $A_{m\times n}\cdot B_{n\times p}=C_{m\times p}$. The product matrix has as many rows as the first matrix and as many columns as the second matrix.

For example, let $A=\begin{bmatrix}2 & -3 & 0\\ 1 & 4 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 4 & -2\\ 3 & 5\end{bmatrix}$.

Then A has order 2×3 and B has order 3×2. Thus AB has order 2×2.

$$AB=\begin{bmatrix}2 & -3 & 0\\ 1 & 4 & -1\end{bmatrix}_{2\times 3}\cdot\begin{bmatrix}1 & 0\\ 4 & -2\\ 3 & 5\end{bmatrix}_{3\times 2}=\begin{bmatrix}2\cdot 1+(-3)\cdot 4+0\cdot 3 & 2\cdot 0+(-3)\cdot(-2)+0\cdot 5\\ 1\cdot 1+4\cdot 4+(-1)\cdot 3 & 1\cdot 0+4\cdot(-2)+(-1)\times 5\end{bmatrix}_{2\times 2}=\begin{bmatrix}-10 & 6\\ 14 & -13\end{bmatrix}$$

Example 2. Find the following products:

a. $\begin{bmatrix}2 & 3\\ -3 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}1 & 2 & -2 & 3\\ -1 & 0 & 3 & -4\end{bmatrix}$ b. $\begin{bmatrix}1 & -1 & 3\\ 2 & 2 & -1\\ 0 & -2 & 3\end{bmatrix}\begin{bmatrix}4 & -2 & 0\\ -1 & 3 & 1\\ 2 & -3 & 1\end{bmatrix}$

Solution

a. $\begin{bmatrix}2 & 3\\ -3 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}1 & 2 & -2 & 3\\ -1 & 0 & 3 & -4\end{bmatrix}=$

$$=\begin{bmatrix}2(1)+3(-1) & 2(2)+3(0) & 2(-2)+3(3) & 2(3)+3(-4)\\ -3(1)+1(1) & -3(2)+1(0) & (-3)(-2)+1(3) & (-3)3+1(4)\\ 1(1)+(-3)(-1) & 1(2)+(-3)0 & 1(-2)+(-3)3 & 1(3)+(-3)(-4)\end{bmatrix}=$$

$$=\begin{bmatrix}-1 & 4 & 5 & -6\\ -4 & -6 & 9 & -13\\ 4 & 2 & -11 & 15\end{bmatrix}.$$

b. $\begin{bmatrix}1 & -1 & 3\\ 2 & 2 & -1\\ 0 & -2 & 3\end{bmatrix}\begin{bmatrix}4 & -2 & 0\\ -1 & 3 & 1\\ 2 & -3 & 1\end{bmatrix}=\begin{bmatrix}4+1+6 & -2+(-)+(-9) & 0+(1)+3\\ 8+(-2)+(-2) & -4+6+3 & 0+2+(1)\\ 0+2+6 & 0+(-6)+(9) & 0+(-2)+3\end{bmatrix}=$

$$=\begin{bmatrix}11 & -14 & 2\\ 4 & 5 & 1\\ 8 & -15 & 1\end{bmatrix}.$$

Generally, matrix multiplication is not communicative. That is, given two matrices A and B, $AB\neq BA$. In some cases, as in Example 2a, if the matrices were reversed, the product would not be defined.

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ -1 & 0 & 3 & -4 \end{bmatrix}_{2\times4} \begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}_{3\times2}$$

$$\text{Columns} \neq \text{Rows}$$

But even in those cases where multiplication is defined, the product AB and BA may not equal. The product of part b of Example 2 with the matrices reversed illustrates this point.

$$\begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & -1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -8 & 14 \\ 5 & 5 & -3 \\ -4 & -10 & 12 \end{bmatrix} \neq \begin{bmatrix} 11 & -14 & 2 \\ 4 & 5 & 1 \\ 8 & -15 & 1 \end{bmatrix}$$

Although matrix multiplication is not commutative, the associative property of multiplication and distributive property do hold for matrices.

Properties of Matrix Multiplication

Associative property. Given matrices A, B and C of orders $m\times n, n\times p$ and $p\times q$, respectively, then $A(BC)=(AB)C$.

Distributive property. Given matrices A_1 and A_2 of order $m\times n$ and matrices B_1 and B_2 of order $n\times p$, then

$$A_1(B_1+B_2)=A_1B_1+A_1B_2 \text{ (Left distributive law)}$$

and

$$(A_1+A_2)B_2=A_1B_2+A_2B_2 \text{ (Right distributive law)}$$

A system of equations can be expressed as the product of matrices. Consider the matrix equation

$$\begin{bmatrix} 2 & -3 & 4 \\ 3 & 0 & 1 \\ 1 & -2 & -5 \end{bmatrix}_{3\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}_{3\times1}.$$

Multiplying the two matrices on the left side of the equation, we have

$$\begin{bmatrix} 2x-3y+4z \\ 3x \qquad + z \\ x-2y-5z \end{bmatrix}_{3\times1} = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}_{3\times1}.$$

Because we multiplied a 3×3 matrix by a 3×1 matrix, the result is a 3×1 matrix. Now using the definition of matrix equality, we have that the given matrix equation is equivalent to the following system of equations:

$$\begin{cases} 2x-3y+4z=9 \\ 3x \qquad +z=4 \\ x-2y-5z=-2 \end{cases}$$

Exercise Set 1

In Exercises 1 to 8, find a. $A+B$; b. $A-B$; c. $2B$; d. $2A-3B$.

1. $A=\begin{bmatrix}2 & -1\\ 3 & 3\end{bmatrix}$, $B=\begin{bmatrix}-1 & 3\\ 2 & 1\end{bmatrix}$.

2. $A=\begin{bmatrix}0 & -2\\ 2 & 3\end{bmatrix}$, $B=\begin{bmatrix}5 & -1\\ 3 & 0\end{bmatrix}$.

3. $A=\begin{bmatrix}0 & -1 & 3\\ 1 & 0 & -2\end{bmatrix}$, $B=\begin{bmatrix}-3 & 1 & 2\\ 2 & 5 & -3\end{bmatrix}$.

4. $A=\begin{bmatrix}2 & -2 & 4\\ 0 & -3 & -4\end{bmatrix}$, $B=\begin{bmatrix}1 & -5 & 6\\ 4 & -2 & -3\end{bmatrix}$.

5. $A=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$.

6. $A=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$.

7. $A=\begin{bmatrix}-2 & 3 & -1\\ 0 & -1 & 2\\ -4 & 3 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}$.

8. $A=\begin{bmatrix}0 & 2 & 0\\ 1 & -3 & 3\\ 5 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}$.

In Exercises 9 to 16, find AB and BA.

9. $A=\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-2 & 4\\ 2 & -3\end{bmatrix}$.

10. $A=\begin{bmatrix}3 & -2\\ 4 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & -1\\ 0 & 4\end{bmatrix}$.

11. $A=\begin{bmatrix}3 & -1\\ 2 & 3\end{bmatrix}$, $B=\begin{bmatrix}4 & 1\\ 2 & -3\end{bmatrix}$.

12. $A=\begin{bmatrix}-3 & 2\\ 2 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & 2\\ -2 & 4\end{bmatrix}$.

13. $A=\begin{bmatrix}2 & -1\\ 0 & 3\\ 1 & -2\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & 3\\ 2 & 0 & 1\end{bmatrix}$.

14. $A=\begin{bmatrix}-1 & 3\\ 2 & 1\\ -3 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & -1 & 2\\ 1 & 2 & -4\end{bmatrix}$.

15. $A=\begin{bmatrix}2 & -1 & 3\\ 0 & 2 & -1\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 0 & 0\\ 1 & -1 & 0\\ 2 & -1 & -2\end{bmatrix}$.

16. $A=\begin{bmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{bmatrix}$.

In Exercises 17 to 24, find AB, if possible.

17. $A=\begin{bmatrix}1 & -2 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 2 & -1\\ 1 & 2\end{bmatrix}$.

18. $A=\begin{bmatrix}-2 & 3\\ 1 & -2\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}3\\ -2\end{bmatrix}$.

21. $A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$, $B=\begin{bmatrix}3 & 6\\ -2 & -4\end{bmatrix}$.

22. $A=\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 3 & 2\\ 2 & -1 & 0\\ 3 & 1 & 2\end{bmatrix}$.

23. $A=\begin{bmatrix}1 & 2 & -2 & 3\\ 0 & -2 & 1 & -3\end{bmatrix}$, $B=\begin{bmatrix}-2 & 0\\ 4 & -2\end{bmatrix}$.

19. $A=\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 0 & -2 \end{bmatrix}$.

20. $A=\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & -3 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 & 0 \\ 2 & 4 & 5 \end{bmatrix}$.

24. $A=\begin{bmatrix} 2 & -2 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 1 & -2 \\ 1 & -1 & 0 & 2 \end{bmatrix}$.

In Exercises 25 to 28, given the matrices $A=\begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$, find the 3×2 matrix X that is a solution of the equation.

25. $3X+A=B$.

26. $2X-A=X+B$.

27. $2A-3X=5B$.

28. $3X+2B=X-2A$.

In Exercises 29 to 32, use the matrices $A=\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, find

29. A^2.

30. B^2.

31. A^3.

32. B^3.

In Exercises 33 to 38, find the system of equations that is equivalent to the given matrix equation.

33. $\begin{bmatrix} 3 & -8 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 11 \\ 1 \end{bmatrix}$.

34. $\begin{bmatrix} 2 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 16 \end{bmatrix}$.

35. $\begin{bmatrix} 1 & -3 & -2 \\ 3 & 1 & 0 \\ 2 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

36. $\begin{bmatrix} 2 & 0 & 5 \\ 3 & -5 & 1 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 7 \\ 14 \end{bmatrix}$.

37. $\begin{bmatrix} 2 & -1 & 0 & 2 \\ 4 & 1 & 2 & -3 \\ 6 & 0 & 1 & -2 \\ 5 & 2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 5 \\ 6 \\ 10 \\ 8 \end{bmatrix}$.

38. $\begin{bmatrix} 5 & -1 & 2 & -3 \\ 4 & 0 & 2 & 0 \\ 2 & -2 & 5 & -4 \\ 3 & 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$.

1.2 Determinants

Associated with each square matrix A is a number called the determinant of A. We will denote the determinant of the matrix A by $\det A$ or by $|A|$. For the remainder of this paragraph, we assume that all matrices are square matrices.

Definition. The determinant of the matrix $A=\left[a_{ij}\right]$ of order 2 is

$$|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}.$$

Caution. Be careful not to confuse the notation for a matrix and that for a determinant. The symbol [ ] (brackets) is used for a matrix; the symbol | | (vertical bars) is used for the determinant of the matrix.

An easy way to remember the formula for the determinant of a 2×2 matrix is to recognize that the determinant is the difference in the products of the diagonal elements.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

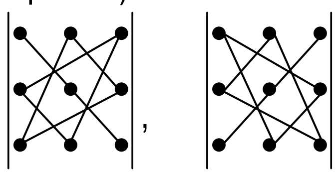
Example 1. Find the Determinant of a Matrix of Order 2 $A = \begin{bmatrix} 5 & 3 \\ 2 & -3 \end{bmatrix}$.

Solution. $|A| = \begin{vmatrix} 5 & 3 \\ 2 & -3 \end{vmatrix} = 5(-3) - 2(3) = -15 - 6 = -21$.

Definition. The determinant of the matrix $A = \left[a_{ij} \right]$ of order 3 is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

An easy way to remember the formula for the determinant of a 3×3 matrix is to recognize that the determinant contains three terms with sign "plus" and three terms with sign "minus" (see the picture):



"+" "–"

To define the determinant of a matrix of order greater than 3, we first need two definitions.

Definition. The minor M_{ij} of the element a_{ij} of a square matrix A of order $n \geq 3$ is the determinant of the matrix of order $n-1$ obtained by deleting the *i*-th row and *j*-th column of A.

Consider the matrix $A = \begin{bmatrix} 2 & -1 & 5 \\ 4 & 3 & -7 \\ 8 & -7 & 6 \end{bmatrix}$. The minor M_{23} is the determinant of matrix A after row 2 and column 3 are deleted from A

$$\begin{vmatrix} 2 & -1 & 5 \\ 4 & 3 & -7 \\ 8 & -7 & 6 \end{vmatrix}, \text{ thus } M_{23} = \begin{vmatrix} 2 & -1 \\ 8 & -7 \end{vmatrix} = 2(-7) - 8(-1) = -14 + 8 = -6.$$

The minor M_{31} is the determinant of matrix A after row 3 and column 1 are deleted from A

$$\begin{vmatrix} 2 & -1 & 5 \\ 4 & 3 & -7 \\ 8 & -7 & 6 \end{vmatrix}, \text{ thus } M_{31} = \begin{vmatrix} -1 & 5 \\ 3 & -7 \end{vmatrix} = (-1)(-7) - 3(5) = 7 - 15 = -8.$$

The second definition we need is the cofactor of a matrix.

Definition. The cofactor C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}

Remark. When $i+j$ is an even integer, $(-1)^{i+j} = 1$. When $i+j$ is an odd integer, $(-1)^{i+j} = -1$. Thus, $C_{ij} = \begin{cases} M_{ij}, & i+j = \text{even number}, \\ -M_{ij}, & i+j = \text{odd number}. \end{cases}$

Example 2. Given $A = \begin{bmatrix} 4 & 3 & -2 \\ 5 & -2 & 4 \\ 3 & -2 & -6 \end{bmatrix}$, find M_{32} and C_{12}.

Solution. $M_{32} = \begin{vmatrix} 4 & -2 \\ 5 & 4 \end{vmatrix} = 4(4) - 5(-2) = 16 + 10 = 26$,

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} 5 & 4 \\ 3 & -6 \end{vmatrix} = -(-30-12) = 42$.

The definition of minors and cofactors are used to define the determinant of a matrix of order 3 or greater.

Statement. Given the square matrix A of order 3 or greater, the determinant of A is the sum of the products of the elements of any row or column and their cofactors. For the *r*-th row of A is

$$|A| = a_{r1}C_{r1} + a_{r2}C_{r2} + a_{r3}C_{r3} + \ldots + a_{rn}C_{rn}.$$

For the *c*-th column of A, the determinant of A is

$$|A| = a_{1c}C_{1c} + a_{2c}C_{2c} + a_{3c}C_{3c} + \ldots + a_{nc}C_{nc}.$$

Example 3. Evaluate the determinant of the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 3 \\ 1 & -3 & 4 \end{bmatrix}$

a. by expanding about row 2;
b. by expanding about column3,
c. by definition.

Solution. By the definition, any row or column can be used in the expansion. To illustrate the method, we arbitrarily choose row 2 and then column 3.

a. $|A| = 4C_{21} + (-2)C_{22} + 3C_{23} = 4(-M_{21}) + (-2)M_{22} + 3(-M_{23}) = -4\begin{vmatrix} 3 & -1 \\ -3 & 4 \end{vmatrix} +$

$+(-2)\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = -4(9) + (-2)(9) + (-3)(-9) = -36 - 18 + 27 = -27$.

b. $|A| = (-1)C_{13} + 3C_{23} + 4C_{33} = (-1)M_{13} + (-3M_{23}) + 4M_{33} = (-1)\begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} +$

$$+4\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = (-1)(-10) + (-3)(-9) + 4(-16) = 10 + 27 - 64 = -27.$$

c.

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 4 & -2 & 3 \\ 1 & -3 & 4 \end{vmatrix} = 2(-2)(4) + 4(-3)(-1) + 3(3)(1) - (-1)(-2)(1) - (-3)(3)(2) - 4(3)(4) =$$

$$= -16 + 12 + 9 - 2 + 18 - 48 = -27.$$

Remark. Example 3 illustrates that whatever row or column is used, expanding by cofactors gives the same value for the determinant. When you are evaluating determinants, choose the most convenient row or column, which usually is row or column containing the most zeros.

Evaluating determinants by expanding by cofactors is very time-consuming for determinants of large orders. For example, a determinant of order 10 has more than 3 million addends, and each addend is the product of ten numbers.

The easiest determinants to evaluate have many zeros in a row or column. It is possible to transform a determinant into one that has many zeros by using elementary row operations.

Effects of Elementary Row Operations on the Determinant of a Matrix

If A is a square matrix of order n, then the following elementary row operations produce the indicated change in the determinant of A.

1. Interchanging any two rows of A changes the sign of $|A|$.
2. Multiplying a row of A by a constant k multiplies the determinant of A by k.
3. Adding a multiple of a row of A to another row does not change the value of the determinant of A.

Remark. The properties of determinants just stated remain true if the word “row” is replaced by “column”. In that case, we would have elementary column operations.

To illustrate these properties, consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$. The determinant of A is $|A| = 2(-2) - 1(3) = -7$. Now consider each of the elementary row operations.

Interchange the rows of A and evaluate the determinant.

$$\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = 1(3) - 2(-2) = 3 + 4 = 7 = -|A|.$$

Multiply row 2 of A by –3 and evaluate the determinant.

$$\begin{vmatrix} 2 & 3 \\ -3 & 6 \end{vmatrix} = 2(6) - (-3)3 = 12 + 9 = 21 = -3|A|.$$

Multiply row 1 of A by –2 and add to row 2. Evaluate the determinant.

$$\begin{vmatrix} 2 & 3 \\ -3 & -8 \end{vmatrix} = 2(-8) - (-3)(3) = -16 + 9 = -7 = |A|.$$

These elementary row operations are used to rewrite a matrix in diagonal form. A matrix is in diagonal form if all elements below (or above) the main diagonal are zero. The matrices

$$A=\begin{bmatrix}2 & -2 & 3 & 1\\ 0 & -2 & 4 & 2\\ 0 & 0 & 6 & 3\\ 0 & 0 & 0 & -5\end{bmatrix} \text{ and } B=\begin{bmatrix}3 & 0 & 0 & 0\\ 2 & -3 & 0 & 0\\ 6 & 4 & -2 & 0\\ 8 & 3 & 4 & 2\end{bmatrix}$$

are in diagonal form.

Statement. Let A be a square matrix of order n in diagonal form. The determinant of A is the product of the elements on the main diagonal.

$$|A| = a_{11}a_{22}a_{33}...a_{nn}.$$

For matrix A given above, $|A| = 2(-2)(6)(-5) = 120.$

Example 4. Evaluate the determinant by rewriting in diagonal form $\begin{vmatrix}2 & 1 & -1 & 3\\ 2 & 2 & 0 & 1\\ 4 & 5 & 4 & -3\\ 2 & 2 & 7 & -3\end{vmatrix}$.

Solution. Rewrite the matrix in diagonal form by using elementary row operations.

$$\begin{vmatrix}2 & 1 & -1 & 3\\ 2 & 2 & 0 & 1\\ 4 & 5 & 4 & -3\\ 2 & 2 & 7 & -3\end{vmatrix} \overset{\substack{-1R_1+R_2\\ -2R_1+R_3\\ -1R_1+R_4}}{=} \begin{vmatrix}2 & 1 & -1 & 3\\ 0 & 1 & 1 & -2\\ 0 & 3 & 6 & -9\\ 0 & 1 & 8 & -6\end{vmatrix} \overset{\substack{\textit{Factor } 3,\\ \textit{from row } 3}}{=} 3\begin{vmatrix}2 & 1 & -1 & 3\\ 0 & 1 & 1 & -2\\ 0 & 1 & 2 & -3\\ 0 & 1 & 8 & -6\end{vmatrix} \overset{\substack{-1R_2+R_3\\ -1R_3+R_4}}{=} 3\begin{vmatrix}2 & 1 & -1 & 3\\ 0 & 1 & 1 & -2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 7 & -4\end{vmatrix} =$$

$$\overset{-7R_3+R_4}{=} 3\begin{vmatrix}2 & 1 & -1 & 3\\ 0 & 1 & 1 & -2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 3\end{vmatrix} = 3(2)(1)(1)(3) = 18.$$

Remark. The last example used only elementary row operations to reduce the matrix to diagonal form. Elementary column operations could also have been used, or a combination of row and column operations could have been used.

A matrix whose determinant is zero is called a singular matrix. In some cases it is possible to recognize when the determinant of a matrix is zero.

Conditions for a Zero Determinant

If A is a square matrix, then $|A| = 0$ when any one of the following is true.

1. A row (column) consists entirely of zeros.
2. Two rows (columns) are identical.
3. One row (column) is a constant multiple of a second row (column).

Statement (Product Property of Determinants). If A and B are square matrices of order n, then

$$|AB| = |A||B|.$$

Exercise Set 2

In Exercises 1 to 8, evaluate the determinants.

1. $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ 2. $\begin{vmatrix} 2 & 9 \\ -6 & 2 \end{vmatrix}$ 3. $\begin{vmatrix} 5 & 0 \\ 2 & -3 \end{vmatrix}$ 4. $\begin{vmatrix} 0 & -8 \\ 3 & 4 \end{vmatrix}$ 5. $\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix}$ 6. $\begin{vmatrix} -3 & 6 \\ 4 & -8 \end{vmatrix}$ 7. $\begin{vmatrix} 0 & 9 \\ 0 & -2 \end{vmatrix}$ 8. $\begin{vmatrix} -3 & 9 \\ 0 & 0 \end{vmatrix}$

In Exercises 9 to 12, evaluate the indicated minor and cofactor for the determinant

$$\begin{vmatrix} 5 & -2 & -3 \\ 2 & 4 & -1 \\ 4 & -5 & 6 \end{vmatrix}.$$

9. M_{11}, C_{11}. 10. M_{21}, C_{21}. 11. M_{32}, C_{32}. 12. M_{33}, C_{33}.

In Exercises 13 to 16, evaluate the indicated minor and cofactor for the determinant

$$\begin{vmatrix} 3 & -2 & 3 \\ 1 & 3 & 0 \\ 6 & -2 & 3 \end{vmatrix}.$$

13. M_{22}, C_{22}. 14. M_{13}, C_{13}. 15. M_{31}, C_{31}. 16. M_{23}, C_{23}.

In Exercises 17 to 26, evaluate the determinant by expanding by cofactors.

17. $\begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{vmatrix}$. 19. $\begin{vmatrix} -2 & 3 & 2 \\ 1 & 2 & -3 \\ -4 & -2 & 1 \end{vmatrix}$. 21. $\begin{vmatrix} 2 & -3 & 10 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix}$. 23. $\begin{vmatrix} 5 & -8 & 0 \\ 2 & 0 & -7 \\ 0 & -2 & -1 \end{vmatrix}$. 25. $\begin{vmatrix} 0 & -2 & 4 \\ 1 & 0 & -7 \\ 5 & -6 & 0 \end{vmatrix}$.

18. $\begin{vmatrix} 3 & 1 & -2 \\ 2 & -5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$. 20. $\begin{vmatrix} 3 & -2 & 0 \\ 2 & -3 & 2 \\ 8 & -2 & 5 \end{vmatrix}$. 22. $\begin{vmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 7 & -8 & 2 \end{vmatrix}$. 24. $\begin{vmatrix} 4 & -3 & 3 \\ 2 & 1 & -4 \\ 6 & -2 & -1 \end{vmatrix}$. 26. $\begin{vmatrix} -2 & 3 & 9 \\ 4 & -2 & -6 \\ 0 & -8 & -24 \end{vmatrix}$.

In Exercises 27 to 40, without expanding, give a reason for each equality.

27. $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 0$.

28. $\begin{vmatrix} 2 & 3 & 0 \\ 1 & -2 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 0$.

29. $\begin{vmatrix} 1 & 4 & -1 \\ 2 & 4 & 12 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$.

30. $\begin{vmatrix} 1 & -3 & 4 \\ 4 & 6 & 1 \\ 0 & -9 & 3 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 & 4 \\ 4 & -2 & 1 \\ 0 & 3 & 3 \end{vmatrix}$.

34. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & 1 \\ -4 & 2 & -6 \end{vmatrix} = 0$.

35. $\begin{vmatrix} 2 & -4 & 5 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{vmatrix} = -12$.

36. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{vmatrix} = -15$.

37. $\begin{vmatrix} 3 & 5 & -2 \\ 2 & 1 & 0 \\ 9 & -2 & -3 \end{vmatrix} = -\begin{vmatrix} 9 & -2 & -3 \\ 2 & 1 & 0 \\ 3 & 5 & -2 \end{vmatrix}$.

31. $\begin{vmatrix} 1 & 5 & -2 \\ 2 & -1 & 4 \\ 3 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -2 \\ 0 & -11 & 8 \\ 3 & 0 & -2 \end{vmatrix}$.

32. $\begin{vmatrix} 1 & 1 & -3 \\ 2 & 2 & 5 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 2 & 5 \\ 0 & -3 & 7 \end{vmatrix}$.

33. $\begin{vmatrix} 4 & -3 & 2 \\ 6 & 2 & 1 \\ -2 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & -3 & 2 \\ 3 & 2 & 1 \\ -1 & 2 & 4 \end{vmatrix}$.

38. $\begin{vmatrix} 6 & 0 & -2 \\ 2 & -1 & -3 \\ 1 & 5 & -7 \end{vmatrix} = -\begin{vmatrix} 0 & 6 & -2 \\ -1 & 2 & -3 \\ 5 & 1 & -7 \end{vmatrix}$.

39. $a^3\begin{vmatrix} 1 & 1 & 1 \\ a & a & a \\ a^2 & a^2 & a^2 \end{vmatrix} = \begin{vmatrix} a & a & a \\ a^2 & a^2 & a^2 \\ a^3 & a^3 & a^3 \end{vmatrix}$.

40. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 0$.

In Exercises 41 to 50, evaluate the determinant by rewriting the determinant in diagonal form by using elementary row or column operations.

41. $\begin{vmatrix} 2 & 4 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix}$.

42. $\begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & 4 \\ 2 & -2 & 3 \end{vmatrix}$.

43. $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix}$.

44. $\begin{vmatrix} 1 & 2 & 5 \\ -1 & 1 & -2 \\ 3 & 1 & 10 \end{vmatrix}$.

45. $\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{vmatrix}$.

46. $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix}$.

47. $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & 1 & 5 \\ -2 & -4 & 1 & 6 \end{vmatrix}$.

48. $\begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 4 & 12 \\ 1 & -1 & 0 & 8 \end{vmatrix}$.

49. $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 6 & 5 & 9 & 8 \\ 2 & 4 & 12 & -1 \\ 1 & 2 & 6 & -1 \end{vmatrix}$.

50. $\begin{vmatrix} 1 & 2 & 0 & -2 \\ -1 & 1 & 3 & 5 \\ 2 & 1 & 4 & 0 \\ -2 & 5 & 2 & 6 \end{vmatrix}$.

In Exercises 51 to 58, evaluate the determinants by using elementary row or column operations.

51. $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 2 & 1 \\ 3 & 8 & 6 \end{vmatrix}$

52. $\begin{vmatrix} 3 & 0 & 10 \\ 3 & -2 & 7 \\ 2 & -1 & 5 \end{vmatrix}$

53. $\begin{vmatrix} 3 & -8 & 7 \\ 2 & -3 & 6 \\ 1 & -3 & 2 \end{vmatrix}$

54. $\begin{vmatrix} 4 & 9 & -11 \\ 2 & 6 & -3 \\ 3 & 7 & -8 \end{vmatrix}$.

55. $\begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 6 & 3 \\ 3 & -1 & 8 & 7 \\ 3 & 0 & 9 & 9 \end{vmatrix}$.

56. $\begin{vmatrix} 1 & 2 & -2 & 3 \\ 3 & 7 & -3 & 11 \\ 2 & 3 & -5 & 11 \\ 2 & 6 & 1 & 8 \end{vmatrix}$.

57. $\begin{vmatrix} 1 & 2 & -2 & 1 \\ 2 & 5 & -3 & 1 \\ 2 & 0 & -10 & 1 \\ 3 & 8 & -4 & 1 \end{vmatrix}$.

1.3 The Inverse of a Matrix

Recall that the multiplicative inverse of a nonzero real number c is $\frac{1}{c}$ the number whose product with c is 1. For example, the multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$ because $\frac{2}{3}\cdot\frac{3}{2}=1$.

For some square matrices we can define a multiplicative inverse. Let us consider square matrix $A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{23} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$.

Definition of Multiplicative Inverse of a Matrix. If A is a square matrix of order n, then the inverse of matrix A, denoted by A^{-1}, has the property that

$$A\cdot A^{-1}=A^{-1}\cdot A=I_n,$$

where I_n is the identity matrix of order n.

Remark. As we will see shortly, not all square matrices have multiplicative inverses.

Definition. The square matrix A is called **nonsingular matrix** if $\det A\neq 0$. If $\det A=0$, then matrix A is called **singular matrix.**

Theorem (Existence of the Inverse of a Square Matrix) If A is a square matrix of order n, then A has a multiplicative inverse if and only if $|A|\neq 0$.

Statement. Inverse matrix A^{-1} is found by formula

$$A^{-1}=\frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{bmatrix},$$

where A_{ij} is the cofactor of a_{ij}.

Example1. Find the inverse of the matrix $A=\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution. Let us find the determinant of given matrix.

$$\det A=\begin{vmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix}=6+0-4-0-8-2=-8\neq 0\Rightarrow A \text{ is nonsingular}.$$

Let us find cofactors A_{ij}:

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}=6-8=-2 \qquad A_{23}=(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}=-(2+1)=-3,$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix}=-(-2-4)=6, \qquad A_{31}=(-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 3 & 4 \end{vmatrix}=-4-0=-4,$$

$A_{13}=(-1)^{1+3}\begin{vmatrix}-1 & 3\\ 1 & 2\end{vmatrix}=-2-3=-5,$ $\quad A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 0\\ -1 & 4\end{vmatrix}=-(4-0)=-4,$

$A_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 0\\ 2 & 2\end{vmatrix}=-(-2-0)=2,$ $\quad A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -1\\ -1 & 3\end{vmatrix}=3-1=2.$

$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}=2-0=2,$

The inverse matrix is $A^{-1}=\dfrac{1}{-8}\begin{bmatrix}-2 & 2 & -4\\ 6 & 2 & -4\\ -5 & -3 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}\\ -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2}\\ \frac{5}{8} & \frac{3}{8} & -\frac{1}{4}\end{bmatrix}.$

The second method for finding the inverse (we will simply use inverse for multiplicative inverse) uses elementary row operations. The procedure will be illustrated by finding the inverse of a 2×2 matrix.

Let $A=\begin{bmatrix}2 & 4\\ 7 & 1\end{bmatrix}$. To the matrix A we will merge the identity matrix I_2 to the right of A and denote this new matrix by $\left[A\middle|I_2\right]=\left[\begin{array}{cc|cc}2 & 7 & 1 & 0\\ 1 & 4 & 0 & 1\end{array}\right].$

$A \longrightarrow$ (left block) $\qquad$ (right block) $\longleftarrow I_2$

Now we use elementary row operations. The goal is to produce

$$\left[I_2\middle|A^{-1}\right]=\left[\begin{array}{cc|cc}1 & 0 & b_{11} & b_{12}\\ 0 & 1 & b_{21} & b_{22}\end{array}\right]$$

In this form, the inverse matrix is the matrix that is to the right of the identity matrix. That is, $A^{-1}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}.$

To find A^{-1}, we use a series of elementary row operations that will result in a 1 in the first row and first column

$$\left[\begin{array}{cc|cc}2 & 7 & 1 & 0\\ 1 & 4 & 0 & 1\end{array}\right]\xrightarrow{(1/2)R_1}\begin{bmatrix}1 & 7/2 & 1/2 & 0\\ 1 & 1/2 & 0 & 1\end{bmatrix}\xrightarrow{-1R_1+R_2}\begin{bmatrix}1 & 7/2 & 1/2 & 0\\ 0 & 1/2 & -1/2 & 1\end{bmatrix}\xrightarrow{2R_2}$$

$$\xrightarrow{2R_2}\begin{bmatrix}1 & 7/2 & 1/2 & 0\\ 0 & 1 & -1 & 2\end{bmatrix}\xrightarrow{(-7/2)R_2+R_1}\begin{bmatrix}1 & 0 & 4 & -7\\ 0 & 1 & -1 & 2\end{bmatrix}.$$

Now that the original matrix has been transformed into the identity matrix, the inverse matrix is the matrix to the right of the identity matrix. Therefore, $A^{-1}=\begin{bmatrix}4 & -7\\ -1 & 2\end{bmatrix}.$

Each elementary row operation is chosen to advance the process of transforming the original matrix into the identity matrix.

Example 2. Find the inverse of a 3×3 matrix $A=\begin{bmatrix}1&-1&2\\2&0&6\\3&-5&7\end{bmatrix}$.

Solution.

<table>
<tr><td>$$\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\2&0&6&0&1&0\\3&-5&7&0&0&1\end{array}\right]$$</td><td>Merge the given matrix with the identity matrix I_3.</td></tr>
<tr><td>$$\xrightarrow[-3R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&2&2&-2&1&0\\0&-2&1&-3&0&1\end{array}\right]$$</td><td>Since a_{11} is already 1, we next position zeros in a_{21} and a_{31}.</td></tr>
<tr><td>$$\xrightarrow{(1/2)R_2}\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&1&1&-1&1/2&0\\0&-2&1&-3&0&1\end{array}\right]$$</td><td>Position a 1 in a_{22}.</td></tr>
<tr><td>$$\xrightarrow{2R_2+R_3}\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&1&1&-1&1/2&0\\0&0&3&-5&1&1\end{array}\right]$$</td><td>Position a 0 in a_{32}.</td></tr>
<tr><td>$$\xrightarrow{(1/3)R_3}\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&1&1&-1&1/2&0\\0&0&1&-5/3&1/3&1/3\end{array}\right]$$</td><td>Position a 1 in a_{33}.</td></tr>
<tr><td>$$\xrightarrow[-2R_3+R_1]{-1R_3+R_2}\left[\begin{array}{ccc|ccc}1&-1&0&13/3&-2/3&-2/3\\0&1&0&2/3&1/6&-1/3\\0&0&1&-5/3&1/3&1/3\end{array}\right]$$</td><td>Now work upward. Position a 0 in a_{23} and a_{13}.</td></tr>
<tr><td>$$\xrightarrow{R_2+R_1}\left[\begin{array}{ccc|ccc}1&0&0&5&-1/2&-1\\0&1&0&2/3&1/6&-1/3\\0&0&1&-5/3&1/3&1/3\end{array}\right]$$</td><td>Position a 0 in a_{12}.</td></tr>
</table>

The inverse matrix is $A^{-1}=\begin{bmatrix}5&-1/2&-1\\2/3&1/6&-1/3\\-5/3&1/3&1/3\end{bmatrix}$.

You should verify that this matrix satisfies the condition of an inverse matrix. That is, show that $A^{-1}\cdot A=A\cdot A^{-1}=I_3$.

Properties of the Inverse Matrix

1. $\det A^{-1}=\dfrac{1}{\det A}$. 2. $\left(A^{-1}\right)^{-1}=A$ 3. $(AB)^{-1}=B^{-1}A^{-1}$

Exercise Set 3

In Exercises 1 to 14, find the inverse of given matrix.

1. $\begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & -2 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 3 & -2 \\ -1 & -5 & 6 \\ 2 & 6 & -3 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 4 \\ 2 & 10 \end{bmatrix}$.

6. $\begin{bmatrix} -2 & 3 \\ -6 & -8 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 1 \\ 3 & 6 & -4 \end{bmatrix}$.

8. $\begin{bmatrix} 2 & 1 & -1 \\ 6 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 4 & -4 \\ 1 & 3 & -4 \\ 2 & 4 & -3 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 3 & -6 & 6 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 5 & 1 \\ 3 & -3 & 7 & 5 \\ -2 & 3 & -4 & -1 \end{bmatrix}$.

12. $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & -1 & 5 \\ 2 & 2 & -1 & 5 \\ 4 & 4 & -4 & 7 \end{bmatrix}$.

13. $\begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 8 \\ 1 & 1 & 6 & 10 \\ -1 & 5 & 5 & 4 \end{bmatrix}$.

14. $\begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -1 & 6 & 6 \\ 3 & -1 & 12 & 12 \\ -2 & -1 & -14 & -10 \end{bmatrix}$.

In Exercises 15 to 26, find the inverse, if it exists, of the given matrix.

15. $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$.

16. $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$.

17. $\begin{bmatrix} -2 & 3 \\ 2 & 4 \end{bmatrix}$.

18. $\begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}$.

19. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ 3 & 8 & 6 \end{bmatrix}$.

20. $\begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 7 \\ 2 & -3 & 6 \end{bmatrix}$.

21. $\begin{bmatrix} 3 & -2 & 7 \\ 2 & -1 & 5 \\ 3 & 0 & 10 \end{bmatrix}$.

22. $\begin{bmatrix} 4 & 9 & -11 \\ 3 & 7 & -8 \\ 2 & 6 & -3 \end{bmatrix}$.

23. $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 6 & 5 \\ 3 & -1 & 9 & 6 \\ 2 & 2 & 4 & 7 \end{bmatrix}$.

24. $\begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 7 & -3 & 1 \\ 2 & 7 & 4 & 3 \\ 1 & 4 & 2 & 4 \end{bmatrix}$.

25. $\begin{bmatrix} 3 & 7 & -1 & 8 \\ 2 & 5 & 0 & 5 \\ 3 & 6 & -4 & 8 \\ 2 & 4 & -4 & 4 \end{bmatrix}$.

26. $\begin{bmatrix} 3 & 1 & 5 & -5 \\ 2 & 1 & 4 & -3 \\ 3 & 0 & 4 & -3 \\ 4 & 1 & 8 & 1 \end{bmatrix}$.

1.4 System of Linear Equations

Recall that an equation of the form $Ax + By = C$ is a linear equation in two variables. A solution of a linear equation in two variables is an ordered pair $(x; y)$, which makes the equation a true statement. The graph of a linear equation, a straight line, is the set of points whose ordered pairs satisfy the equation.

A system of equations is two or more equations considered together. The solution of a system of equations is an ordered pair that is the solution of each equation.

The system of equations is called a **consistent** system of equations when the system has unique solution.

The system of equations is called a **dependent** system of equations when the system has infinite set of solutions.

The system of equations is called a in**consistent** system of equations when the system has no solution.

1.4.1 Gauss-Jordan Elimination Method

A matrix can be created from a system of linear equations. Consider the system of linear equations $\begin{cases} 2x-3y+z=2 \\ x \qquad -3z=4. \\ 4x-y+4z=3 \end{cases}$

Using only the coefficients and constants of this system, we can write the 3×4 matrix

$$A/B=\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 1 & 0 & -3 & 4 \\ 4 & -1 & 4 & 3 \end{array}\right].$$

This matrix is called the **augmented matrix** of system of equations. The matrix formed by the coefficients of the system is the **coefficient matrix**. The matrix formed from the constants is the **constant matrix** for the system. The coefficient matrix and constant matrix for the given system are:

$$\text{Coefficient matrix: } A=\begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -3 \\ 4 & -1 & 4 \end{bmatrix} \quad \text{Constant matrix: } B=\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

Remark. When a term is missing one of the equations of system (as in the second equation), the coefficient of that term is 0 and a 0 is entered in the matrix.

In general case for the system

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=b_1 \\ a_{21}x+a_{22}y+a_{23}z=b_2 \\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{cases} \leftrightarrow A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } A/B=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right].$$

We can write a system of equations from an augmented matrix:

$$\text{Augmented matrix: } A/B=\left[\begin{array}{ccc|c} 2 & -1 & 4 & 3 \\ 1 & 1 & 0 & 2 \\ 3 & -2 & -1 & 2 \end{array}\right] \xrightarrow{\text{system}} \begin{cases} 2x-y+4z=3 \\ x+y \qquad =2 \\ 3x+2y-z=2 \end{cases}$$

In certain cases, an augmented matrix represents a system of equations that we can solve by back substitution. Consider the following augmented matrix and equivalent system of equations:

$$A/B=\left[\begin{array}{ccc|c} 1 & -3 & 4 & 5 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{equivalent system}} \begin{cases} x-3y+4z=5 \\ \qquad y+2z=-4 \\ \qquad\qquad z=-1 \end{cases}$$

Solving this system by using back substitution, we find that the solution is (3, -2, -1). An augmented matrix is said to be in echelon form if the equivalent system of equations can be solved by back substitution.

Echelon Form

An augmented matrix is in echelon form if all the following conditions are satisfied:

1. The first nonzero number in any row is 1.
2. Rows are arranged so that the column containing the first nonzero number is to the left of the column containing the first nonzero number of the next row.
3. All rows consisting entirely of zeros appear at the bottom of the matrix.

Following are three examples of matrices in echelon form:

$$\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

We can write an augmented matrix in echelon form by using what are called **elementary row operations**. These operations are a rewording, using matrix terminology, of the operations that produce equivalent equations.

Elementary Row Operations

Given the augmented matrix for a system of linear equations, each of the following elementary row operations produces a matrix of an equivalent system of equations. ERO are:

1. Interchanging two rows (Interchange the i-th and j-th rows: $R_i \leftrightarrow R_j$.).

2. Multiplying all the elements in a row by the same nonzero number (Multiply the i-th row by k, a nonzero constant: kR_i).

3. Replacing a row by the sum of that row and a nonzero multiple of any other row (Replace the j-th row by the sum of that row a nonzero multiple of the i-th row: $kR_i + R_j$).

To demonstrate these operations, we will use the 3×3 matrix $\begin{bmatrix} 2 & 1 & -2 \\ 3 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix}$.

$\begin{bmatrix} 2 & 1 & -2 \\ 3 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 3 \\ 3 & -2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$	Interchange row 1 and row 3.
$\begin{bmatrix} 2 & 1 & -2 \\ 3 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow{-3R_2} \begin{bmatrix} 2 & 1 & -2 \\ -9 & 6 & -6 \\ 1 & -2 & 3 \end{bmatrix}$	Multiply row 2 by –3.
$\begin{bmatrix} 2 & 1 & -2 \\ 3 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow{-2R_3 + R_1} \begin{bmatrix} 0 & 5 & -8 \\ 3 & -2 & 2 \\ 1 & -2 & 3 \end{bmatrix}$	Multiply row 3 by –2 and add to row 1. Replace row 1 by the sum.

The **Gauss-Jordan** elimination method is an algorithm that uses elementary row operations to solve a system of linear equations. The goal of this method is rewrite an augmented matrix in echelon form.

We will now demonstrate how to solve a system of two equations in two variables by the Gauss-Jordan method. Consider the system of equations $\begin{cases} 2x+5y=-1 \\ 3x-2y=8 \end{cases}$.

The augmented matrix for this system is $A/B=\left[\begin{array}{cc|c} 2 & 5 & -1 \\ 3 & -2 & 8 \end{array}\right]$.

The goal of the Gauss-Jordan method is to rewrite the augmented matrix in echelon form by using elementary row operations which are chosen so that first, there is a 1 as a_{11}, second, there is a 0 as a_{21}; third, there is a 1 as a_{22}.

$\left[\begin{array}{cc\|c} 2 & 5 & -1 \\ 3 & -2 & 8 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{cc\|c} 1 & 5/2 & -1/2 \\ 3 & -2 & 8 \end{array}\right]$	Multiply row 1 by $\frac{1}{2}$. The result is a 1 as a_{11}.
$\left[\begin{array}{cc\|c} 1 & 5/2 & -1/2 \\ 3 & -2 & 8 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc\|c} 1 & 5/2 & -1/2 \\ 0 & -19/2 & 19/2 \end{array}\right]$	Multiply row 1 by –3 and add the result to row 2. Replace row 2. The result is a 0 as a_{21}.
$\left[\begin{array}{cc\|c} 1 & 5/2 & -1/2 \\ 0 & -19/2 & 19/2 \end{array}\right] \xrightarrow{(-2/19)R_2} \left[\begin{array}{cc\|c} 1 & 5/2 & -1/2 \\ 0 & 1 & -1 \end{array}\right]$	Multiply row 2 by $-2/19$. The result is 1 as a_{22}. The matrix is now in row echelon form.

The system of equations written from the echelon form of the matrix is $\begin{cases} x+\frac{5}{2}y=-\frac{1}{2} \\ y=-1 \end{cases}$.

Solving by back substitution, replace y in the first equation by –1 and solve for x.

$$x+\left(\frac{5}{2}\right)(-1)=-\frac{1}{2} \Rightarrow x=2.$$

The solution of the system of equations is $(2;-1)$.

Remark. The order in which a matrix is reduced to echelon form is important. Using elementary row operations, change a_{11} to a 1 and change the remaining elements in the first column to 0. Move to a_{22}. Change a_{22} to 1 and change the remaining elements below a_{22} in column 2 to 0. Move to the next column and repeat the procedure. Continue moving down the main diagonal, repeating the procedure until you reach a_{mm} or all remaining elements on the main diagonal are zero.

To conserve space, we will occasionally perform more than one elementary row operation in one step. For example, the notation $\xrightarrow[-2R_1+R_3]{3R_1+R_2}$ means that two elementary row operations were performed. First multiply row 1 by 3 and add to row 2. Replace row 2. Now multiply row 1 by –2 and add to row 3. Replace row 3.

Example 1. Solve the system of equations using the Gauss-Jordan method

$$\begin{cases} x-3y+z=5 \\ 3x-7y+2z=12 \\ 2x-4y+z=3 \end{cases}$$

Solution. Write the augmented matrix and then use elementary row operations to write the matrix in echelon form.

$$A/B=\left[\begin{array}{ccc|c}1&-3&1&5\\3&-7&2&12\\2&-4&1&3\end{array}\right]\xrightarrow[-2R_1+R_3]{3R_1+R_2}\left[\begin{array}{ccc|c}1&-3&1&5\\0&2&-1&-3\\0&2&-1&-7\end{array}\right]\xrightarrow{(1/2)R_2}\left[\begin{array}{ccc|c}1&-3&1&5\\0&1&-1/2&-3/2\\0&2&-1&-7\end{array}\right]$$

$$\xrightarrow{-2R_2+R_3}\left[\begin{array}{ccc|c}1&-3&1&5\\0&1&-1/2&-3/2\\0&0&0&-4\end{array}\right].$$

Equivalent system $\begin{cases}x-3y+z=5\\ \quad y-\dfrac{1}{2}z=-\dfrac{3}{2}.\\ \quad 0z=-4\end{cases}$

Because the equation 0z = -4 has no solution, the system has no solution.

Example 2. Solve the system of equations using the Gauss-Jordan method

$$\begin{cases}x-3y+4z=1\\2x-5y+3z=6.\\x-2y-z=5\end{cases}$$

Solution. Write the augmented matrix and then use elementary row operations to reduce the matrix to echelon form.

$$A/B=\left[\begin{array}{ccc|c}1&-3&4&1\\2&-5&3&6\\1&-2&-1&5\end{array}\right]\xrightarrow[-R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|c}1&-3&4&1\\0&1&-5&4\\0&1&-5&4\end{array}\right]\xrightarrow{-R_2+R_3}\left[\begin{array}{ccc|c}1&-3&4&1\\0&1&-5&4\\0&0&0&0\end{array}\right].$$

Equivalent system $\begin{cases}x-3y+4z=1\\ \quad y-5z=4\end{cases}.$

Any solution of the system of equations is a solution of $y-5x=4$. Solve this equation for y then substitute into the first equation.

$$y=5x+4.$$

$$x-3y+4z=1.$$

$$x-3(5z+4)+4z=1$$

or

$$x=11z+13.$$

Both x and y are expressed in terms of z. Let z be any real number c. The solutions of the system of equations are $(11c+13;5c+4;c)$.

Example 3. Solve a system of equations using the Gauss-Jordan method

$$\begin{cases}x_1-2x_2-3x_3-2x_4=1\\2x_1-3x_2-4x_3-2x_4=3.\\x_1+x_2+x_3-7x_4=-7\end{cases}$$

Solution. Write the augmented matrix and use elementary row operations to reduce the matrix to echelon form.

$$A/B=\left[\begin{array}{cccc|c}1&-2&-3&-2&1\\2&-3&-4&-2&3\\1&1&1&-7&-7\end{array}\right]\xrightarrow[-1R_1+R_3]{-2R_1+R_2}\left[\begin{array}{cccc|c}1&-2&-3&-2&1\\0&1&2&2&1\\0&3&4&-5&-8\end{array}\right]\xrightarrow{-3R_2-\frac{1}{2}R_3}$$

$$\xrightarrow{-3R_2-\frac{1}{2}R_3}\left[\begin{array}{cccc|c}1&-2&-3&-2&1\\0&1&2&2&1\\0&0&-1&-11/2&-11/2\end{array}\right].$$

Equivalent system $\begin{cases}x_1-2x_2-3x_3-2x_4=1\\ \quad x_2+2x_3+2x_4=1\\ \quad\quad x_3+\dfrac{11}{2}x_4=\dfrac{11}{2}\end{cases}.$

We now express each of the variables in terms of x_4. Solve the third equation for x_3.

$$x_3=-\frac{11}{2}x_4+\frac{11}{2}$$

Substitute this value into the second equation and solve for x_2.

$$x_2+2\left(-\frac{11}{2}x_4+\frac{11}{2}\right)+2x_4=1$$

Simplifying, we have $x_2=9x_4-10$. Substitute the values for x_2 and x_3 into the first equation and solve for x_1.

$$x_1-2(9x_4-10)-3\left(-\frac{11}{2}x_4+\frac{11}{2}\right)-2x_4=1.$$

Simplifying, we have $x_1=\frac{7}{2}x_4-\frac{5}{2}$. If x_4 is any real number c, the solution is of the form $\left(\frac{7}{2}c-\frac{5}{2},9c-10,-\frac{11}{2}c+\frac{11}{2},c\right)$.

1.4.2 Method of Inverse Matrix

Systems of linear equations can be solved by finding the inverse of the coefficient matrix. Consider the system of linear equations

$$\begin{cases}a_{11}x+a_{12}y+a_{13}z=b_1\\a_{21}x+a_{22}y+a_{23}z=b_2\\a_{31}x+a_{32}y+a_{33}z=b_3\end{cases}\qquad(1)$$

Using matrix multiplication and the concept of equality of matrices, this system can be written as a matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad (2)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

then Equation (1) can be written as

$$AX = B. \qquad (3)$$

The inverse of the coefficient matrix A is A^{-1}. To solve the system of equations, multiply each side of the equation (3) by the inverse A^{-1}.

$$A^{-1}AX = A^{-1}B \overset{A^{-1}A=I}{\Leftrightarrow} X = A^{-1}B. \qquad (4)$$

Thus, we can find solution of the system (1) using formula (4).

Example 4. Find the solution of the system $\begin{cases} x_1 + \quad\quad 7x_3 = 20 \\ 2x_1 + x_2 - x_3 = -3 \\ 7x_1 + 3x_2 + x_3 = 2 \end{cases}$ of equations by using the inverse of the coefficient matrix.

Solution. Write the system as a matrix equation $\begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & -1 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -3 \\ 2 \end{bmatrix}$.

The inverse of the coefficient matrix is $A^{-1} = \begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix}$.

Multiplying each side of the matrix equation by the inverse, we have

$$\begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & -1 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & -7 & \frac{7}{3} \\ 3 & 16 & -5 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 20 \\ -3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Thus $x_1 = -1$, $x_2 = 2$ and $x_3 = 3$. The answer is (-1,2,3).

1.4.3 Cramer's Rule

An application of determinants is used to solve a system of linear equations. Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}.$$

To eliminate x_2 from this system, we first multiply the top equation by a_{22} and the bottom equation by a_{12}. Then subtract.

$$\begin{array}{c} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 \\ \underline{a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2} \\ (a_{22}a_{11} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2 \end{array}$$

This can also be written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} x_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

or

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

In a similar manner we can find x_2. The results are given in Cramer′s Rule for a system of two linear equations.

Statement (Cramer′s Rule for a System of Two Linear Equations). Let $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ be the system of equations for which the determinant of the coefficient matrix is not zero. The solution of the system of equations is the ordered pair whose coordinates are

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \text{and} \quad x_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Note that the denominator Δ - is the determinant of the coefficient matrix of the variables. The denominator Δ_1 is formed by replacing column 1 of the coefficient determinant with the constants b_1 and b_2. The determinant Δ_2 is formed by replacing column 2 of the coefficient determinant by the constants b_1 and b_2.

Example 5. Solve the following system of equations using Cramer′s Rule:

$$\begin{cases} 5x_1 - 3x_2 = 6 \\ 2x_1 - 4x_2 = -7 \end{cases}.$$

Solution. $x_1 = \frac{\begin{vmatrix} 6 & -3 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix}} = \frac{3}{26}$, $x_2 = \frac{\begin{vmatrix} 5 & 6 \\ 2 & -7 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix}} = -\frac{47}{26}$. Answer: $\left(\frac{3}{26}, -\frac{47}{26}\right)$.

Cramer′s Rule can be used for a system of three linear equations in three variables. For example, consider the system of equations

$$\begin{cases} 2x-3y+z=2 \\ 4x \qquad +2z=-3 \\ 3x+y-2z=1 \end{cases}$$

To solve this system of equations, we extend the concepts behind the solution for a system of two linear equations. The solution of the system has the form $(x;y;z)$ where

$$x=\frac{\Delta_1}{\Delta},\ y=\frac{\Delta_2}{\Delta},\ z=\frac{\Delta_3}{\Delta}.$$

The determinant Δ is the determinant of the coefficient matrix. The determinants Δ_1, Δ_2 and Δ_3 are the determinants of the matrices formed by replacing the first, second, and third columns, respectively, by the constants.

For the given system of equations $x=\frac{\Delta_1}{\Delta},\ y=\frac{\Delta_2}{\Delta},\ z=\frac{\Delta_3}{\Delta}$, where

$$\Delta=\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix}=-42,\ \Delta_1=\begin{vmatrix} 2 & -3 & 1 \\ -3 & 0 & 2 \\ 1 & 1 & -2 \end{vmatrix}=5,\ \Delta_2=\begin{vmatrix} 2 & 2 & 1 \\ 4 & -3 & 2 \\ 3 & 1 & -2 \end{vmatrix}=49,\ \Delta_3=\begin{vmatrix} 2 & -3 & 2 \\ 4 & 0 & -3 \\ 3 & 1 & 1 \end{vmatrix}=53.$$

Thus $x=-\frac{5}{42},\ y=-\frac{49}{42},\ z=-\frac{53}{42}$. Answer: $\left(-\frac{5}{42},-\frac{49}{42},-\frac{53}{42}\right)$.

Cramer′s Rule can be extended to a system of n linear equations in n variables.

Let $\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots+a_{2n}x_n=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\ldots+a_{3n}x_n=b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots+a_{nn}x_n=b_n \end{cases}$ be a system of n equations in n variables.

The solution of the system is given by $(x_1,x_2,x_3,\ldots,x_n)$ where

$$x_1=\frac{\Delta_1}{\Delta},\ x_2=\frac{\Delta_2}{\Delta},\ x_3=\frac{\Delta_3}{\Delta},\ldots,\ x_n=\frac{\Delta_n}{\Delta}.$$

Where Δ is the determinant of the coefficient matrix and $\Delta \neq 0$. Δ_i is the determinant formed by replacing the *i*-th column of the coefficient matrix with the column of constants $b_1, b_2, b_3, \ldots, b_n$.

Because the determinant of the coefficient matrix must be nonzero to use Cramer′s Rule, this method is not appropriate for systems of linear equations with no solution or infinitely many solutions. In fact, the only time a system of linear equations will have a unique solution is when the coefficient determinant is not zero, a fact summarized in the following theorem.

Theorem (Systems of Linear Equations with Unique Solutions) A system of n linear equations in n variables has a unique solution if and only if the determinant of the coefficient matrix is not zero.

Cramer′s Rule is also useful when we want to determine only a value of a single variable in a system of equations.

Example 6. Find x_3 for the system of equations $\begin{cases} 4x_1 + 3x_3 - 2x_4 = 2 \\ 3x_1 + x_2 + 2x_3 - x_4 = 4 \\ x_1 - 6x_2 - 2x_3 + 2x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = -1 \end{cases}$.

Solution. Find Δ and Δ_3: $\Delta = \begin{vmatrix} 4 & 0 & 3 & -2 \\ 3 & 1 & 2 & -1 \\ 1 & -6 & -2 & 2 \\ 2 & 2 & 0 & -1 \end{vmatrix} = 39$, $\Delta_3 = \begin{vmatrix} 4 & 0 & 2 & -2 \\ 3 & 1 & 4 & -1 \\ 1 & -6 & 0 & 2 \\ 2 & 2 & -1 & -1 \end{vmatrix} = 96$.

Thus, $x_3 = \dfrac{96}{39} = \dfrac{32}{13}$.

Exercise Set 4

In Exercises 1 to 4, write the augmented matrix, the coefficient matrix, and the constant matrix.

1. $\begin{cases} 2x - 3y + z = 1 \\ 3x - 2y + 3z = 0 \\ x + 5z = 4 \end{cases}$

2. $\begin{cases} -3y + 2z = 3 \\ 2x - y = -1 \\ 3x - 2y + 3z = 4 \end{cases}$

3. $\begin{cases} 2x - 3y - 4z + \omega = 2 \\ 2y + z = 2 \\ x - y + 2z = 4 \\ 3x - 3y - 2z = 1 \end{cases}$

4. $\begin{cases} x - y + 2z + 3\omega = -2 \\ 2x + z - 2\omega = 1 \\ 3x - 2\omega = 3 \\ -x + 3y - z = 3 \end{cases}$

In Exercises 5 to 12, use elementary row operations to write the matrix in echelon form.

5. $\begin{bmatrix} 2 & -1 & 3 & -2 \\ 1 & -1 & 2 & 2 \\ 3 & 2 & -1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 7 & 3 \\ 3 & 6 & 8 & -1 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -5 & -1 & 2 \\ 3 & -4 & 1 & -2 \\ 1 & -2 & -1 & 3 \end{bmatrix}$

8. $\begin{bmatrix} -2 & 1 & -1 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 1 & -1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 3 & -6 & 10 & -14 \\ 5 & -8 & 19 & -21 \\ 2 & -4 & 7 & -10 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & 5 & -2 & 2 \\ 4 & 3 & 1 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -3 & 4 & 2 & 1 \\ 2 & -3 & 5 & -2 & -1 \\ -1 & 2 & -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -1 & 3 & 2 & 2 \\ 1 & -2 & 2 & 1 & -1 \\ 3 & -5 & -1 & -2 & 3 \end{bmatrix}$

In Exercises 13 to 38, solve the system of equations by the Gauss-Jordan method.

13. $\begin{cases} x+2y-2z=-2 \\ 5x+9y-4z=-3 \\ 3x+4y-5z=-3 \end{cases}$

14. $\begin{cases} x-3y+z=8 \\ 2x-5y-3z=2 \\ x+4y+z=1 \end{cases}$

15. $\begin{cases} 3x+7y-7z=-4 \\ x+2y-3z=0 \\ 5x+6y+z=-8 \end{cases}$

16. $\begin{cases} 2x-3y+2z=13 \\ 3x-4y-3z=1 \\ 3x+y-z=2 \end{cases}$

17. $\begin{cases} x+2y-2z=3 \\ 5x+8y-6z=14 \\ 3x+4y-2z=8 \end{cases}$

18. $\begin{cases} 3x-5y+2z=4 \\ x-3y+2z=4 \\ 5x-11y+6z=12 \end{cases}$

19. $\begin{cases} 3x+2y-z=1 \\ 2x+3y-z=1 \\ x-y+2z=3 \end{cases}$

20. $\begin{cases} 2x+5y+2z=-1 \\ x+2y-3z=5 \\ 5x+12y+z=10 \end{cases}$

21. $\begin{cases} x-3y+2z=0 \\ 2x-5y-2z=0 \\ 4x-11y+2z=0 \end{cases}$

22. $\begin{cases} x+y-2z=0 \\ 3x+4y-z=0 \\ 5x+6y-5z=0 \end{cases}$

23. $\begin{cases} 2x+y-3z=4 \\ 3x+2y+z=2 \end{cases}$

24. $\begin{cases} 3x-6y+2z=2 \\ 2x+5y-3z=2 \end{cases}$

25. $\begin{cases} 2x+2y-4z=4 \\ 2x+3y-5z=4 \\ 4x+5y-9z=8 \end{cases}$

26. $\begin{cases} 3x-10y+2z=34 \\ x-4y+z=13 \\ 5x-2y+7z=31 \end{cases}$

27. $\begin{cases} x+3y+4z=11 \\ 2x+3y+2z=7 \\ 4x+9y+10z=20 \\ 3x-2y+z=1 \end{cases}$

28. $\begin{cases} x-4y+3z=4 \\ 3x-10y+3z=4 \\ 5x-18y+9z=10 \\ 2x+2y-3z=-11 \end{cases}$

29. $\begin{cases} t+2a-3\nu+w=-7 \\ 3t+5u-8\nu+5w=-8 \\ 2t+3a-7\nu+3w=-11 \\ 4t+8a-10\nu+7w=-10 \end{cases}$

30. $\begin{cases} t+4u+2v-3w=11 \\ 2t+10u+3v-5w=17 \\ 4t+16u+7v-9w=34 \\ t+4u+v-w=4 \end{cases}$

31. $\begin{cases} 2t-u+3v+2w=2 \\ t-u+2v+w=2 \\ 3t \qquad -2v-3w=13 \\ 2t+2u \qquad -2w=6 \end{cases}$

32. $\begin{cases} 4t+7u-10v+3w=-29 \\ 3t+5u-7v+2w=-20 \\ t+2u-3v+w=-9 \\ 2t-u+2v-4w=15 \end{cases}$

33. $\begin{cases} 3t+10u+7v-6w=7 \\ 2t+8u+6v-5w=-5 \\ t+4u+2v-3w=2 \\ 4t+14u+9v-8w=8 \end{cases}$

34. $\begin{cases} t-u+3v-5w=10 \\ 2t-3u+4v+w=7 \\ 3t+u-2v-2w=6 \end{cases}$

35. $\begin{cases} t-3u+2v+4w=13 \\ 3t-8u+4v+13w=35 \\ 2t-7u+8v+5w=28 \\ 4t-11u+6v+17w=56 \end{cases}$

36. $\begin{cases} t-u+2v-3w=9 \\ 4t \qquad +11v-10w=46 \\ 3t-u+8v-6w=27 \end{cases}$

37. $\begin{cases} 3t-4u+v \qquad =2 \\ t+u-2v+3w=1 \end{cases}$

38. $\begin{cases} 2t \qquad +3v-4w=2 \\ t+2u-4v+w=-3 \end{cases}$

In Exercises 23 to 32, solve the system of equations by using inverse matrix methods.

23. $\begin{cases} x+4y=6 \\ 2x+7y=11 \end{cases}$

24. $\begin{cases} 2x+3y=5 \\ x+2y=4 \end{cases}$

25. $\begin{cases} x-2y=8 \\ 3x+2y=-1 \end{cases}$

26. $\begin{cases} 3x-5y=-18 \\ 2x-3y=-11 \end{cases}$

27. $\begin{cases} x+y+2z=4 \\ 2x+3y+3z=5 \\ 3x+3y+7z=14 \end{cases}$

28. $\begin{cases} x+2y-z=5 \\ 2x+3y-z=8 \\ 3x+6y-2z=14 \end{cases}$

29. $\begin{cases} x+2y+2z=5 \\ -2x-5y-2z=8 \\ 2x+4y+7z=19 \end{cases}$

30. $\begin{cases} x-y+3z=5 \\ 3x-y+10z=16 \\ 2x-2y+5z=9 \end{cases}$

31. $\begin{cases} w+2x+z=6 \\ 2w+5x+y+2z=10 \\ 2w+4x+y+4z=8 \\ 3w+6x+z=16 \end{cases}$

32. $\begin{cases} w-x+2y=5 \\ 2w-x+6y+2z=16 \\ 3w-2x+9y+4z=28 \\ w-2x-z=2 \end{cases}$

In Exercises 33 to 52, solve each system of equations by using Cramer's Rule.

33. $\begin{cases} 3x_1+4x_2=8 \\ 4x_1-5x_2=1 \end{cases}$

34. $\begin{cases} x_1-3x_2=9 \\ 2x_1-4x_2=-3 \end{cases}$

35. $\begin{cases} 5x_1+4x_2=-1 \\ 3x_1-6x_2=5 \end{cases}$

36. $\begin{cases} 2x_1+5x_2=9 \\ 5x_1+7x_2=8 \end{cases}$

37. $\begin{cases} 7x_1+2x_2=0 \\ 2x_1+x_2=-3 \end{cases}$

38. $\begin{cases} 3x_1-8x_2=1 \\ 4x_1+5x_2=-2 \end{cases}$

39. $\begin{cases} 3x_1-7x_2=0 \\ 2x_1+4x_2=0 \end{cases}$

40. $\begin{cases} 5x_1+4x_2=-3 \\ 2x_1-4x_2=0 \end{cases}$

41. $\begin{cases} 1{,}2x_1+0{,}3x_2=2{,}1 \\ 0{,}8x_1-1{,}4x_2=-1{,}6 \end{cases}$

42. $\begin{cases} 3{,}2x_1-4{,}2x_2=1{,}1 \\ 0{,}7x_1+3{,}2x_2=-3{,}4 \end{cases}$

43. $\begin{cases} 3x_1-4x_2+2x_3=1 \\ x_1-x_2+2x_3=-2 \\ 2x_1+2x_2+3x_3=-3 \end{cases}$

44. $\begin{cases} 5x_1-2x_2+3x_3=-2 \\ 3x_1+x_2-2x_3=3 \\ x_1-2x_2+3x_3=-1 \end{cases}$

45. $\begin{cases} x_1+4x_2-2x_3=0 \\ 3x_1-2x_2+3x_3=4 \\ 2x_1+x_2-3x_3=-1 \end{cases}$

46. $\begin{cases} 4x_1-x_2+2x_3=6 \\ x_1+3x_2-x_3=-1 \\ 2x_1+3x_2-2x_3=5 \end{cases}$

47. $\begin{cases} 2x_2-3x_3=1 \\ 3x_1-5x_2+x_3=0 \\ 4x_1+2x_3=-3 \end{cases}$

48. $\begin{cases} 2x_1+5x_2=1 \\ x_1-3x_3=-2 \\ 2x_1-x_2+2x_3=4 \end{cases}$

49. $\begin{cases} 4x_1-5x_2+x_3=-2 \\ 3x_1+x_2=4 \\ x_1-x_2+3x_3=0 \end{cases}$

50. $\begin{cases} 3x_1-x_2+x_3=5 \\ x_1+3x_3=-2 \\ 2x_1+2x_2-5x_3=0 \end{cases}$

51. $\begin{cases} 2x_1+2x_2-3x_3=0 \\ x_1-3x_2+2x_3=0 \\ 4x_1-x_2+3x_3=0 \end{cases}$

52. $\begin{cases} x_1+3x_2=-2 \\ 2x_1-3x_2+x_3=1 \\ 4x_1+5x_2-2x_3=0 \end{cases}$

In Exercises 53 to 58, solve for the indicated variable.

53. $\begin{cases} 2x_1-3x_2+4x_3-x_4=1 \\ x_1+2x_2+2x_4=-1 \\ 3x_1+x_2-2x_4=2 \\ x_1-3x_2+2x_3-x_4=3 \end{cases}$

56. $\begin{cases} 2x_1+5x_2-5x_3-3x_4=-3 \\ x_1+7x_2+8x_3-x_4=4 \\ 4x_1+x_3+x_4=3 \\ 3x_1+2x_2-x_3=0 \end{cases}$

Solve for x_2.

54. $\begin{cases} 3x_1 + x_2 + 3x_4 = 4 \\ 2x_1 - 3x_2 = -2 \\ x_1 + x_2 + 2x_4 = 3 \\ 2x_1 \quad 3x_2 - 2x_4 = 4 \end{cases}$

Solve for x_4.

55. $\begin{cases} x_1 - 3x_2 + 2x_3 + 4x_4 = 0 \\ 3x_1 + 5x_2 - 6x_3 + 2x_4 = -2 \\ 2x_1 - x_2 + 9x_3 + 8x_4 = 0 \\ x_1 + x_2 + x_3 - 8x_4 = -3 \end{cases}$

Solve for x_1.

Solve for x_3.

57. $\begin{cases} 3x_2 - x_3 + 2x_4 = 1 \\ 5x_1 + x_2 + 3x_3 - x_4 = -4 \\ x_1 - 2x_2 + 9x_4 = 5 \\ 2x_1 + 2x_3 = 3 \end{cases}$

Solve for x_4.

58. $\begin{cases} 4x_1 + x_2 - 3x_4 = 4 \\ 5x_1 + 2x_2 - 2x_3 + x_4 = 7 \\ x_1 - 3x_2 + 2x_3 - 2x_4 = -6 \\ 3x_3 - 4x_4 = -7 \end{cases}$

Solve for x_1.

1.5 Review

Gauss-Jordan Elimination Method

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is of order $m \times n$ or dimension $m \times n$.

For a system of equations, it is possible to form a coefficient matrix, an augmented matrix, and a constant matrix.

Echelon Form. An augmented matrix is in echelon form if all the following conditions are satisfied:

1. The first nonzero number in any row is a 1.
2. Rows are arranged so that the column containing the first nonzero number is to the left of the column containing the first nonzero number of the next row.
3. All rows consisting entirely of zeros appear at the bottom of the matrix.

The Gauss-Jordan elimination method is used to solve a system of equations.

Elementary Row Operations. The elementary row operations for a matrix are:

1. Interchanging two rows.
2. Multiplying all the elements in a row by the same nonzero number.
3. Replacing a row by the sum of that row and a multiple of any other row.

The Algebra of Matrices

Two matrices A and B are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

The sum of two matrices of the same order is the matrix whose elements are the sum of the corresponding elements of the two matrices.

The $m \times n$ zero matrix is the matrix whose elements are all zeros.

The product of real number and a matrix is called scalar multiplication.

To multiply two matrices, the number of columns of the first matrix must equal the number of rows of the second matrix.

In general, matrix multiplication is not commutative.

The multiplicative identity matrix is the matrix with 1s on the main diagonal and zeros everywhere else.

The Inverse of Matrix

The multiplicative inverse of a square matrix A, denoted by A^{-1}, has the property that

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n,$$

where I_n is the multiplicative identity matrix.

A singular matrix is one that does not have a multiplicative inverse.

Determinants

Associated with each square matrix is a number called the determinant of the matrix. The minor of the element a_{ij} of a square matrix is the determinant of the matrix obtained by deleting the *i*-th row and *j*-th column of A.

The cofactor of the element a_{ij} of a square matrix is $(-1)^{i+j} M_{ij}$, where M_{ij} is the minor of the element.

The value of a determinant can be found by multiplying the elements of any row or column by their respective cofactors and then adding the results. This is called expanding by cofactors.

Cramer′s Rule

Cramer′s Rule is a method of solving a system of n equations in n variables.

Challenge Exercises

In Exercises 1 to 15, answer true or false. If the answer is false, give an example

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 4 & 9 \\ 1 & 16 \end{bmatrix}$.
2. Every matrix has an additive inverse.
3. Every square matrix has a multiplicative inverse.
4. Let the matrices A, B and C be square matrices of order n. If $AB = AC$, then $B = C$.
5. It is possible to find the determinant of every square matrix.
6. If A and B are square matrices of order n, then det(*A*+*B*)= det(*A*)+det(*B*).
7. Cramer′s Rule can be used to solve any system of three equations in three variables.
8. If A and B are matrices of order n, then $AB - BA = 0$.
9. A nonsingular matrix has a multiplicative inverse.
10. If A, B and C are square matrix of order n, then the product ABC depends on which two matrices are multiplied first. That is $(AB)C$ produces a different result than $A(BC)$.
11. The Gauss-Jordan method for solving a system of linear equations can be applied only to systems of equations that have the same number of variables as equations.
12. If A is a square matrix of order n, then $\det(2A) = 2\det A$.
13. If A and B are matrices, then the product AB is defined when the number of columns of A equals the number of rows B.
14. If A and B are squared matrices of order n and $AB = 0$ (the zero matrix), then $A = 0$ or $B = 0$.
15. If $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, then $A^5 = A$.

Review Exercises

In Exercises 1 to 18, perform the indicated operations. Let

$$A=\begin{bmatrix}2 & -1 & 3\\ 3 & 2 & -1\end{bmatrix}, B=\begin{bmatrix}0 & -2\\ 4 & 2\\ 1 & -3\end{bmatrix}, C=\begin{bmatrix}2 & 6 & 1\\ 1 & 2 & -1\\ 2 & 4 & -1\end{bmatrix} \text{ and } D=\begin{bmatrix}-3 & 4 & 2\\ 4 & -2 & 5\end{bmatrix}.$$

1. $3A$.
2. $-2B$.
3. $-A+D$.
4. $2A-3D$.
5. AB.
6. DB.
7. BA.
8. BD.
9. C^2.
10. C^3.
11. BAC.
12. ADB.
13. $AB-BA$.
14. $DB-BD$.
15. $(A-D)C$.
16. $AC-DC$.
17. C^{-1}.
18. $|C|$.

In Exercises 19 to 34, solve the system of equations by using the Gauss-Jordan method.

19. $\begin{cases}2x-3y=7\\ 3x-4y=10\end{cases}$
20. $\begin{cases}3x+4y=-9\\ 2x+3y=-7\end{cases}$
21. $\begin{cases}4x-5y=12\\ 3x+y=9\end{cases}$
22. $\begin{cases}2x-5y=10\\ 5x+2y=4\end{cases}$
23. $\begin{cases}x+2y+3z=3\\ 3x+8y+11z=17\\ 2x+6y+7z=12\end{cases}$
24. $\begin{cases}x-y+3z=10\\ 2x-y+7z=24\\ 3x-6y+7z=21\end{cases}$
25. $\begin{cases}2x-y-z=4\\ x-2y-2z=5\\ 3x-3y-8z=19\end{cases}$
26. $\begin{cases}3x-7y+8z=10\\ x-3y+2z=0\\ 2x-8y+7z=5\end{cases}$
27. $\begin{cases}4x-9y+6z=54\\ 3x-8y+8z=49\\ x-3y+2z=17\end{cases}$
28. $\begin{cases}3x+8y-5z=6\\ 2x+9y-z=-8\\ x-4y-2z=16\end{cases}$
29. $\begin{cases}x+y+2z=-5\\ 2x+3y+5z=-13\\ 2x+5y+7z=-19\end{cases}$
30. $\begin{cases}x-2y+3z=9\\ 3x-5y+8z=25\\ x-z=5\end{cases}$
31. $\begin{cases}w+2x-y+2z=1\\ 3w+8x+y+4z=1\\ 2w+7x+3y+2z=0\\ w+3x-2y+5z=6\end{cases}$
32. $\begin{cases}w-3x-2y+z=-1\\ 2w-5x+3z=1\\ 3w-7x+3y=-18\\ 2w-3x-5y-2z=-8\end{cases}$
33. $\begin{cases}w+3x+y-4z=3\\ w+4x+3y-6z=5\\ 2w+8x+7y-5z=11\\ 2w+5x-6z=4\end{cases}$

In Exercises 35 to 38, solve the given system of equations for each set of constants. Use the inverse matrix method.

35. $\begin{cases}3x+4y=b_1\\ 2x+3y=b_2\end{cases}$

a. $b_1=2, b_2=-3$ b. $b_1=-2, b_2=4$

36. $\begin{cases}2x+y-z=b_1\\ 4x+4y+z=b_2\\ 2x+2y-3z=b_3\end{cases}$

a. $b_1=-1, b_2=2, b_3=4$ b. $b_1=-2, b_2=3, b_3=0$

37. $\begin{cases}2x-5y=b_1\\ 3x-7y=b_2\end{cases}$

a. $b_1=-3, b_2=4$ b. $b_1=2, b_2=-5$

38. $\begin{cases}3x-2y+z=b_1\\ 3x-y+3z=b_2\\ 6x-4y+z=b_3\end{cases}$

a. $b_1=0, b_2=3, b_3=-2$ b. $b_1=1, b_2=2, b_3=-4$

In Exercises 39 to 44, solve the system of equations by using Cramer′s Rule.

39. $\begin{cases} 2x_1 - 3x_2 = 2 \\ 3x_1 + 5x_2 = 2 \end{cases}$

40. $\begin{cases} 2x_1 + x_2 - 3x_3 = 2 \\ 3x_1 + 2x_2 + x_3 = 1 \\ x_1 - 3x_2 + 4x_3 = -2 \end{cases}$

41. $\begin{cases} 2x_2 + 5x_3 = 2 \\ 2x_1 - 5x_2 + x_3 = 4 \\ 4x_1 + 3x_2 = 2 \end{cases}$

42. $\begin{cases} 3x_1 + 4x_2 = -3 \\ 5x_1 - 2x_2 = 2 \end{cases}$

43. $\begin{cases} 3x_1 + 2x_2 - x_3 = 0 \\ x_1 + 3x_2 - 2x_3 = 3 \\ 4x_1 - x_2 - 5x_3 = -1 \end{cases}$

44. $\begin{cases} 2x_1 - 3x_2 - 4x_3 = 2 \\ x_1 - 2x_2 + 2x_3 = -1 \\ 2x_1 + 7x_2 - x_3 = 2 \end{cases}$

In Exercises 45 and 46, solve for the indicated variable.

45. $\begin{cases} x_1 - 3x_2 + x_3 + 2x_4 = 3 \\ 2x_1 + 7x_2 - 3x_3 + x_4 = 2 \\ -x_1 + 4x_2 + 2x_3 - 3x_4 = -1 \\ 3x_1 + x_2 - x_3 - 2x_4 = 0 \end{cases}$

Solve for x_3.

46. $\begin{cases} 2x_1 + 3x_2 - 2x_3 + x_4 = -2 \\ x_1 - x_2 - 3x_3 + 2x_4 = 2 \\ 3x_1 + 3x_2 - 4x_3 - x_4 = 4 \\ 5x_1 - 5x_2 - x_3 + 2x_4 = 7 \end{cases}$

Solve for x_2.

In Exercises 47 and 48, solve the input-output problems.

47. An electronics conglomerate has three divisions, which produce computers, monitors, and disk drivers. For each $ 1 of output, the computer division needs $.05 worth of computers, $.02 worth of monitors, and $.03 of disk drivers. For each $1 of output, the monitor division needs $.06 worth of computers, $.04 worth of monitors, and $.03 worth of disk drivers. For each $1 of output, the disk drive division requires $.08 worth of computers, $.04 worth of monitors, and $.05 worth of disk drivers. If sales level estimates are $30 million for the computer division, $12 million for the monitor division, and $21 million for the disk drive division, at what level should each division produce to satisfy this demand?

48. A manufacturing conglomerate has three divisions, which produce paper, lumber, and prefabricated walls. For each $1 of output, the lumber division needs $.07 worth of lumber, $.03 worth of paper, and $.03 of prefabricated walls. For each $1 of output, the paper division needs $.04 worth of lumber, $.07 worth of paper, and $.03 worth of prefabricated wall. For each $1 of output, the prefabricated walls division requires $.07 worth of lumber, $.04 worth of paper, and $.02 worth of prefabricated walls. If sales estimates are $27 million for the lumber division, $18 million for the paper division, and $10 million for the prefabricated walls division, at what level should each division produce to satisfy this demand?

II ANALYTIC GEOMETRY

2.1 Algebraic Operations on Vectors

2.1.1 The Algebra of Vectors

Wind, fluid flow, and force introduce the mathematical notion of a **vector,** which may be represented by an arrow that records both magnitude and a direction.

Definition. A vector is a directed line segment $\overrightarrow{AB}$ from A to B, as shown in Fig. 1. A is the "tail" and B is the "head" of the vector.

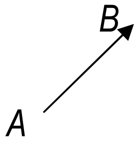


Figure 1

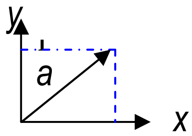


Figure 2

It is useful to introduce the **zero vector**. It has length 0 and no direction. In print letters $\vec{a}, \vec{b}, \ldots$ or $\overrightarrow{AB}, \overrightarrow{CD}, \ldots$ are used to denote vectors. The length of $\vec{a}$ is denoted by $|\vec{a}|$ and also is called the **magnitude of** $\vec{a}$. Any vector of length 1 is called a **unit vector.**

If the origin of a rectangular coordinate system is at the tail of $\vec{a}$, then the head of $\vec{a}$ has coordinates $(x;y;z)$, as shown in Fig. 2. The numbers x, y and z are called the **scalar components of** $\vec{a}$ relative to the coordinate system.

If $A(x_1;y_1;z_1)$ and $B(x_2;y_2;z_2)$ then a vector $\overrightarrow{AB}$ has **scalar components:**

$$\overrightarrow{AB} = (x_2 - x_1; y_2 - y_1; z_2 - z_1)$$

The magnitude of $\vec{a} = (x;y;z)$ is $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$.

Example 1. Find the length of the vector represented by $\overrightarrow{PQ}$ if $P(4;1)$ and $Q(7;-3)$.

Solution. The scalar components of $\overrightarrow{PQ} = (7-4; -3-1) = (3;-4)$. Thus $|\overrightarrow{PQ}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$.

Adding and Subtracting Vectors

1. Addition of vectors. The sum of two vectors $\vec{a}$ and $\vec{b}$ is defined as follows. Place $\vec{b}$ in such a way that its tail is at the head of $\vec{a}$. Then the vector $\vec{c} = \vec{a} + \vec{b}$ goes from the tail of $\vec{a}$ to the head of $\vec{b}$. Observe that $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, since both sums lie on the diagonal of the parallelogram, as shown in Fig. 3.

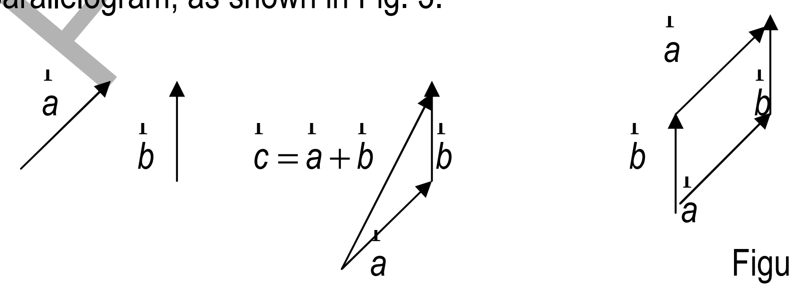


Figure 3

If $\vec{a}=(x_a;y_a;z_a)$ and $\vec{b}=(x_b;y_b;z_b)$ then $\vec{c}=\vec{a}+\vec{b}=(x_a+x_b;y_a+y_b;z_a+z_b)$.

Example 1. Find the sum of the vectors $\vec{a}=(2;3)$ and $\vec{b}=(4;1)$.

Solution. $\vec{c}=\vec{a}+\vec{b}=(2+4;3+1)=(6;4)$.

2. Subtraction of vectors. Let $\vec{a}$ and $\vec{b}$ be vectors. The vector $\vec{c}$ such that $\vec{c}+\vec{b}=\vec{a}$ is called the difference of $\vec{a}$ and $\vec{b}$ is denoted $\vec{c}=\vec{a}-\vec{b}$. Thus, if $\vec{a}=(x_a;y_a;z_a)$ and $\vec{b}=(x_b;y_b;z_b)$ then $\vec{c}=\vec{a}-\vec{b}=(x_a-x_b;y_a-y_b;z_a-z_b)$ (see Fig 4).

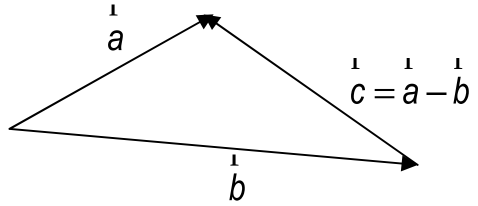


Figure 4

3. The **negative** of the vector $\vec{a}$ is defined as the vector having the same magnitude as $\vec{a}$ but the opposite direction. It is denoted $-\vec{a}$. If $\vec{a}=\overrightarrow{AB}$, then $-\vec{a}=\overrightarrow{BA}$. Observe that $\vec{a}+(-\vec{a})=\vec{0}$, just as with scalars.

Example 2. Show that $\vec{a}=\left(\frac{4}{9};-\frac{8}{9};\frac{1}{9}\right)$ is a unit vector.

Solution. $|\vec{a}|=\sqrt{\left(\frac{4}{9}\right)^2+\left(-\frac{8}{9}\right)^2+\left(\frac{1}{9}\right)^2}=\sqrt{\frac{16+64+1}{81}}=\sqrt{\frac{81}{81}}=1$.

Definition. If c is a scalar and $\vec{a}$ a vector, the product $c\vec{a}$ is the vector whose length is $|c|$ times the length of $\vec{a}$ and whose direction is the same as that of $\vec{a}$ if c is positive and opposite to that of $\vec{a}$ if c is negative.

Observe that $0\vec{a}$ has length 0 and thus is the zero vector $\vec{0}$ to which no direction is assigned. The vector $c\vec{a}$ is called a **scalar multiple** of the vector $\vec{a}$. If $\vec{a}=(x;y;z)$, then $c\vec{a}=(cx;cy;cz)$.

Example 2. Compute $3\vec{a}$ when $\vec{a}=(-1;0;2)$.

Solution. $3\vec{a}=3(-1;0;2)=(-3;0;6)$.

Theorem. For any vector $\vec{a}$ not equal to $\vec{0}$, the vector $\frac{\vec{a}}{|\vec{a}|}$ is the unit vector in the direction of $\vec{a}$.

Example 3. Compute $\frac{\vec{a}}{|\vec{a}|}$ when $\vec{a}=(4;-5;20)$.

Solution. We have $\left|\vec{a}\right| = \sqrt{4^2 + (-5)^2 + 20^2} = \sqrt{441} = 21$. Thus $\dfrac{\vec{a}}{\left|\vec{a}\right|} = \left(\dfrac{4}{21}; -\dfrac{5}{21}; \dfrac{20}{21}\right)$.

The next definition introduces and names three special vectors which will be used often.

Definition. The vectors $\vec{i} = (1;0;0)$, $\vec{j} = (0;1;0)$ and $\vec{k} = (0;0;1)$ are called the **basic vectors**.

Example 4. Express the vector $3\vec{i} + 4\vec{j} + 5\vec{k}$ in the form $\overrightarrow{(x;y;z)}$.

Solution.

$$3\vec{i} + 4\vec{j} + 5\vec{k} = 2\overrightarrow{(1;0;0)} + 4\overrightarrow{(0;1;0)} + 5\overrightarrow{(0;0;1)} = \overrightarrow{(2;0;0)} + \overrightarrow{(0;4;0)} + \overrightarrow{(0;0;5)} = \overrightarrow{(2;4;5)}.$$

As Example 4 suggests, $x\vec{i} + y\vec{j} + z\vec{k} = \overrightarrow{(x;y;z)}$. Generally, the notation $x\vec{i} + y\vec{j} + z\vec{k}$ is preferable to $\overrightarrow{(x;y;z)}$. First of all, it is more geometric. Second, when x, y and z are messy expressions, the notation $x\vec{i} + y\vec{j} + z\vec{k}$ is easier to read.

When representing a plane vector $\vec{a} = \overrightarrow{(x;y)}$, we do not need the vector $\vec{k}$; we have simply $\vec{a} = x\vec{i} + y\vec{j}$, since the coefficient of $\vec{k}$ is 0.

Exercise Set 5

1. Draw the vector $(2;3)$, placing its tail at:

   a) $(0;0)$ b) $(-1;2)$ c) $(1;1)$

2. Draw the vector $(1;2)$, placing its tail at:

   a) $(0;0)$ b) $(-1;2)$ c) $(3;3)$.

3. Find $\left|\vec{a}\right|$ if $\vec{a}$ is:

   a) $\vec{a} = (4;0)$ b) $\vec{a} = (-5;12)$ c) $\vec{a} = (5;12)$ d) $\vec{a} = (-8;-6)$

   f) $\vec{a} = \overrightarrow{(1;3;-2)}$ j) $\vec{a} = \overrightarrow{(2;1;-5)}$

4. Find the magnitude of $\overrightarrow{AB}$ if:

   a) $A = (2;1), B = (1;4)$ b) $A = (1;-3), B = (-2;4)$

5. Find $\vec{a} + \vec{b}$ if:

   a) $|y| = 10\left(\dfrac{7}{2}; \dfrac{13}{2}\right), \vec{b} = (2;0)$ b) $\vec{a} = (2;2), \vec{b} = (1;-1)$

6. Find $\vec{a} - \vec{b}$ if:

   a) $\vec{a} = (4;3), \vec{b} = (2;0)$, b) $\vec{a} = (3;4), \vec{b} = (5;1)$, c) $\vec{a} = (1;1), \vec{b} = (-2;4)$,

   d) $\vec{a} = (2;3;4), \vec{b} = (1;5;0)$, e) $\vec{a} = (3;4;2), \vec{b} = (0;0;0)$.

7. Find the scalar components of $\vec{a}$ if:

   a) its tail is at $\left(\dfrac{1}{2}; 5\right)$ and its head at $\left(\dfrac{7}{2}; \dfrac{13}{2}\right)$;

   b) its tail is at $(2;7)$ and its head at $(2;4)$;

c) its tail is at $(2;4)$ and its head at $(2;7)$;

d) its tail is at $(5;3)$ and its head at $\left(-\frac{1}{2};\frac{1}{3}\right)$.

8. Find the scalar components of $\vec{a}$ if:

a) $|\vec{a}| = 10$, and $\vec{a}$ points to the northwest;

b) $|\vec{a}| = 6$, and $\vec{a}$ points to the south;

c) $|\vec{a}| = 9$, and $\vec{a}$ points to the southeast;

d) $|\vec{a}| = 5$, and $\vec{a}$ points to the east.

(North is indicated to the positive y axis).

9. Plot the given points P:

a) $P(2,3,1)$; b) $P(0,4,-1)$; c) $P(1,-1,2)$; d) $P(-1,-1,2)$; f) $P(-1,-2,-3)$.

10. Find the distance between the points:

a) $(1,4,2)$ and $(2,1,5)$; b) $(-3,2,1)$ and $(4,0,-2)$.

11. Find $|\vec{a}+\vec{b}|$ if:

a) $\vec{a} = (1;2;0)$, $\vec{b} = (2;3;5)$; b) $\vec{a} = (3;-2;1)$, $\vec{b} = (-4;3;2)$.

12. Compute and sketch $c\vec{a}$ if $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$ and c is:

a) 2; b) -2; c) $\frac{1}{2}$; d) $-\frac{1}{2}$.

2.1.2 The Dot Product of Two Vectors

Definition. Let $\vec{a}$ and $\vec{b}$ be two nonparallel and nonzero vectors. They determine a triangle and an angle φ, shown in Figure 1. The angle between $\vec{a}$ and $\vec{b}$ is φ. Note that $0 < \varphi < \pi$.

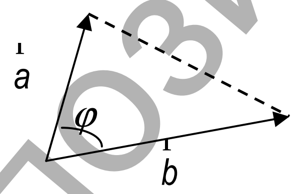


Figure 1.

If $\vec{a}$ and $\vec{b}$ are nonparallel, the angle between them is 0 (if they have the same direction) or π (if they have opposite directions). The angle between $\vec{0}$ and any other vector is not defined.

The angle between $\vec{i}$ and $\vec{j}$ is $\frac{\pi}{2}$; the angle between $\vec{k}$ and $-\vec{k}$ is π; the angle between $2\vec{i} + 2\vec{j} + 2\vec{k}$ and $5\vec{i} + 5\vec{j} + 5\vec{k}$ is 0.

The dot product of vectors can now be defined.

Definition. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Their **dot product** is the number

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\varphi,$$

where φ is the angle between $\vec{a}$ and $\vec{b}$. If $\vec{a}$ and $\vec{b}$ is $\vec{0}$, their dot product is 0. The dot product is a scalar and is also called the **scalar product** of $\vec{a}$ and $\vec{b}$.

Example 1. Compute the dot product $\vec{a}\cdot\vec{b}$ if $\vec{a}=3\vec{i}+3\vec{j}$ and $\vec{b}=5\vec{i}$.

Solution. Inspection of Fig.2 shows that φ, the angle between $\vec{a}$ and $\vec{b}$, is $\dfrac{\pi}{4}$.

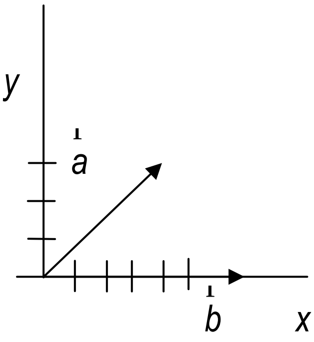


Figure 2.

Also, $\left|\vec{a}\right|=\sqrt{3^2+3^2}=\sqrt{18}$, $\left|\vec{b}\right|=\sqrt{5^2+0^2}=5$.

Thus

$$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cos\varphi=\sqrt{18}\cdot 5\cos\frac{\pi}{4}=\sqrt{18}\cdot 5\cdot\frac{\sqrt{2}}{2}=15.$$

Properties of Dot Product:

1. For any two vectors $\vec{a}$ and $\vec{b}$: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.
2. For any vector $\vec{a}$: $\vec{a}\cdot\vec{a}=\left|\vec{a}\right|^2$.
3. If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}\cdot\vec{b}=0$. Moreover, its converse is valid: If $\vec{a}\cdot\vec{b}=0$ and neither $\vec{a}$ nor $\vec{b}$ is $\vec{0}$, then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is 0. Thus $\vec{a}$ and $\vec{b}$ are perpendicular. *Consequently, the vanishing of the dot product is a test for perpendicularity.*
4. *Formula for dot product in scalar components.* To find the dot product just add the products of corresponding components: if $\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ and $\vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$, then $\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2$.
5. The angle between two vectors $\vec{a}$ and $\vec{b}$ can be determined by the formula: $\cos\varphi=\dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}$.

Example 2. Are the vectors $3\vec{i}+7\vec{j}$ and $9\vec{i}-4\vec{j}$ perpendicular?

Solution. Compute the dot product: $\left(3\vec{i}+7\vec{j}\right)\cdot\left(9\vec{i}-4\vec{j}\right)=3\cdot 9+7\cdot(-4)=27-28=-1$. Since the dot product is not 0, the vectors are not perpendicular.

Example 3. Find the angle φ between two vectors $\vec{a}=2\vec{i}-\vec{j}+3\vec{k}$ and $\vec{b}=\vec{i}+\vec{j}+2\vec{k}$.

Solution. Compute $\vec{a}\cdot\vec{b}$ in two ways:

$$\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cos\varphi=\sqrt{2^2+(-1)^2+3^2}\cdot\sqrt{1^2+1^2+1^2}\cos\varphi=\sqrt{14}\cdot\sqrt{6}=\sqrt{84}.$$

Also: $\vec{a}\cdot\vec{b}=2\cdot 1+(-1)\cdot 1+3\cdot 2=7$. Then $\cos\varphi=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\dfrac{7}{\sqrt{84}}=\dfrac{\sqrt{21}}{6}\approx 0{,}764$. A calculator or trigonometric table shows that φ is about $40{,}2^{\circ}$ or 0,702 radians.

2.1.3 The Cross Product of Two Vectors

It is frequently necessary in applications of vectors in space to construct a nonzero vector perpendicular to two given vectors $\vec{a}$ and $\vec{b}$.

If $\vec{a}$ and $\vec{b}$ are not parallel and are drawn with their tails at a single point, they determine a plane. Any vector $\vec{c}$ perpendicular to this plane is perpendicular to both $\vec{a}$ and $\vec{b}$. There are many such vectors, all parallel to each other and having various lengths. For instance, any vector perpendicular to both $\vec{i}$ and $\vec{j}$ is of the form $\alpha\vec{k}$, where α is an arbitrary scalar, positive or negative.

In the following definition, vectors appear as entries in a matrix.

Definition Cross product (vector product). Let $\vec{a}=x_a\vec{i}+y_a\vec{j}+z_a\vec{k}$ and $\vec{b}=x_b\vec{i}+y_b\vec{j}+z_b\vec{k}$. The vector

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x_a & y_a & z_a\\ x_b & y_b & z_b\end{vmatrix}=\vec{i}\begin{vmatrix}y_a & z_a\\ y_b & z_b\end{vmatrix}-\vec{j}\begin{vmatrix}x_a & z_a\\ x_b & z_b\end{vmatrix}+\vec{k}\begin{vmatrix}x_a & y_a\\ x_b & y_b\end{vmatrix}=$$

$$=(y_az_b-z_ay_b)\vec{i}-(x_az_b-z_ax_b)\vec{j}+(x_ay_b-y_ax_b)\vec{k}$$

Is called the *cross product* (or *vector product*) of $\vec{a}$ and $\vec{b}$. It is denoted $\vec{a}\times\vec{b}$. The determinant of $\vec{a}\times\vec{b}$ is expanded along its first row.

Example 1. Compute $\vec{a}\times\vec{b}$ if $\vec{a}=2\vec{i}-\vec{j}+3\vec{k}$ and $\vec{b}=3\vec{i}+4\vec{j}+\vec{k}$.

Solution. By definition,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 3\\ 3 & 4 & 1\end{vmatrix}=\vec{i}\begin{vmatrix}-1 & 3\\ 4 & z_b\end{vmatrix}-\vec{j}\begin{vmatrix}2 & 3\\ 3 & 1\end{vmatrix}+\vec{k}\begin{vmatrix}2 & -1\\ 3 & 4\end{vmatrix}=-13\vec{i}+7\vec{j}+11\vec{k}.$$

Note that $\vec{a}\times\vec{b}$ is a vector, while $\vec{a}\cdot\vec{b}$ is a scalar.

Properties of Cross Product:

1. $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.
2. The order of the factors in the vector product is critical: $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$. This property corresponds to the fact that when two rows of a matrix are interchanged, its determinant changes sign.
3. If $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a}\times\vec{b}=\vec{0}$. This corresponds to the fact that if two rows of a matrix are identical, then its determinant is 0.

4. For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$: $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$. This distributive law can be established by a straightforward computation.
5. A scalar can be factored out of a cross product: $\left(\alpha\vec{a}\right)\times\vec{b}=\alpha\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(\alpha\vec{b}\right)$.
6. The magnitude of $\vec{a}\times\vec{b}$ is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$: $\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\varphi$, where $\varphi=\left(\widehat{\vec{a};\vec{b}}\right)$ - the angle between $\vec{a}$ and $\vec{b}$.

Example 2. A parallelogram in the plane has the vertices $A(0;0),B(2;4),C(5;1);D(7;5)$. Find its area.

Solution. The parallelogram is spanned by the vectors:

$\vec{a}=\overrightarrow{AB}=(2-0;4-0)=(2;4)=2\vec{i}+4\vec{j}+0\vec{k}$;

$\vec{b}=\overrightarrow{AC}=(5-0;5-0)=(5;1)=5\vec{i}+\vec{j}+0\vec{k}$.

Consequently, its area is the magnitude of the vector:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 4 & 0\\ 5 & 1 & 0\end{vmatrix}=0\vec{i}-0\vec{j}+\vec{k}\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=-18\vec{k}, \text{ thus } S=\left|\vec{a}\times\vec{b}\right|=\left|-18\vec{k}\right|=18.$$

Example 3. Find a vector perpendicular to the plane determined by the three points $A(1;3;2),B(4;-1;1),C(3;0;2)$.

Solution. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in a plane. The vector $\vec{c}=\overrightarrow{AB}\times\overrightarrow{AC}$, being perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, is perpendicular to the plane. Now, $\overrightarrow{AB}=3\vec{i}-4\vec{j}-\vec{k}$ and $\overrightarrow{AC}=2\vec{i}-3\vec{j}+0\vec{k}$. Thus

$$\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -4 & -1\\ 2 & -3 & 0\end{vmatrix}=-3\vec{i}-2\vec{j}-\vec{k}=(-3;-2;-1).$$

Exercise Set 6

In Exercises 1 to 6 compute $\vec{a}\cdot\vec{b}$.

1. $\vec{a}$ has length 3, $\vec{b}$ has length 4, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.
2. $\vec{a}$ has length 2, $\vec{b}$ has length 2, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{3\pi}{4}$.
3. $\vec{a}$ has length 5, $\vec{b}$ has length 1/2, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.
4. $\vec{a}$ is the zero vector, $\vec{b}$ has length 5.
5. $\vec{a}=2\vec{i}-3\vec{j}+5\vec{k}$ and $\vec{b}=\vec{i}-\vec{j}-\vec{k}$.
6. $\vec{a}=\overrightarrow{PQ}$ and $\vec{b}=\overrightarrow{PR}$, where $P(1,0,2),Q(1,1,-1),R(2,3,5)$.

7. a) Draw the vectors $7\vec{i}+12\vec{j}$ and $9\vec{i}-5\vec{j}$.
   b) Do they seem to be perpendicular?
   c) Determine whether they are perpendicular by examining their dot product.

8. a) Draw the vectors $\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$.
   b) Do they seem to be perpendicular?
   c) Determine whether they are perpendicular by examining their dot product.

9. a) Estimate the angle between $\vec{a}=3\vec{i}+4\vec{j}$ and $\bar{b}=5\vec{i}+12\vec{j}$ by drawing them.
   b) Find the angle between $\vec{a}$ and $\vec{b}$.

10. Find the cosine of the angle between $2\vec{i}-4\vec{j}+6\vec{k}$ and $\vec{i}+2\vec{j}+3\vec{k}$.

11. a) Show that scalar component of $\vec{a}$ on $\vec{b}$ is given by the formula $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.
   b) Find the scalar component of $2\vec{i}-\vec{j}$ on $\vec{j}+3\vec{k}$.
   c) Find the scalar component of $2\vec{i}-\vec{j}$ on $\dfrac{\vec{j}+3\vec{k}}{|\vec{j}+3\vec{k}|}$.

12. Find the scalar component of $2\vec{i}-\vec{j}$ on: a) $3\vec{j}+2\vec{k}$; b) $-3\vec{j}-2\vec{k}$.

13. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A(1,3), B(7,4), C(2,8)$ and $D(1,-5)$.

14. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if $A(1,2,-5), B(1,0,1), C(0,-1,3)$ and $D(2,1,4)$.

In Exercises 15 to 18 find the vector components of $\vec{a}$ parallel and perpendicular to $\vec{b}$.

15. $\vec{a}=2\vec{i}+3\vec{j}+4\vec{k}$ and $\vec{b}=\vec{i}+\vec{j}+\vec{k}$.
16. $\vec{a}=\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}-\vec{j}$.
17. $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{b}=2\vec{j}+3\vec{k}$.
18. $\vec{a}=\vec{j}$ and $\vec{b}=2\vec{i}+3\vec{j}-\vec{k}$.

In Exercises 19 to 22 compute $\vec{a}\times\vec{b}$:

19. $\vec{a}=\vec{k}$; $\vec{b}=\vec{j}$.
20. $\vec{a}=\vec{i}+\vec{j}$; $\vec{b}=\vec{i}-\vec{j}$
21. $\vec{a}=\vec{i}+\vec{j}+\vec{k}$; $\vec{b}=\vec{i}+\vec{j}$
22. $\vec{a}=\vec{k}$; $\vec{b}=\vec{i}+\vec{j}$

In Exercises 23 to 22 compute $\vec{a}\times\vec{b}$ and check that it is perpendicular to both $\vec{a}$ and to $\vec{b}$:

23. $\vec{a}=2\vec{i}-3\vec{j}+\vec{k}$; $\vec{b}=\vec{i}+\vec{j}+2\vec{k}$
24. $\vec{a}=\vec{i}-\vec{j}$; $\vec{b}=\vec{j}+4\vec{k}$

25. Find the area of a parallelogram three of whose vertices are $A(0;0;0)$, $B(1;5;4)$, $C(2;-1;3)$.

26. Find the area of a parallelogram three of whose vertices are $A(1;2;-1)$, $B(2;1;4)$, $C(3;5;2)$.
27. Find a vector perpendicular to the plane determined by the three points $A(1;2;1), B(2;1;-3), C(0;1;5)$.
28. Find a vector perpendicular to the line through $A(1;2;1)$ and $B(4;1;0)$ and also to the line through $C(3;5;2)$ and $D(2;6;-3)$.
29. Find the area of the triangles whose vertices are:
    a) $A(0;0)$, $B(3;5)$, $C(2;-1)$
    b) $A(1;4)$, $B(3;0)$, $C(-1;2)$
    c) $A(1;1;1)$, $B(2;0;1)$, $C(3;-1;4)$

2.2 Lines and Planes

2.2.1 Lines in the plane

A line (i.e., a straight line) is a geometric object. When it is placed in a coordinate plane, the points (in the plane) through which the line passes, satisfy certain geometric conditions. For example, any two distinct points $M_1(x_1;y_1)$ and $M_2(x_2;y_2)$ on the line, determine it completely.

Equations of a line.

1. General Linear Equation. Let $\vec{n} = A\vec{i} + B\vec{j} = (A;B)$ be a nonzero vector and $(x_0;y_0)$ be a point in the xy plane. There is a unique line through $(x_0;y_0)$ that is perpendicular to $\vec{n} = (A;B)$, as shown in Fig. 1. Vector $\vec{n} = (A;B)$ is called a **normal** to the line.

Theorem 1. An equation of the line (in the xy plane) to the nonzero vector $\vec{n} = (A;B)$ is given by:

$$Ax + By + C = 0, \tag{1}$$

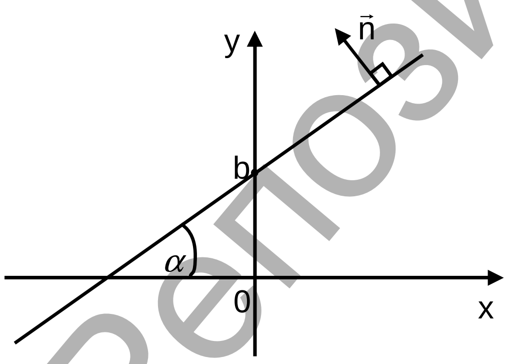


Fig.1

where A, B, C are constants, with the condition that both A and B are not zero simultaneously $\left(A^2 + B^2 \neq 0\right)$.

Case 1. If $B = 0$ (but $A \neq 0$), then the equation (1) becomes $Ax + C = 0$ or $x = -\frac{C}{A}$, which represents a vertical line (i.e., a line parallel to $y-axis$).

Case 2. If $A = 0$ (but $B \neq 0$), then the equation (1) becomes $By + C = 0$ or $y = -\frac{C}{B}$, which represents a horizontal line (i.e., a line parallel to $x-axis$).

2. Slope Form of the Equation of a Line.

$$y = kx + b \tag{2}$$

where $k = tg\alpha$ - is the slope of the line; α - the angle of inclination (or simply inclination) of a line (the smallest positive angle between line and $x-axis$); the line makes an intercept b on the $y-axis$ (see Fig. 1).

Let a line with angle inclination α, have the slope k, then:

1. Value of k is given by $tg\alpha$.
2. If α is acute, k is positive.
3. If α is obtuse, k is negative.
4. If $\alpha = 0$ then $k = 0$. The line is parallel to $x - axis$.
5. If $\alpha = \frac{\pi}{2}$, the line is vertical and k is not defined.

3. Point-Slope Form of the Equation of a Line.

$$y - y_0 = k(x - x_0), \qquad (3)$$

where the point $M_0(x_0;y_0)$ lies on the line.

4. Two-Point Equation of a Line.

$$\frac{y - y_2}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}, \qquad (4)$$

where points $M_1(x_1;y_1)$ and $M_2(x_2;y_2)$ are any two distinct fixed points on the line.

5. Equation of a Nonvertical Line in the Intercept Form.

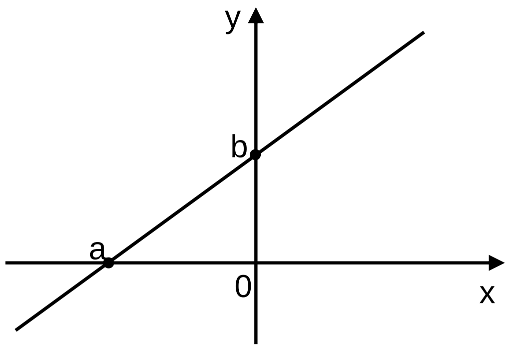


Fig. 2

Let l be any nonvertical line, which makes an intercept "a" on the $x - axis$ and an intercept "b" on the $y - axis$ $(a \neq 0, b \neq 0)$. Equation

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (5)$$

is called the "intercept form" of the equation of a line (See Fig 2).

6. Equation of the Line Passing through the Point $M_0(x_0;y_0)$ and Perpendicular to $\vec{n} = (A;B)$:

$$A(x - x_0) + B(y - y_0) = 0 \qquad (6)$$

Example 1. Find the equation of the line passing through $(2;1)$ with the given slope $k = \frac{1}{2}$.

Solution. Let us use point-slope form of the equation. At (3) above is, $y - y_0 = k(x - x_0)$. Here, $x_0 = 2$, $y_0 = 1$ and $k = \frac{1}{2}$. So that equation becomes $y - 1 = \frac{1}{2}(x - 2) \Rightarrow$ $\Rightarrow y - 1 = \frac{1}{2}x - 2 \Rightarrow y = \frac{1}{2}x - 1$.

Example 2. Find the equation of the line through $(-5;-3)$ and $(6;1)$.

Solution. Let us use two-point equation of a line. At (4) above is, $\frac{y - y_2}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$. Here, $x_1 = -5, y_1 = -3, x_2 = 6, y_2 = 1$. So that equation becomes

$$\frac{y-1}{-3-1}=\frac{x-6}{-5-6}\Rightarrow\frac{y-1}{-4}=\frac{x-6}{-11}\Rightarrow -11(y-1)=-4(x-6)\Rightarrow -11y+11=-4x+24$$
$$\Rightarrow 4x-11y-13=0.$$

Example 3. Find the equation of the line through $(2;-7)$ and perpendicular to the vector $4\vec{i}+\vec{j}$.

Solution. Let us use the equation $A(x-x_0)+B(y-y_0)=0$, where $x_0=2, y_0=-7$ and $A=4, B=1$. We have $4(x-2)+1(y-(-7))=0\Rightarrow 4x-8+y+7=0\Rightarrow 4x+y-1=0$.

Relations between the Lines

Let us consider two lines on the plane in the form:

$$l_1:\ A_1x+B_1y+C_1=0,\qquad l_2:\ A_2x+B_2y+C_2=0.$$

Definition. The angle from l_1 to l_2 is the angle φ through which l_1 must be rotated counter clockwise about the point of intersection in order to coincide with l_2 (See Fig. 3).

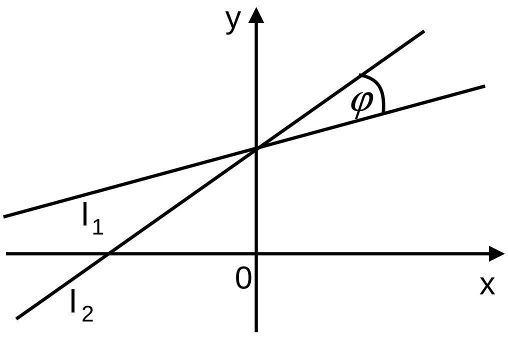


Fig. 3

Case 1. The angle from l_1 to l_2 is given by:

$$\cos\varphi=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}.$$

Case 2. Perpendicular lines ($l_1\perp l_2$):

$$A_1A_2+B_1B_2=0.$$

Case 3. Parallel Lines ($l_1 \parallel l_2$):

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}\neq\frac{C_1}{C_2}.$$

Let us consider two lines on the plane in the form:

$$l_1:\ y=k_1x+b_1,\qquad l_2:\ y=k_2x+b_2.$$

Case 1. The angle from l_1 to l_2 is given by:

$$tg\varphi=\frac{k_2-k_1}{1+k_1k_2}.$$

Case 2. Perpendicular lines ($l_1\perp l_2$):

$$k_1k_2=-1 \text{ or } k_1=-\frac{1}{k_2}.$$

Case 3. Parallel Lines ($l_1 \parallel l_2$):

$$k_1=k_2.$$

Theorem 2. The distance from the point $P_1(x_1;y_1)$ to the line l whose equation is $Ax+By+C=0$ is

$$d(P;l)=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

Example 4. Find the equation of the line through $(6;8)$ which is parallel to the line with equation $3x-5y=11$.

Solution. We solve $3x-5y=11$ for y, and we get $y=\frac{3}{5}x-\frac{11}{5}$. This equation shows that the slope of this line is $k=\frac{3}{5}$. Since the desired line passes through the point $(6;8)$, its equation must be, $y-8=\frac{3}{5}(x-6)\Rightarrow 5y-40=3x-18\Rightarrow 3x-5y+22=0$.

Example 5. Let the equations of l_1 and l_2 be $y-2x=2$ and $2y+5x=17$. Find the tangent of the angle φ from l_1 to l_2.

Solution. From the equations of l_1 and l_2, we find that $k_1=2$ and $k_2=-\frac{5}{2}$, then

$$tg\varphi=\frac{k_2-k_1}{1+k_1k_2}=\frac{-\frac{5}{2}-2}{1+2\cdot\left(-\frac{5}{2}\right)}=\frac{-\frac{9}{2}}{-4}=\frac{9}{8}.$$

Example 6. Find the distance from the point $P(3;7)$ to the line $2x-4y+5=0$.

Solution. By Theorem 2, the distance is

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}=\frac{|2\cdot3-4\cdot7+5|}{\sqrt{2^2+4^2}}=\frac{|-17|}{\sqrt{20}}=\frac{17}{\sqrt{20}}.$$

2.2.2 Planes

Definition. A vector $\vec{n}$ is said to be perpendicular to a plane if $\vec{n}$ is perpendicular to every line situated in the plane.

Equations of a plane.

1. Equation of the Plane Passing through the Point $M_0(x_0;y_0)$ and Perpendicular to $\vec{n}=(A;B;C)$:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

Example 1. Find the equation of the plane through $(1;-2;4)$ that is perpendicular to the vector $\vec{n}=(5;3;6)$.

Solution. An equation of the plane is $5(x-1)+3(y-(-2))+6(z-4)=0$, which is simplified to $5x+3y+6z-23=0$.

2. General Linear Equation.

Theorem 1. Let A,B,C, and D be constants such that not all A,B and C are 0. Then the equation

$$Ax+By+Cz+D=0$$

describes a plane. Moreover, the vector $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}=(A;B;C)$ is perpendicular to this plane.

3. Three-Point Equation of a Plane.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0,$$

where points $M_1(x_1;y_1;z_1)$, $M_2(x_2;y_2;z_2)$ and $M_3(x_3;y_3;z_3)$ are any three distinct fixed points on the plane.

4. Equation of a Plane in the Intercept Form. Let Δ be any plane, which makes an intercept "a" on the $x-axis$, an intercept "b" on the $y-axis$ and an intercept "c" on the $z-axis$ $(a \neq 0, b \neq 0, c \neq 0)$. Equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

is called the "intercept form" of the equation of a plane.

The Distance from Point to the Plane.

Theorem 2. The distance from the point $M_1(x_1;y_1;z_1)$ to the plane $\Delta:\ Ax+By+Cz+D=0$ is

$$d(M_1;\Delta)=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}.$$

Example 2. Find the distance from the point $M(2;1;5)$ to the plane $x-3y+4z+8=0$.

Solution. By Theorem 2, the desired distance is $d=\dfrac{|1\cdot 2-3\cdot 1+4\cdot 5+8|}{\sqrt{1^2+(-3)^2+4^2}}=\dfrac{27}{\sqrt{26}}$.

Relations between the Planes

Let us consider two planes in the form:

$$\Delta_1:\ A_1x+B_1y+C_1z+D=0, \quad \Delta_2:\ A_2x+B_2y+C_2z+D=0.$$

Definition. The angle between two planes is the angle between their normal vectors.

Case 1. The angle between Δ_1 to Δ_2 is given by:

$$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}.$$

Case 2. Perpendicular planes ($\Delta_1 \perp \Delta_2$): $A_1A_2+B_1B_2+C_1C_2=0$.

Case 3. Parallel planes ($\Delta_1 \parallel \Delta_2$): $\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}\neq\dfrac{D_1}{D_2}$.

2.2.3 Lines in Space

Equations of a line.

1. Parametric equation. Consider the line L through point $M_0(x_0;y_0;z_0)$ and parallel to the vector $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$. Equation

$$\begin{cases} x = x_0 + a_1 t \\ y = y_0 + a_2 t \\ z = z_0 + a_3 t \end{cases}$$

is the parametric equation for the line trough $M_0(x_0; y_0; z_0)$ parallel to $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

2. Symmetric Equation. The equation of the line through point $M_0(x_0; y_0; z_0)$ and parallel to the vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ can be given in the form

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3},$$

if none a_1, a_2, a_3 is 0.

Example 1. Write parametric equations for the line through point $M(1;2;2)$ parallel to the vector $\vec{a} = 3\vec{i} - \vec{j} + 5\vec{k}$. Does the point $(10;-1;16)$ lie on the line?

Solution. Parametric equations of the line are given by $\begin{cases} x = 1 + 3t \\ y = 2 - t \\ z = 2 + 5t \end{cases}$. Does the point $(10;-1;16)$ lie on the line? To find out, determine if there is a number t such that these three equations are simultaneously satisfied: $\begin{cases} 10 = 1 + 3t \\ -1 = 2 - t \\ 16 = 2 + 5t \end{cases}$. The first equation has the solution $t = 3$. This value does not satisfy the second equation, since $-1 = 2 - 3$. But it does not satisfy the third equation, since $16 \neq 2 + 5 \cdot 3$. Hence $(10;-1;16)$ is not on the line.

Definition. If the vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is parallel to the line L, then the numbers a_1, a_2, a_3 are called *direction numbers* of L. (Note that direction numbers are not unique.)

The next definition is closely related to the preceding one.

The direction of a vector in the plane is described by a single angle, the angle it makes with the positive x axis. The direction of a vector in space involves three angles, two of which almost determine the third.

Definition. Let $\vec{a}$ be a nonzero vector in space. The angle between $\vec{a}$ and $\vec{i}$ is denoted α; $\vec{a}$ and $\vec{j}$ is denoted β; $\vec{a}$ and $\vec{k}$ is denoted γ. The angles α, β and γ are called the *direction angles of* $\vec{a}$.

The numbers $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called *direction cosines of* $\vec{a}$.

Theorem 1. If α, β and γ are the direction angles of the vector $\vec{a}$, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 2. The vector $\vec{a}$ makes angles of 60° with the x and y axes. What angle does it make with the z axis?

Solution. Here $\alpha = 60^\circ$ and $\beta = 60^\circ$; hence $\cos\alpha = \cos\beta = \frac{1}{2}$. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, it follows that $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$, $\cos^2\gamma = \frac{1}{2}$. Thus $\cos\gamma = \frac{\sqrt{2}}{2}$ or $\cos\gamma = -\frac{\sqrt{2}}{2}$. Hence $\gamma = 45^\circ$ or $\gamma = 135^\circ$.

Exercise Set 7

1. Find the equation of the line passing through $(2;1)$ with the given slope, and sketch the line:

a) $k = 0$; b) $k = -3$; c) $k = \frac{2}{3}$.

In Exercises 2 to 5 find the line through the given point and perpendicular to the given vector:

2. $M(2;3)$, $\vec{n} = (4;5)$;
3. $M(1;0)$, $\vec{n} = (2;-1)$;
4. $M(4;5)$, $\vec{n} = (2;3)$;
5. $M(2;-1)$, $\vec{n} = (1;3)$.

In Exercises 6 to 9 find a vector perpendicular to the given line:

6. $2x - 3y + 8 = 0$;
7. $\pi x - \sqrt{2}y = 7$;
8. $y = 3x + 7$;
9. $2(x-1) + 5(y+2) = 0$.

10. Find the distance from the point $M(0;0)$ to the line $3x + 4y - 10 = 0$.

11. Find the distance from the point $M\left(\frac{3}{2};\frac{2}{3}\right)$ to the line $2x - y + 5 = 0$.

12. Find the distance from the point $M(0;0;0)$ to the plane $2x - 4y + 3z + 2 = 0$.

13. Find the distance from the point $M(1;2;3)$ to the plane $x + 2y - 3z + 5 = 0$.

14. Find the distance from the point $M(2;2;-1)$ to the plane that passes through $(1;4;3)$ and has a normal $2\vec{i} - 7\vec{j} + 2\vec{k}$.

15. Find the distance from the point $M(0;0;0)$ to the plane that passes through $(4;1;0)$ and is perpendicular to the vector $\vec{i} + \vec{j} + \vec{k}$.

Find two points on the planes 16 – 21 and a normal vector:

16. $x + 2y + 3z = 0$;
17. $x + 2y + 3z = 6$;
18. the yz plane.
19. The plane through $(0;0;0)$ perpendicular to $\vec{i} + \vec{j} - \vec{k}$.
20. The plane through $(1;1;1)$ perpendicular to $\vec{i} + \vec{j} - \vec{k}$.
21. The plane through $(0;0;0)$ and $(1;0;0)$ and $(0;1;1)$.

In Exercises 22 to 25 find general linear equation for planes.

22. The plane through $(1;2;-1)$ perpendicular to $\vec{n} = \vec{i} + \vec{j}$.
23. The plane through $(1;2;-1)$ perpendicular to $\vec{n} = \vec{i} + 2\vec{j} - \vec{k}$.
24. The plane through $(1;0;1)$ parallel to $x + 2y + z = 0$.

25. The plane through $(1;2;-1)$ parallel to $x+y+z=1$.

26. Explain why a plane cannot:

a) contain $(1;2;3)$ and $(2;3;4)$ and be perpendicular to $\vec{n}=\vec{i}+\vec{j}$;

b) be perpendicular to $\vec{n}=\vec{i}+\vec{j}$ and parallel to $\vec{m}=\vec{i}+\vec{k}$;

c) contain $(1;0;0)$, $(0;1;0)$, $(0;0;1)$ and $(1;1;1)$;

d) contain $(1;1;-1)$ if it has $\vec{n}=\vec{i}+\vec{j}-\vec{k}$;

e) go through origin and have the equation $ax+by+cz=1$.

27. Find the angle between $x+2y+2z=0$ and: a) $x+2z=0$; b) $x+2z=5$; c) $x=0$.

28. How far is the plane $x+y-z=1$ from $(0;0;0)$ and also from $(1;1;-1)$? Find the nearest points.

29. Find the distance between planes $2x-2y+z=1$, $2x-2y+z=3$.

30. Find the distance between planes $x+y+5z=7$, $3x+2y+z=1$.

31. Find the direction cosines of the vector $2\vec{i}+3\vec{j}+4\vec{k}$.

32. Find direction cosines of the line through the points $(1;3;2)$ and $(4;-1;5)$.

33. A vector $\vec{a}$ has direction angles $\alpha=70^{\circ}$ and $\beta=80^{\circ}$. Find the third direction angle γ and show the possibilities on a diagram.

34. Suppose that the three direction angles of a vector are equal. What must they be?

35. Give parametric equations for the line through $\left(\frac{1}{2};\frac{1}{3};\frac{1}{2}\right)$ and with direction numbers 2, 5 and 8.

36. Give parametric equations for the line through $(1;2;3)$ and $(4;5;7)$.

37. Find the equation of the plane determined by the points $(0;0;0)$, $(4;1;2)$ and $(2;5;0)$.

38. Find the equation of the plane determined by the points $(1;-1;2)$, $(2;1;3)$ and $(3;3;5)$.

39. How far is the point $(3;1;5)$ from the plane determined by the points $(1;3;3)$, $(2;-1;7)$ and $(1;2;4)$?

40. The planes $2x+3y+5z=8$ and $x-2y+4z=9$ meet in a line. Find a vector parallel to that line.

41. Give symmetric equations for the line through the points $(1;0;3)$ and $(2;1;-1)$.

42. Give symmetric equations for the line through the points $(7;-1;5)$ and $(4;3;2)$.

43. a) How far is the point $(1;1;1)$ from the line through the points $(2;1;3)$ and $(1;4;5)$?

b) Find the point on the line nearest to $(1;1;1)$.

44. Let L be the line in which the planes $x+y+3z=5$ and $2x-y+z=2$ intersect:

a) find the vector parallel to L;

b) find a point on L;

c) find parametric equations for L.

45. Where does the line of intersection of the planes $x+2y+z=4$ and $2x-y+z=1$ meet the plane $3x+2y+z=6$?

46. The planes $x+2y+3z=6$ and $2x-3y+4z=8$ intersect in a line L:

a) Find the vector parallel to L;

b) Find a point on L;

c) Find parametric equations for L;

d) Find symmetric equations for L.

47. A plane passes through the points $(1;1;2)$, $(1;3;4)$ and $(2;1;-1)$. A second plane passes through $(2;1;-1)$, $(1;0;2)$ and $(3;4;1)$:

a) Find a normal to each plane;

b) Find the cosine of the angle between the two planes;

c) Find the angle between the planes.

2.3 Review

Algebra of Vectors.

A vector may be pictured as an arrow. Two arrows that point in the same direction and have the same length represent the same vector. The following table summarizes the basic concepts of vectors in space. For plane vectors disregard the third component.

Symbol	*Name*	*Algebraic formula if* $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ *and* $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$
$\vec{a}$	Vector	$a_1\vec{i}+a_2\vec{j}+a_3\vec{k}=(a_1;a_2;a_3)$
$\lvert\vec{a}\rvert$	Magnitude of $\vec{a}$	$\sqrt{a_1^2+a_2^2+a_3^2}$
$-\vec{a}$	Negative or opposite of $\vec{a}$	$-a_1\vec{i}-a_2\vec{j}-a_3\vec{k}$
$\vec{a}+\vec{b}$	Sum of $\vec{a}$ and $\vec{b}$	$(a_1+b_1)\vec{i}+(a_2+b_2)\vec{j}+(a_3+b_3)\vec{k}$
$\vec{a}-\vec{b}$	Difference of $\vec{a}$ and $\vec{b}$	$(a_1-b_1)\vec{i}+(a_2-b_2)\vec{j}+(a_3-b_3)\vec{k}$
$c\vec{a}$	Scalar multiple of $\vec{a}$	$ca_1\vec{i}+ca_2\vec{j}+ca_3\vec{k}=(ca_1;ca_2;ca_3)$
$\vec{a}\cdot\vec{b}$	Dot or scalar product	$a_1\cdot b_1+a_2\cdot b_2+a_3\cdot b_3$
$\vec{a}\times\vec{b}$	Cross or vector product	$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Line in the Plane

Equations of a line.

1. General Linear Equation. An equation of the line (in the xy plane) to the nonzero vector $\vec{n}=(A;B)$ is given by $Ax+By+C=0$, where A,B,C are constants, with the condition that both A and B are not zero simultaneously $\left(A^2+B^2\neq 0\right)$.

Case 1. If $B=0$ (but $A\neq 0$), then the equation (1) becomes $Ax+C=0$ or $x=-\dfrac{C}{A}$, which represents a vertical line (i.e., a line parallel to $y-axis$).

Case 2. If $A=0$ (but $B\neq 0$), then the equation (1) becomes $By+C=0$ or $y=-\dfrac{C}{B}$, which represents a horizontal line (i.e., a line parallel to $x-axis$).

2. Slope Form of the Equation of a Line. $y=kx+b$, where $k=tg\alpha$ - is the slope of the line; α - the angle of inclination (or simply inclination) of a line (the smallest positive angle between line and $x-axis$); the line makes an intercept b on the $y-axis$.

3. Point-Slope Form of the Equation of a Line. $y-y_0=k(x-x_0)$, where the point $M_0(x_0;y_0)$ lies on the line.

4. Two-Point Equation of a Line. $\dfrac{y-y_2}{y_1-y_2}=\dfrac{x-x_1}{x_1-x_2}$, where points $M_1(x_1;y_1)$ and $M_2(x_2;y_2)$ are any two distinct fixed points on the line.

5. Equation of a Nonvertical Line in the Intercept Form. $\dfrac{x}{a}+\dfrac{y}{b}=1$.

6. Equation of the Line Passing through the Point $M_0(x_0;y_0)$ and Perpendicular to $\vec{n}=(A;B)$: $A(x-x_0)+B(y-y_0)=0$.

Relations between the Lines

Let us consider two lines on the plane in the form: $l_1: A_1x+B_1y+C_1=0$, $l_2: A_2x+B_2y+C_2=0$.

Case 1. The angle from l_1 to l_2 is given by: $\cos\varphi=\dfrac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}$.

Case 2. Perpendicular lines ($l_1\perp l_2$): $A_1A_2+B_1B_2=0$.

Case 3. Parallel Lines ($l_1 \,||\, l_2$): $\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}\neq\dfrac{C_1}{C_2}$.

Let us consider two lines on the plane in the form: $l_1: y=k_1x+b_1$, $l_2: y=k_2x+b_2$.

Case 1. The angle from l_1 to l_2 is given by: $tg\varphi=\dfrac{k_2-k_1}{1+k_1k_2}$.

Case 2. Perpendicular lines ($l_1\perp l_2$): $k_1k_2=-1$ *or* $k_1=-\dfrac{1}{k_2}$.

Case 3. Parallel Lines ($l_1 \,||\, l_2$): $k_1=k_2$.

The distance from the point $P_1(x_1;y_1)$ to the line l whose equation is $Ax+By+C=0$ is

$$d(P;l)=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

Planes

Equations of a plane.

1. Equation of the Plane Passing through the Point $M_0(x_0;y_0)$ and Perpendicular to $\vec{n}=(A;B;C)$: $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

2. General Linear Equation $Ax+By+Cz+D=0$, where the vector $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}=(A;B;C)$ is perpendicular to this plane.

3. Three-Point Equation of a Plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0,$$

where points $M_1(x_1;y_1;z_1)$, $M_2(x_2;y_2;z_2)$ and $M_3(x_3;y_3;z_3)$ are any three distinct fixed points on the plane.

4. Equation of a Plane in the Intercept Form. Let Δ be any plane, which makes an intercept "a" on the $x-axis$, an intercept "b" on the $y-axis$ and an intercept "c" on the $z-axis$ ($a\neq 0, b\neq 0, c\neq 0$). Equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

is called the "intercept form" of the equation of a plane.

The distance from the point $M_1(x_1;y_1;z_1)$ to the plane Δ: $Ax+By+Cz+D=0$ is

$$d(M_1;\Delta)=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}.$$

Relations between the Planes

Let us consider two planes in the form: Δ_1: $A_1x+B_1y+C_1z+D=0$, Δ_2: $A_2x+B_2y+C_2z+D=0$.

Case 1. The angle between Δ_1 and Δ_2 is given by:

$$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}.$$

Case 2. Perpendicular planes ($\Delta_1\perp\Delta_2$): $A_1A_2+B_1B_2+C_1C_2=0$.

Case 3. Parallel planes ($\Delta_1 \parallel \Delta_2$): $\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}\neq\dfrac{D_1}{D_2}$.

Lines in Space

Equations of a line

1. Parametric equation. Equation $\begin{cases} x=x_0+a_1t \\ y=y_0+a_2t \\ z=z_0+a_3t \end{cases}$ is the parametric equation for the line trough $M_0(x_0;y_0;z_0)$ parallel to $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$.

2. Symmetric Equation. The equation of the line through point $M_0(x_0;y_0;z_0)$ and parallel to the vector $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ can be given in the form

$$\frac{x-x_0}{a_1}=\frac{y-y_0}{a_2}=\frac{z-z_0}{a_3},$$

if none a_1,a_2,a_3 is 0.

The direction angles α, β and γ of the vector $\vec{a}$ are the angles it makes with $\vec{i}$, $\vec{j}$ and $\vec{k}$; $\cos\alpha,\cos\beta,\cos\gamma$ are the direction cosines of the vector. They are related by the equation:

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

Review Exercises

1. Given $\vec{a}=\vec{i}+2\vec{j}-\vec{k}$ and $\vec{b}=2\vec{i}-\vec{j}+3\vec{k}$, find
   a) $\vec{a}\cdot\vec{b}$,
   b) $|\vec{a}|$,
   c) a unit vector in the direction of $\vec{a}$,
   d) the scalar component of $\vec{b}$ on $\vec{a}$,
   e) the vector component of $\vec{b}$ on $\vec{a}$,
   f) the scalar component of $\vec{a}$ on $\vec{b}$,
   g) the vector component of $\vec{b}$ perpendicular to $\vec{a}$,
   h) the cosine of the angle between $\vec{a}$ and $\vec{b}$,
   i) the angle between $\vec{a}$ and $\vec{b}$,
   j) $\vec{a}\times\vec{b}$,
   k) $\vec{b}\times\vec{a}$,
   l) a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$,
   m) the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.
2. Find the direction cosines of a vector normal to the plane $x-2y+2z=16$.
3. Find the volume of the parallelepiped spanned by $\vec{a}=(2;3;1)$, $\vec{b}=(1;2;3)$ and $\vec{c}=(1;4;5)$.
4. Where does the line through $(1;2;1)$ and $(3;1;1)$ meet the plane determined by the three points $(2;-1;1)$, $(5;2;3)$, and $(4;1;3)$?
5. Find the point on the plane $2x-y+3z+12=0$ that is nearest the origin. Use vectors, not calculus.
6. Use determinants to find x and y such that $x\left(2\vec{i}+\vec{j}\right)+y\left(\vec{i}+3\vec{j}\right)=4\vec{i}-2\vec{j}$.
7. Use determinants to find scalars x, y and z such that $x\left(3\vec{i}+\vec{j}+2\vec{k}\right)+y\left(\vec{i}-\vec{j}+2\vec{k}\right)+z\left(2\vec{i}+2\vec{j}+\vec{k}\right)=3\vec{i}+3\vec{j}+3\vec{k}$.

8. Suppose that vector from $(1;3;3)$ to the plane $x-4y+5z+4=0$ makes an angle of 45° with that plane. Find the length of the vector.

9. The planes $2x+5y+z-10=0$ and $3x-y+4z-11=0$ meet in a line. For this line find

a) direction numbers,
b) direction cosines,
c) direction angles,
d) a point on the line.

10. Find parametric equations of the line through the $(1;1;2)$ that is parallel to the planes $x+2y+3z=0$ and $2x-y+3z+4=0$.

11. Does the plane through the $(1;1;-1)$, perpendicular to $2\vec{i}+4\vec{j}+5\vec{k}$, pass through the point $(4;5;-7)$?

12. Is the line through $(1;4;7)$ and $(5;10;15)$ perpendicular to plane $2x+3y+4z=17$?

13. Two planes that intersect in a line determine a dihedral angle. Find the dihedral angle between $x-3y+4z=10$ and $2x+y+z=11$.

14. Find the point on the plane $\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=1$

a) nearest the origin,
b) nearest the point $(1,2,3)$.

15. Let $\vec{a}=2\vec{i}+3\vec{j}+4\vec{k}$. Find

a) the scalar component of $\vec{a}$ in the direction $\vec{i}$,
b) the vector component of $\vec{a}$ in the direction $\vec{i}$,
c) the scalar component of $\vec{a}$ in the direction $-\vec{i}$,
d) the vector component of $\vec{a}$ in the direction $-\vec{i}$,
e) the projection of $\vec{a}$ on $\vec{i}+\vec{j}$.

16. A parallelogram is spanned by the three vectors $\vec{a}=(1;2;3)$, $\vec{b}=(2;1;1)$ and $\vec{c}=(3;3;1)$:

a) find the volume of parallelepiped;
b) find the area of the face spanned by $\vec{a}$ and $\vec{c}$;
c) find the angle between $\vec{a}$ and the face spanned by $\vec{b}$ and $\vec{c}$.

17. Find the angle between the line through $(0;0;0)$ and $(1;1;1)$ and the plane through $(1;2;3)$, $(4;1;5)$ and $(2;0;6)$.

18. Find the area of triangle whose vertices are $(1;1;2)$, $(2;1;4)$, and $(3;0;5)$.

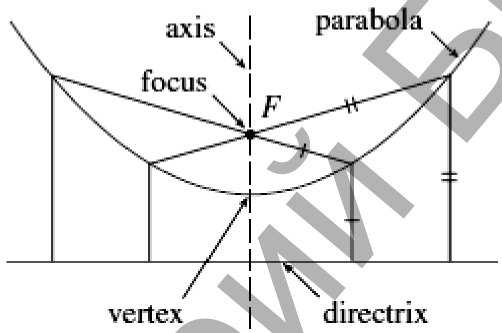
III CONIC SECTION

3.1 Parabolas

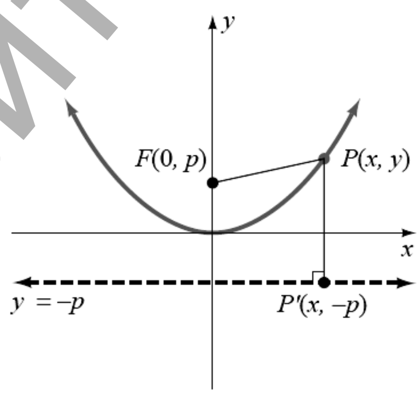
Definition of a Parabola. A **parabola** is the set of points in the plane that are equidistant from a fixed line (the **directrix**) and a fixed point (the **focus**) not on the directrix.

The line that passes through the focus and is perpendicular to the directrix is called the axis of symmetry of the parabola. The midpoint of the line segment between the focus and directrix on the axis of symmetry is the **vertex** of the parabola.

Using the definition of parabola, we can determine an equation of a parabola. Suppose that the coordinates of the vertex of a parabola are $(0;0)$ and the axis of symmetry is the *y*-axis. The equation of the directrix is $y=-p, p>0$. The focus lies on the axis of symmetry and is the same distance from the vertex as the vertex is from the directrix. Thus the coordinates of the focus are $(0;p)$.



Let $P(x;y)$ be any point on the parabola. Then, using the distance formula and the fact that the distance between any point on the parabola and the focus is equal to the distance from the point P to the directrix, we can write the equation $d(P,F)=d(P,D)$.



By the distance formula,

$$\sqrt{(x-0)^2+(y-p)^2}=y+p$$

Now squaring each side and simplifying,

$$\left(\sqrt{(x-0)^2+(y-p)^2}\right)^2=(y+p)^2$$

$$x^2+y^2-2py+p^2=y^2+2py+p^2$$

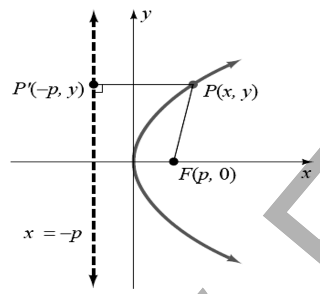
$$x^2=4py.$$

This is an equation of a parabola with vertex at the origin and a vertical axis of symmetry. The equation of a parabola with a horizontal axis of a symmetry is derived in a similar manner.

Standard Form of the Equation of a Parabola with Vertex at the Origin:

1. Vertical Axis of Symmetry. The standard form of the equation of a parabola with vertex $(0;0)$ and vertical axis of symmetry is $x^2 = 4py$. The focus is $(0;p)$, and the equation of the directrix is $y = -p$.

2. Horizontal Axis of Symmetry. The standard form of the equation of a parabola with vertex $(0;0)$ and horizontal axis of symmetry is $y^2 = 4px$. The focus is $(p;0)$, and the equation of the directrix is $x = -p$.



Standard Form of the Equation of a Parabola with Vertex at $(a;b)$:

1. Vertical Axis of Symmetry. The standard form of the equation of a parabola with vertex $(a;b)$ and vertical axis of symmetry is $(x-a)^2 = 4p(y-b)$. The focus is $(a;b+p)$, and the equation of the directrix is $y = b - p$.

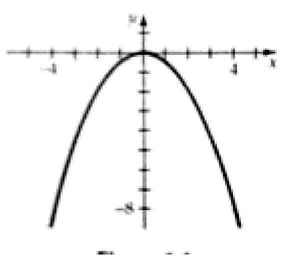
2. Horizontal Axis of Symmetry. The standard form of the equation of a parabola with vertex $(a;b)$ and horizontal axis of symmetry is $(y-b)^2 = 4p(x-a)$. The focus is $(a+p;b)$, and the equation of the directrix is $x = a - p$.

Example 1. Find the focus and directrix of the parabola given by the equation $y = -\frac{1}{2}x^2$.

Solution. Because the x term is squared, the standard form of the equation is $x^2 = 4py$. Write the given equation in standard form:

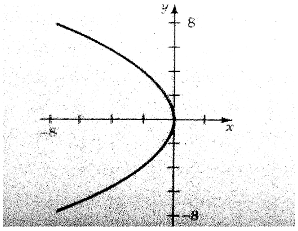
$$y = -\frac{1}{2}x^2 \Rightarrow x^2 = -2y.$$

Comparing this equation in standard form gives $4p = -2$ or $p = -\frac{1}{2}$. Because p is negative, the parabola will open down and the focus will be below the vertex $(0;0)$. The coordinates of the focus are $\left(0;-\frac{1}{2}\right)$. The equation of the directrix is $y = \frac{1}{2}$.



Example 2. Find the equation of the parabola in standard form with vertex at the origin and focus at $(-2;0)$.

Solution. Because the vertex is at $(0;0)$ and the focus is at $(-2;0)$, $p=-2$. The graph of the parabola opens toward the focus and thus, in this case, the parabola is opening to the left. The equation of the parabola in standard form that opens to the left is $y^2=4px$. Substitute -2 for p in this equation and simplify: $y^2=4(-2)x=-8x$. The equation is $y^2=-8x$.



Example 3. Find the equation of the directrix and the coordinates of the vertex and focus of the parabola given by the equation $3x-2y^2+8y-4=0$.

Solution. Rewrite the equation and then complete the square.

$$3x+2y^2+8y-4=0$$

$$2y^2+8y=-3x+4$$

$$2\left(y^2+4y\right)=-3x+4$$

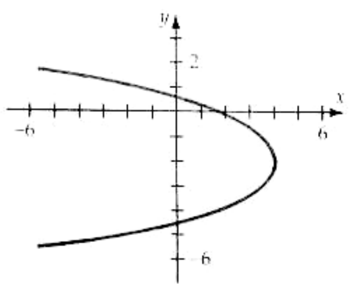
$$2\left(y^2+4y+4\right)=-3x+4+8$$ Complete the square. Note that 8 is added to each side.

$$2(y+2)^2=-3(x-4)$$

$$(y+2)^2=-\frac{3}{2}(x-4)$$ Write the equation in standard form.

Comparing this equation to $(y-b)^2=4p(x-a)$, we have a parabola that opens to the left with vertex $(4;-2)$ and $4p=-\frac{3}{2}$. Thus $p=-\frac{3}{8}$. The coordinates of the focus are $\left(4+\left(\frac{-3}{8}\right),-2\right)=\left(\frac{29}{8},-2\right)$. The equation of the directrix is $x=4-\left(\frac{-3}{8}\right)=\frac{35}{8}$.

Choosing some values for y and finding the corresponding values for x, we plot a few points. We use the fact that the line $y=-2$ is the axis of symmetry. Thus, for a point on one side of the axis of symmetry, there is a corresponding point on the other side. Two points are $(-2;1)$ and $(-2;-5)$.



Example 4. Find the equation in standard form of the parabola with directrix $x=-1$ and focus $(3;2)$.

Solution. The vertex is the midpoint of the line segment joining $(3;2)$ and the point $(-1;2)$ on the directrix.

$$(a;b)=\left(\frac{-1+3}{2},\frac{2+2}{2}\right)=(1,2).$$

The standard form of the equation will be of the form $(y-b)^2=4p(x-a)$. The distance from the vertex to the focus is 2. Thus $4p=8$ and the equation of the parabola is $(y-2)^2=8(x-1)$.

Exercise Set 8

In Exercises 1 to 26, find the vertex, focus, and directrix of the parabola given by each equation. Sketch the graph.

1. $x^2=-4y$
2. $2y^2=x$
3. $y^2=\frac{1}{3}x$
4. $x^2=-\frac{1}{4}y$
5. $(x-2)^2=8(y+3)$
6. $(y+1)^2=6(x-1)$
7. $(y+4)^2=-4(x-2)$
8. $(x-3)^2=-(y+2)$
9. $(y-1)^2=2x+8$
10. $(x+2)^2=3y-6$
11. $(2x-4)^2=8y-16$
12. $x^2+8x-y+6=0$
13. $(3x+6)^2=18y-36$
14. $x^2-6x+y+10=0$
15. $x+y^2-3y+4=0$
16. $x-y^2-4y+9=0$
17. $2x-y^2-6y+1=0$
18. $3x+y^2+8y+4=0$
19. $x^2+3x+3y-1=0$
20. $x^2+5x-4y-1=0$
21. $2x^2-8x-4y+3=0$
22. $6x-3y^2-12y+4=0$
23. $2x+4y^2+8y-5=0$
24. $4x^2-12x+12y+7=0$
25. $3x^2-6x-9y+4=0$
26. $6x-3y^2+9y+5=0$

27. Find the equation in standard form of the parabola with vertex at the origin and focus $(0;-4)$.

28. Find the equation in standard form of the parabola with vertex at the origin and focus $(5;0)$.

29. Find the equation in standard form of the parabola with vertex at $(-1;2)$ and focus $(-1;3)$.

30. Find the equation in standard form of the parabola with vertex at $(2;-3)$ and focus $(0;-3)$.

31. Find the equation in standard form of the parabola with focus $(3;-3)$ and directix $y=-5$.

32. Find the equation in standard form of the parabola with focus $(-2;4)$ and directix $x=4$.

33. Find the equation in standard form of the parabola with vertex $(-4;1)$, axis of symmetry parallel to the y-axis, and passing through the point $(-2;2)$.

34. Find the equation in standard form of the parabola with vertex $(3;-5)$, axis of symmetry parallel to the x-axis, and passing through the point $(4;3)$.

Supplemental Exercises

In Exercises 35 to 37, use the following definition of latus rectum: the line segment with endpoints on the parabola, through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum** of the parabola.

35. Find the length of the latus rectum for the parabola $x^2 = 4y$.

36. Find the length of the latus rectum for the parabola $y^2 = -8x$.

37. Find the length of the latus rectum for any parabola in terms of $|p|$, the distance from the vertex of the parabola to the focus.

The result of Exercise 37 can be stated as the following theorem.

Theorem. Two points on a parabola will be $2|p|$ units on each side of the axis of symmetry on the line through the focus and perpendicular to that axis.

38. Use the theorem to sketch a graph of the parabola given by the equation $(x-3)^2 = 2(y+1)$.

39. Use the theorem to sketch a graph of the parabola given by the equation $(y+4)^2 = -(x-1)$.

40. Use the theorem to sketch a graph of the parabola given by the equation $4x - y^2 + 8y = 0$.

41. Show that the point on the parabola closest to the focus is the vertex. (Hint: Consider the parabola $x^2 = 4py$ and a point on the parabola $(a;b)$. Find the square of the distance between the point $(a;b)$ and the focus. You may want to review the technique of minimizing a quadratic expression.)

42. By using the definition for a parabola, find the equation in standard form of the parabola with $V(0;0)$), $F(-c;0)$, and directrix $x = c$.

43. Sketch a graph of $4(y-2) = (x|x|-1)$.

44. Find the equation of the directrix of the parabola with vertex at the origin and whose focus is the point $(1;1)$.

45. Find the equation of the parabola with vertex at the origin and focus at the point $(1;1)$. (Hint: You will need the answer to Exercise 44 and the definition of a parabola.)

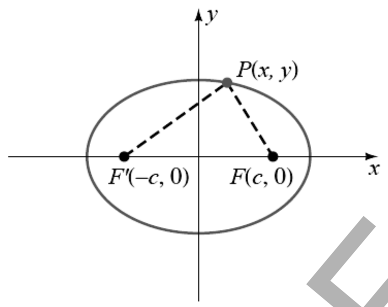
3.2 Ellipses

Definition. An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points (foci) is a positive constant.

This definition can be used to draw an ellipse using a piece of string and two. Tack the ends of the string to the foci, and trace a curve with a pencil held tight against the string. The resulting curve is an ellipse. The positive constant is the length of the string.

Ellipse with Center at $(0;0)$

The graph of an ellipse is oval-shaped, with two axes of symmetry. The longer axis is called the **major axis**. The foci of the ellipse are on the major axis. The shorter axis is called the **minor axis**. It is customary to denote the length of the major axis as $2a$ and the length of the minor axis as $2b$. The length of the **semiaxes** are one-half the axes. Thus the length of the semimajor axis is denoted by a and the length of the semiminor axis by b. The **center** of the ellipse is the midpoint of the major axis. The endpoints of the major axis are the **vertices** (plural of vertex) of the ellipse.



Consider the point $(a;0)$, which is one vertex of an ellipse, and the point $(c;0)$ and $(-c;0)$, which are the foci of the ellipse. The distance from $(a;0)$ to $(c;0)$ is $a-c$. Similarly, the distance from $(a;0)$ to $(-c;0)$ is $a+c$. From the definition of an ellipse, the sum of distances from any point on the ellipse to the foci is a constant. By adding the expression $a-c$ and $a+c$, we have

$$(a-c)+(a+c)=2a.$$

Thus the constant is precisely the length of the major axis.

Now let $P(x;y)$ be any point on the ellipse. By using the definition of an ellipse, we have

$$d(P,F_1)+d(P,F_2)=2a$$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

Subtract the second radical from each side of the equation and then square each side.

$$\left[\sqrt{(x+c)^2+y^2}\right]^2=\left[2a-\sqrt{(x-c)^2+y^2}\right]^2$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$$

Divide each side by -4 and then square each side again.

$$\left[-cx+a^2\right]^2=\left[a\sqrt{(x-c)^2+y^2}\right]^2$$

$$c^2x^2=2cxa^2+a^2c^2+a^2y^2$$

Simplify and then rewrite with x and y terms on the left side.

$$-a^2x^2 + c^2x^2 - a^2y^2 = -a^4 + a^2c^2$$

$$-\left(a^2 - c^2\right)x^2 - a^2y^2 = -a^2\left(a^2 - c^2\right) \qquad \text{Factor and let } b^2 = a^2 - c^2.$$

$$-b^2x^2 - a^2y^2 = -a^2b^2 \qquad \text{Divide each side by } -b^2a^2.$$

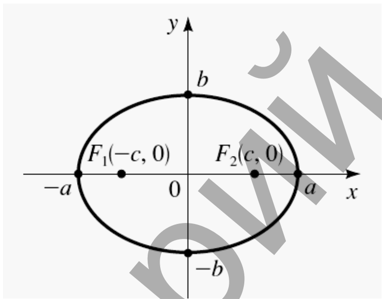
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{An equation of an ellipse}$$

Standard Forms of the Equation of an Ellipse with Center at the Origin

1. Major Axis on the *x*-axis. The standard form of the equation of an ellipse with the center at the origin and major axis on the *x*-axis is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b.$$

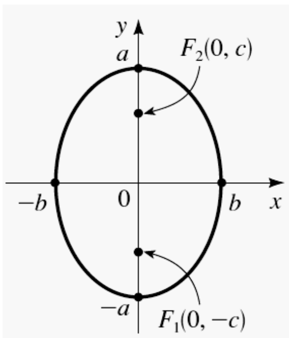
The coordinates of the vertices are $(a;0)$ and $(-a;0)$, and the coordinates of the foci are $(c;0)$ and $(-c;0)$, where $c^2 = a^2 - b^2$.



2. Major Axis on the *y*-axis. The standard form of the equation of an ellipse with the center at the origin and major axis on the y-axis is given by

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b.$$

The coordinates of the vertices are $(0;a)$ and $(0;-a)$, and the coordinates of the foci are $(0;c)$ and $(0;-c)$, where $c^2 = a^2 - b^2$.



Remark. By looking at the standard form of the equation of an ellipse and noting that $a > b$, observe that the orientation of the major axis is determined by the larger denominator. When the x^2 term has larger denominator, the major axis is on the x-axis. When the y^2 term has the larger denominator, the major axis is on the y-axis.

Example 1. Find the vertices and foci of the ellipse given by the equation $\frac{x^2}{25}+\frac{y^2}{49}=1$. Sketch the graph.

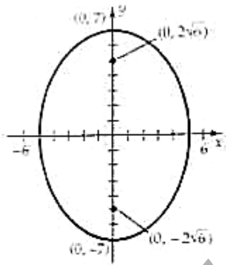
Solution. Because the y^2 term has the larger denominator, the major axis is on the *y*-axis.

$$a^2=49 \quad b^2=25 \quad c^2=a^2-b^2$$

$$a=7 \quad b=5 \quad c^2=49-25=24$$

$$c=\sqrt{24}=2\sqrt{6}$$

The vertices are $(0;7)$ and $(0;-7)$. The foci are $(0;2\sqrt{6})$ and $(0;-2\sqrt{6})$.



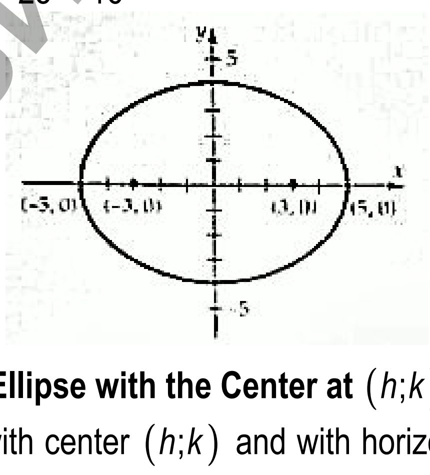
Example 2. Find the equation in standard form of the ellipse with foci $(3;0)$ and $(-3;0)$ and major axis of length 10. Sketch the graph.

Solution. Because the foci are on the major axis, the major axis is on the x-axis. The length of the major axis is 2*a*. Thus $2a=10$. Solving for a, we have $a=5$.

Because the foci are $(3;0)$ and $(-3;0)$ and the center of the ellipse is the midpoint between the two foci, the distance from the center of the ellipse to a focus is 3. Therefore, $c=3$.

To find b^2, use the equation $c^2=a^2-b^2$: $9=25-b^2 \Rightarrow b^2=16$.

The equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$.



Ellipse with the Center at $(h;k)$

The equation of an ellipse with center $(h;k)$ and with horizontal or vertical major axes can be found by using a translation of coordinates. Given a coordinate system with axes labeled x' and y', the standard form of the equation of an ellipse with center at the origin is

$$\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}=1.$$

Now place the origin of the $x'y'$ -coordinate system at $(h;k)$ in an *xy*-coordinate system.

The relationship between an ordered pair in the $x'y'$ -coordinate system and the *xy*-coordinate system is given by transformation equations $\begin{cases} x' = x - h \\ y' = y - k \end{cases}$. Substitute the expressions for x' and y' into the equation of an ellipse. The equation of the ellipse with center at $(h;k)$ is

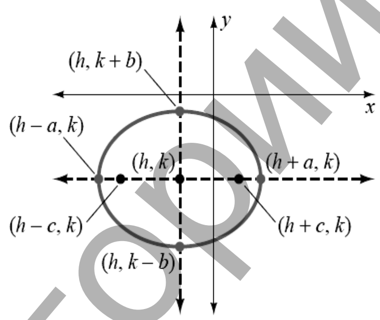
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Standard Form of Ellipse with Center at $(h;k)$

1. Major Axis Parallel to the x -axis. The standard form of the equation of an ellipse with the center at $(h;k)$ and major axis parallel to the *x*-axis is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \ a > b.$$

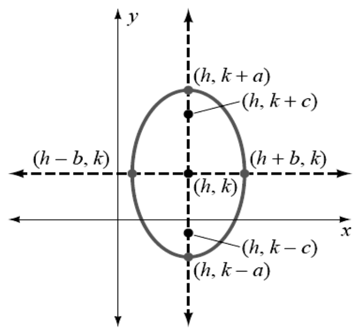
The coordinates of the vertices are $(h+a;k)$ and $(h-a;k)$, where $c^2 = a^2 - b^2$.



2. Major Axis Parallel to the *y*-axis The standard form of the equation of an ellipse with the center at $(h;k)$ and major axis parallel to the *y*-axis is given by

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \ a > b.$$

The coordinates of the vertices are $(h;k+a)$ and $(h;k-a)$, and the coordinates of the foci are $(h;k+c)$ and $(h;k-c)$, where $c^2 = a^2 - b^2$.



Example 3. Find the vertices and foci of the ellipse $4x^2 + 9y^2 - 8x + 36y + 4 = 0$. Sketch the graph.

Solution. Write the equation of the ellipse in standard form by completing the square.

$$4x^2+9y^2-8x+36y+4=0$$

$$4x^2-8x+9y^2+36y=-4 \qquad \text{Rearrange terms.}$$

$$4\left(x^2-2x\right)+9\left(y^2+4y\right)=-4 \qquad \text{Factor.}$$

$$4\left(x^2-2x-1\right)+9\left(y^2+4y+4\right)=-4+4+36 \qquad \text{Complete the square.}$$

$$4(x-1)^2+9(y+2)^2=36 \qquad \text{Factor.}$$

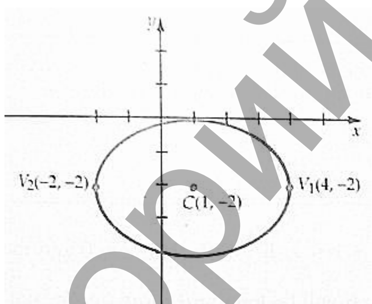
$$\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1 \qquad \text{Divide by 36.}$$

From the equation of the ellipse in standard form, we see that the coordinates of the center of the ellipse are $(1;-2)$. Because the larger denominator is 9, the major axis is parallel to the x-axis and $a^2=9$. Thus $a=3$. The vertices are $(4;-2)$ and $(-2;-2)$.

To find the coordinates of the foci, we find c.

$$c^2=a^2-b^2=9-4=5 \Rightarrow c=\sqrt{5}.$$

The foci are $\left(1+\sqrt{5},-2\right)$ and $\left(1-\sqrt{5},-2\right)$



Example 4. Find the standard form of the equation of the ellipse with center at $(4;-2)$, foci $(4;1)$ and $(4;-5)$, and minor axis of length 10.

*Solution***.** Because the foci are on the major axis, the major axis is parallel to the *y*-axis. The distance from the center of the ellipse to a focus is c. The distance between $(4;-2)$ and $(4;1)$ is 3. Therefore $c=3$.

Recall that the length of the minor axis is $2b$. Thus $2b=10$. Solving for b, we have $b=5$. To find a^2, use the equation $c^2=a^2-b^2$.

$$9=a^2-25 \Rightarrow a^2=34.$$

Thus, the equation is

$$\frac{(x-4)^2}{25}+\frac{(y+2)^2}{34}=1.$$

Eccentricity of an Ellipse

The graph of an ellipse can be very long and thin, or it can be much like a circle. The eccentricity of an ellipse is a measure of its « roundness».

Definition. The **eccentricity** e of an ellipse is the ratio of c to a, where c is the distance from the center to the focus and a is the length of the semimajor axis. That is,

$$e = \frac{c}{a}.$$

Because $c < a$, for an ellipse, $0 < e < 1$. If $c \approx 0$, then $e \approx 0$ and the graph will be almost like a circle.

If $c \approx a$, then $e \approx 1$ and the graph will be long and thin.

Example 5. Find the eccentricity of the ellipse $8x^2 + 9y^2 = 18$.

Solution. First, write the equation of the ellipse in standard form. Divide each side of the equation by 18.

$$\frac{8x^2}{18} + \frac{9y^2}{18} = 1 \Rightarrow \frac{4x^2}{9} + \frac{y^2}{2} = 1 \Rightarrow \frac{x^2}{9/4} + \frac{y^2}{2} = 1.$$

The last step is necessary because the standard form of the equation has coefficients of 1 in the numerator. Thus we have $a^2 = \frac{9}{4}$, $a = \frac{3}{2}$.

Use the equation $c^2 = a^2 - b^2$ to find c: $c^2 = \frac{9}{4} - 2 = \frac{1}{4}$, $c = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

Now we can find the eccentricity: $e = \frac{c}{a} = \frac{1/2}{3/2} = \frac{1}{3}$. The eccentricity is $\frac{1}{3}$.

Exercise Set 9

In Exercises 1 to 20, find the vertices and foci of the ellipse given by each equation. Sketch the graph.

1. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
2. $\frac{x^2}{49} + \frac{y^2}{36} = 1$
3. $\frac{x^2}{9} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{64} + \frac{y^2}{25} = 1$
5. $3x^2 + 4y^2 = 12$
6. $5x^2 + 4y^2 = 20$
7. $25x^2 + 16y^2 = 400$
8. $25x^2 + 12y^2 = 300$
9. $64x^2 + 25y^2 = 400$
10. $9x^2 + 64y^2 = 144$
11. $4x^2 + y^2 - 24x - 8y + 48 = 0$
12. $x^2 + 9y^2 + 6x - 36y + 36 = 0$
13. $5x^2 + 9y^2 - 20x + 54y + 56 = 0$
14. $9x^2 + 16y^2 + 36x - 16y - 104 = 0$
15. $16x^2 + 9y^2 - 64x - 80 = 0$
16. $16x^2 + 9y^2 + 36y - 108 = 0$
17. $25x^2 + 16y^2 + 50x - 32y - 359 = 0$
18. $16x^2 + 9y^2 - 64x - 54y + 1 = 0$
19. $8x^2 + 25y^2 - 48x + 50y + 47 = 0$
20. $4x^2 + 9y^2 + 24x + 18y + 44 = 0$

In Exercises 21 to 32, find the equation in standard form of each ellipse, given the information provided.

21. Center $(0;0)$, major axis of length 10, foci at $(4;0)$ and $(-4;0)$.
22. Center $(0;0)$, minor axis of length 6, foci at $(0;4)$ and $(0;-4)$.
23. Vertices $(6;0)$, $(-6;0)$; ellipse passes through $(0;4)$, and $(0;-4)$.

24. Vertices $(5;0)$, $(-5;0)$, ellipse passes through $(0;7)$, and $(0;-7)$.

25. Major axis of length 12 on the x-axis, center at $(0;0)$, and passing through $(2;-3)$.

26. Minor axis of length 8, center at $(0;0)$, and passing through (-2, 2).

27. Center (-2, 4), vertices (-6, 4) and (2, 4), foci (-5, 4) and (1, 4).

28. Center (0, 3), minor axis of length 4, foci (0, 0) and (0, 6).

29. Center $(2;4)$, major axis parallel to the y-axis and of length 10, the ellipse passes through the point (3, 3).

30. Center (-4, 1), minor axis parallel to the y-axis of length 8, and the ellipse passes through the point (0, 4).

31. Vertices (5, 6) and (5, -4), foci (5, 4) and (5, -2).

32. Vertices (-7, -1) and (5, -1), foci (-5, -1) and (3, -1).

In Exercises 33 to 40, use the eccentricity of the ellipse to find the equation in standard form of each of the following ellipse.

33. Eccentricity 2/5, major axis on the x-axis of length 10, and center at (0, 0).

34. Eccentricity 3/4, foci at (-9, 0).

35. Foci at (0, -4) and (0, 4), eccentricity 2/3.

36. Foci at (0, -3) and (0, 3), eccentricity ¼.

37. Eccentricity 2/5, foci (-1, 3) and (3, 3).

38. Eccentricity 1/4, foci (-2, 4) and (-2, -2).

39. Eccentricity 2/3, major axis of length 24 on the y-axis, center at (0, 0).

40. Eccentricity 3/5, major axis of length 15 on the x-axis, center at (0, 0).

Supplemental Exercises

41. Explain why the graph of the equation $4x^2+9y^2-8x+36y+76=0$ is or is not an ellipse.

42. Explain why the graph of the equation $4x^2+9y-16x-2=0$ is or is not an ellipse. Sketch the graph of this equation.

In Exercises 43 to 46, find the equation in standard form of an ellipse by using the definition of an ellipse.

43. Find the equation of the ellipse with foci at (-3, 0) and (3, 0) that passes through the point $\left(3;\frac{9}{2}\right)$.

44. Find the equation of the ellipse with foci at (0, 4) and (0, -4) that passes through the point $\left(\frac{9}{5};4\right)$.

45. Find the equation of the ellipse with foci at (-1, 2) and (3, 2) that passes through the point $(3;5)$.

46. Find the equation of the ellipse with foci at (-1, 1) and (-1, 7) that passes through the point $\left(\frac{3}{4};1\right)$.

In Exercises 47 and 48, find the latus rectum of the given ellipse. The line segment with endpoints on the ellipse that is perpendicular to the major axis and passes through the focus is the **latus rectum** of the ellipse.

47. Find the length of the latus rectum of the ellipse given by $\frac{(x-1)^2}{9}+\frac{(y+1)^2}{16}=1$.

48. Find the length of the latus rectum of the ellipse given by $9x^2+16y^2-36x+96y+36=0$.

49. Show that for any ellipse, the length of the latus rectum is $\frac{2b^2}{a}$.

50. Use the definition of an ellipse to find the equation of an ellipse with center at (0, 0) and foci (0, c) and (0, -c).

Recall that a parabola has a directrix that is a line perpendicular to the axis of symmetry. An ellipse has two directrixes, both of which are perpendicular to the major axis and outside the ellipse. For an ellipse with center at the origin and whose major axis is the x-axis, the equations of the directrixes are $x=\frac{a^2}{c}$ and $x=-\frac{a^2}{c}$.

51. Find the directrix of the ellipse in Exercise 3.

52. Find the diretctrix of the ellipse in Exercise 4.

53. Let $P(x;y)$ be a point on the ellipse $\frac{x^2}{12}+\frac{y^2}{8}=1$. Show that the distance from the point P to the focus (2, 0) divided by the distance from the point P to the directrix $x=6$ equals the eccentricity.

54. Let $P(x;y)$ be a point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. Show that the distance from the point P to the focus (3, 0) divided by the distance from the point to the directrix $x=\frac{25}{3}$ equals the eccentricity.

55. Generalize the result of Exercises 53 and 54. That is, show that if $P(x;y))$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $F(c;0)$ is the focus and $x=\frac{a^2}{c}$ is the directrix, then the following equation is true: $e=\frac{d(P,F)}{d(P,D)}$.

3.3 Hyperbolas

Definition of a Hyperbola. A hyperbola is the set of all points in the plane, the difference of whose distances from two fixed points (foci) is a positive constant.

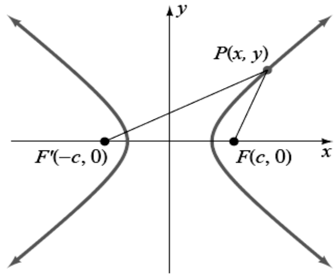
Remark. This definition differs from that of an ellipse in that the ellipse was defined in terms of the sum of the distances, whereas the hyperbola is defined in terms of the difference of two distances.

Hyperbolas with Center at $(0;0)$

The **transverse axis** is the line segment joining the intercepts through the foci of a hyperbola. The midpoint of the transverse axis is called the **center** of the hyperbola. The **conjugate** axis passes through the center of the hyperbola and is perpendicular to the transverse axis.

The length of the transverse axis is customarily denoted $2a$, and the distance between the two foci is denoted $2c$. The length of the conjugate axis is denoted $2b$.

The **vertices** of a hyperbola are the points where the hyperbola intersects the transverse axis.



To determine the positive constant stated in the definition of hyperbola, consider the point $V(a;0)$, which is one vertex of a hyperbola, and the points $F_1(c,0)$ and $F_2(-c,0)$ which are the foci of the hyperbola. The difference of the distance from V to F_1, $c-a$ and the distance from $V(a;0)$ to $F_2(-c,0)$, $c+a$, must be a constant. By subtracting these distances, we find

$$\left|(c-a)-(c+a)\right| = |-2a| = 2a.$$

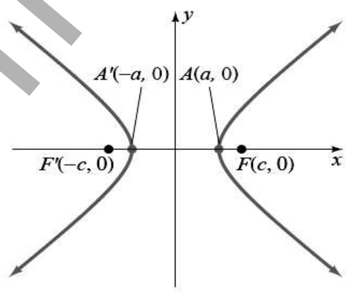
Thus the constant is $2a$ and is the length of the transverse axis. The absolute value was used to ensure that the distance is a positive number.

Standard Forms of the Equation of a Hyperbola with Center at the Origin

1. Transverse axis on the x-axis. The standard form of the equation of a hyperbola with the center at the origin and transverse axis on the x-axis is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The coordinates of the vertices are $(a;0)$ and $(-a;0)$ and the coordinates of the foci are $(c;0)$ and $(-c;0)$, where $c^2 = a^2 + b^2$.



2. Transverse axis on the y-axis. The standard form of the equation of a hyperbola with the center at the origin and transverse axis on the y-axis is given by

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

The coordinates of the vertices are $(0;a)$ and $(0;-a)$ and the coordinates of the foci are $(0;c)$ and $(0;-c)$, where $c^2 = a^2 + b^2$.

Remark. By looking at the equations, note that it is possible to determine the transverse axis by finding which term in the equation is positive. If the x^2 term is positive, then the transverse axis is on the x-axis. When the y^2 term is positive, the transverse axis is on the y-axis.

Consider the hyperbola given by the equation $\frac{x^2}{16}-\frac{y^2}{9}=1$. Because the x^2 term is positive, the transverse axis is on the x-axis, $a^2=16$, thus $a=4$. The vertices are $(4;0)$ and $(-4;0)$. To find the foci, we determine c: $c^2=a^2+b^2=16+9=25 \Rightarrow c=\sqrt{25}=5$. The foci are $(5;0)$ and $(-5;0)$.

The asymptotes of the hyperbola are a useful guide to sketching the graph of the hyperbola. Each hyperbola has two asymptotes that pass through the center of the hyperbola.

Asymptotes of a Hyperbola with Center at the Origin

The **asymptotes** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by the equations $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$. The asymptotes of the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are given by the equations $y=\frac{a}{b}x$ and $y=-\frac{a}{b}x$.

We can outline a proof for the equations of the asymptotes by using the equation of a hyperbola in standard form.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$y^2=b^2\left(\frac{x^2}{a^2}-1\right) \qquad \text{Solve for } y^2.$$

$$=\frac{b^2}{a^2}\left(x^2-a^2\right) \qquad \text{Factor out } \frac{1}{a^2}.$$

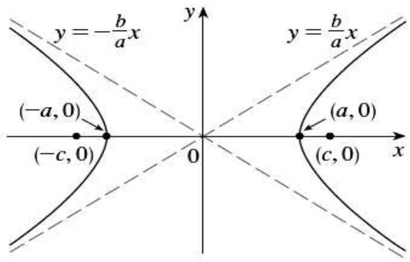
$$=\frac{b^2}{a^2}x^2\left(1-\frac{a^2}{x^2}\right) \qquad \text{Factor out } x^2.$$

$$y=\pm\frac{b}{a}x\sqrt{1-\frac{a^2}{x^2}} \qquad \text{Take the square root of each side.}$$

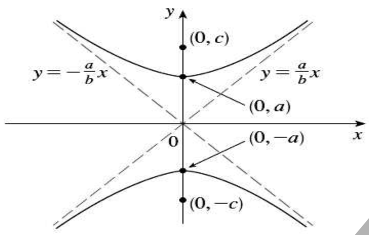
As $|x|$ becomes larger and larger, $1-\frac{a^2}{x^2}$ approaches 1. For large values of $|x|$, $y\approx\pm\frac{b}{a}x$ and thus $y=\pm\frac{b}{a}x$ are asymptotes for hyperbolas with the transverse axis on the y-axis.

Remark. One method for remembering the equations of the asymptotes is to write the equation of a hyperbola in standard form but replace 1 by 0 and then solve for y.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \text{ thus } y^2=\frac{b^2}{a^2}x^2 \text{ or } y=\pm\frac{b}{a}x$$



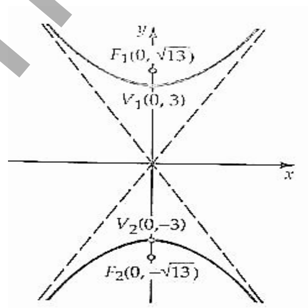
$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=0 \quad \text{thus } y^2=\frac{a^2}{b^2}x^2 \text{ or } y=\pm\frac{a}{b}x$$



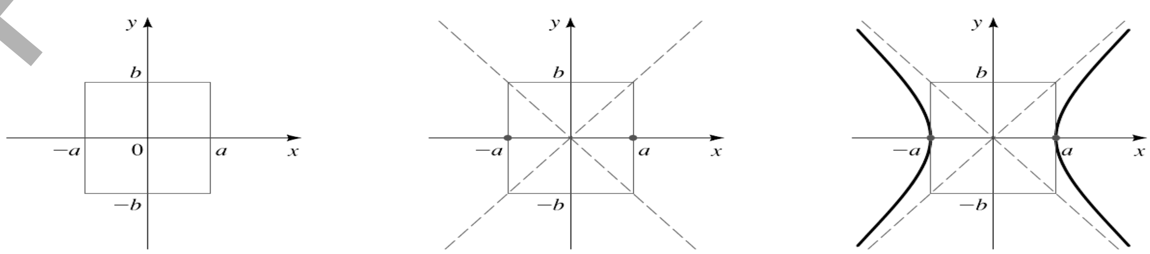
Example 1. Find the foci, vertices, and asymptotes of the hyperbola given by the equation $\frac{y^2}{9}-\frac{x^2}{4}=1$. Sketch the graph.

Solution. Because the y^2 term is positive, the transverse axis is the y-axis. We know $a^2=9$; thus $a=3$. The vertices are $V_1(0,3)$ and $V_2(0,-3)$. $c^2=a^2+b^2=9+4\Rightarrow c=\sqrt{13}$. The foci are $F_1(0,\sqrt{13})$ and $F_2(0,-\sqrt{13})$.

Because $a=3$. and $b=2$ ($b^2=4$), the equations of the asymptotes are $y=\frac{3}{2}x$ and $y=-\frac{3}{2}x$.



To sketch the graph, we draw a rectangle with its center at the origin that has dimensions equal to the lengths of the transverse and conjugate axes. The asymptotes are extensions of the diagonals of the rectangle.



(a) Central box (b) Asymptotes (c) Hyperbola

Hyperbola with the Center at the Point $(h;k)$

Using a translation of coordinates similar to that used for ellipse, we can write the equation of a hyperbola with its center at the point $(h;k)$. Given coordinates axes labeled x' and y', an equation of a hyperbola with center at the origin is

$$\frac{(x')^2}{a^2}-\frac{(y')^2}{b^2}=1.$$

Now place the origin of this coordinate system at the point $(h;k)$ of the xy-coordinate system. The relationship between an ordered pair in the $x'y'$-coordinate system and the xy-coordinate system is given by the transformation equations $\begin{cases} x'=x-h \\ y'=y-k \end{cases}$. Substitute the expressions for x' and y' into the equation of a hyperbola. The equation of the hyperbola with center at $(h;k)$ is

$$\frac{(x-k)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$

Standard Form of Hyperbolas with Center at $(h;k)$

1. Transverse Axis Parallel to the x-axis. The standard form of the equation of a hyperbola with center $(h;k)$ and transverse axis parallel to x-axis is given by

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$

The coordinates of the vertices are $V_1(h+a,k)$ and $V_2(h-a,k)$. The coordinates of the foci are $F_1(h+c,k)$ and $F_2(h-c,k)$. The equations of the asymptotes are $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$.

2. Transverse Axis Parallel to the y-axis. The standard form of the equation of a hyperbola with center $(h;k)$ and transverse axis parallel to the y-axis is given by

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

The coordinates of the vertices are $V_1(h,k+a)$ and $V_2(h,k-a)$. The coordinates of the foci are $F_1(h,k+a)$ and $F_2(h,k-c)$. The equations of the asymptotes are $y-k=\frac{a}{b}(x-h)$ and $y-k=-\frac{a}{b}(x-h)$.

Example 2. Find the vertices, foci, and asymptotes of the hyperbola given by the equation $4x^2-9y^2-16x+54y-29=0$. Sketch the graph.

Solution. Write the equation of the hyperbola in standard form by completing the square.

$$4x^2 - 9y^2 - 16x + 54y - 29 = 0$$

$$4x^2 - 16x - 9y^2 + 54y = 29 \qquad \text{Rearrange terms.}$$

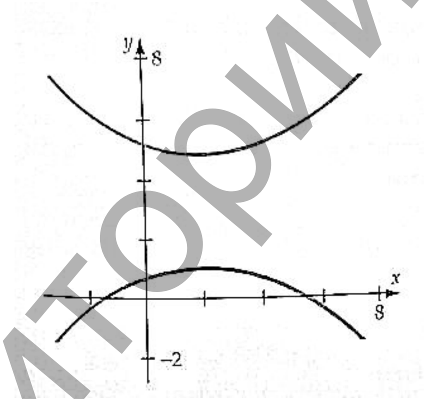
$$4(x^2 - 4x) - 9\left(y^2 - 6y\right) = 29 \qquad \text{Factor.}$$

$$4\left(x^2 - 4x + 4\right) - 9\left(y^2 - 6y + 9\right) = 29 + 16 - 81 \qquad \text{Complete the square.}$$

$$4(x-2)^2 - 9(y-3)^2 = -36 \qquad \text{Factor.}$$

$$\frac{(y-3)^2}{4} - \frac{(x-2)^2}{9} = 1 \qquad \text{Divide by } -36.$$

The coordinates of the center are (2, 3). Because the term containing $(y-3)^2$ is positive, the transverse axis is parallel to the y-axis. We know $a^2 = 4$; thus $a = 2$. The vertices are (2, 5) and (2, 1). $c^2 = a^2 + b^2 = 4 + 9 \Rightarrow c = \sqrt{13}$. The foci are $\left(2, 3+\sqrt{13}\right)$ and $\left(2, 3-\sqrt{13}\right)$. We know $b^2 = 9$; thus $b = 3$. The equations of the asymptotes are $y = \frac{2}{3}x + \frac{5}{3}$ and $y = -\frac{2}{3}x + \frac{13}{3}$.



Eccentricity of a Hyperbola

The graph of a hyperbola can be very wide or very narrow. The **eccentricity** of a hyperbola is a measure of its “wideness”.

Definition. The **eccentricity** e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is the length of the semi-transverse axis: $e = \frac{c}{a}$.

For a hyperbola, $c > a$ and therefore $e > 1$. As the eccentricity of the hyperbola increases, the graph becomes wider and wider.

Example 3. Find the standard form of the equation of the hyperbola that has eccentricity $\frac{3}{2}$, center at the origin, and a focus (6, 0).

Solution. Because the focus is located at (6, 0) and the center is at the origin, $c = 6$. An extension of the transverse axis contains the foci, and thus the transverse axis is on the x-axis.

$$e=\frac{3}{2}=\frac{c}{a}$$

$$\frac{3}{2}=\frac{6}{a}$$ Substitute the value for c.

$$a=4$$ Solve for a.

To find b, use the equation $c^2=a^2+b^2$ and the values for c and : $c^2=a^2+b^2 \Rightarrow 36=16+b^2 \Rightarrow b^2=20$. The equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{20}=1$.

Exercise Set 10

In Exercises 1 to 24, find the center, vertices, foci, and asymptotes for the hyperbola given by each equation. Sketch the graph.

1. $\frac{x^2}{16}-\frac{y^2}{25}=1$
2. $\frac{x^2}{16}-\frac{y^2}{9}=1$
3. $\frac{y^2}{4}-\frac{x^2}{25}=1$
4. $\frac{y^2}{25}-\frac{x^2}{36}=1$
5. $\frac{(x-3)^2}{16}-\frac{(y+4)^2}{9}=1$
6. $\frac{(x+3)^2}{35}-\frac{y^2}{4}=1$
7. $\frac{(y+2)^2}{4}-\frac{(x-1)^2}{16}=1$
8. $\frac{(y-2)^2}{36}-\frac{(x+1)^2}{49}=1$
9. $x^2-y^2=9$
10. $4x^2-y^2=16$
11. $16y^2-9x^2=144$
12. $9y^2-25x^2=225$
13. $9y^2-36x^2=4$
14. $16x^2-25y^2=9$
15. $x^2-y^2-6x+8y-3=0$
16. $4x^2-25y^2+16x+50y-109=0$
17. $9x^2-4y^2+36x-8y+68=0$
18. $16x^2-9y^2-32x-54y+79=0$
19. $4x^2-y^2+32x+6y+39=0$
20. $x^2-16y^2+8x-64y+16=0$
21. $9x^2-16y^2-36x-64y+116=0$
22. $2x^2-9y^2+12x-18y+18=0$
23. $4x^2-9y^2+8x-18y-6=0$
24. $2x^2-9y^2-8x+36y-46=0$

In Exercises 25 to 38, find the equation in standard form of the hyperbola satisfying the stated conditions.

25. Vertices $(3,0)$ and $(-3,0)$, foci $(4,0)$ and $(-4,0)$.
26. Vertices $(0,2)$ and $(0,-2)$ foci $(0,3)$ and $(0,-3)$.
27. Foci $(0,5)$ and $(0,-5)$ asymptotes $y=2x$.
28. Foci $(4,0)$ and $(-4,0)$, asymptotes $y=x$ and $y=-x$.
29. Vertices $(0,3)$ and $(0,-3)$ and passing through (2, 4).
30. Vertices (5, 0) and (-5, 0) and passing through (-1, 3).
31. Asymptotes $y=\frac{1}{2}x$ and $y=-\frac{1}{2}x$, vertices (0, 4) and (0, -4).

32. Asymptotes $y=\frac{2}{3}x$ and $y=-\frac{2}{3}x$, vertices (6, 0) and (-6, 0).

33. Vertices (6, 3) and (2, 3), foci (7, 3) and (1, 3).

34. Vertices (-1, 5) and (-1, -1), foci (-1, 7) and (-1, -3).

35. Foci (1, -2) and (7, -2), slope of an asymptote $\frac{5}{4}$.

36. Foci (-3, -6) and (-3, -2), slope of an asymptote 1.

37. Passing through (9, 4), slope of an asymptote $\frac{1}{2}$, center (7, 2), transverse axis parallel to the y-axis

38. Passing through (6, 1), slope of an asymptote 2, center (3, 3), transverse axis parallel to the x-axis

In Exercises 39 to 44, use the eccentricity to find the equation in standard form of a hyperbola.

39. Vertices (1, 6) and (1, 8), eccentricity 2.

40. Vertices (2, 3) and (-2, 3), eccentricity $\frac{5}{2}$.

41. Eccentricity 2, foci (4, 0) and (-4, 0).

42. Eccentricity 4/3, foci (0, 6) and (0, -6).

43. Center (4, 1), conjugate axis length 4, eccentricity $\frac{4}{3}$.

44. Center (-3, -3), conjugate axis length 6, eccentricity 2.

In Exercises 45 to 52, identify the graph of each equation as a parabola, ellipse, or hyperbola. Sketch the graph.

45. $4x^2+9y^2-16x-36y+16=0$.

46. $2x^2+3y-8x+2=0$.

47. $5x-4y^2+24y-11=0$.

48. $9x^2-25y^2-18x+50y=0$.

49. $x^2+2y-8x=0$.

50. $9x^2+16y^2+36x-64y-44=0$.

51. $25x^2+9y^2-50x-72y-56=0$.

52. $(x-3)^2+(y-4)^2=(x+1)^2$.

Supplemental Exercises

In Exercises 53 to 56, use the definition for a hyperbola to find the equation of the hyperbola in standard form.

53. Foci (2, 0) and (-2, 0) and passes through the point (2,3).

54. Foci (0, 3) and (0, -3) and passes through the point $\left(\frac{5}{2};3\right)$.

55. Foci (0, 4) and (0, -4) and passes through the point $\left(\frac{7}{3};4\right)$.

56. Foci (5, 0) and (-5,0) and passes through the point $\left(5;\frac{9}{4}\right)$.

Recall that an ellipse has two directrixes that are lines perpendicular to the line containing the foci. A hyperbola also has two directrixes that are perpendicular to the transverse axis and outside the hyperbola. For a hyperbola with center at the origin and transverse axis on the x-axis, the equations of the directrixes are $x=\frac{a^2}{c}$ and $x=-\frac{a^2}{c}$.

In Exercises 57 to 61, use this information to solve each exercise.

57. Find the directrixes for the hyperbola in Exercise 21.

58. Find the directrixes for the hyperbola in Exercise 22.

59. Let $P(x;y)$ be a point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$. Show that the distance from the point P to the focus (5, 0) divided by the distance from the point P to the directrix $x=\frac{9}{5}$ equals the eccentricity.

60. Let $P(x;y)$ be a point on the hyperbola $\frac{x^2}{7}-\frac{y^2}{9}=1$. Show that the distance from the point P to the focus (4, 0) divided by the distance from the point to the directrix $x=\frac{7}{4}$ equals the eccentricity.

3.4 Review

Conic Sections

A parabola is the set of points in the plane that are equidistant from a fixed line (the directrix) and a fixed point (the focus) not on the directrix.

The equations of a parabola with vertex at $(a;b)$ and axis of symmetry parallel to a coordinate axis are given by

$$(x-a)^2=4p(y-b) \text{ Focus: } (a,b+p), \text{ directrix: } y=b-p$$

$$(y-b)^2=4p(x-a) \text{ Focus: } (a+p,b), \text{ directrix: } x=a-p.$$

Ellipse

An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points (foci) is a positive constant.

The equations of an ellipse with center at $(h;k)$ and major axis parallel to the coordinate axes are given by

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \text{ Foci: } (h\pm c,k), \text{ vertices: } (h\pm a,k)$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \text{ Foci: } (h,k\pm c), \text{ vertices: } (h,k\pm a)$$

For each equation, $a>b$ and $c^2=a^2-b^2$.

The eccentricity e of an ellipse is given by $e=\frac{c}{a}$.

Hyperbolas

A hyperbola is the set of all points in the plane, the difference of whose distances from two fixed points (foci) is a positive constant.

The equations of a hyperbola with center at $(h;k)$ and transverse axis parallel to a coordinate axis are given by

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ Foci: } (h\pm c,k)\text{, vertices: } (h\pm a,k)$$

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \text{ Foci: } (h,k\pm c)\text{, vertices: } (h,k\pm a).$$

For each equation, $c^2=a^2+b^2$.

The eccentricity e of a hyperbola is given by $e=\dfrac{c}{a}$.

Challenge Exercises

In Exercises 1 to 10, answer true or false. If the answer is false, give an example.

1. The graph of a parabola is the same shape as one branch of a hyperbola.
2. For the two axes of an ellipse, the major axis and the minor axis, the major axis is always the longer axis.
3. For the two axes of a hyperbola, the transverse axis and the conjugate axis, the transverse axis is always the longer axis.
4. If two ellipses have the same foci, they will have the same graph.
5. A hyperbola is similar to a parabola in that both curves have asymptotes.
6. If a hyperbola with center at the origin and a parabola with vertex at the origin have the same focus, (0, c), the two graphs will always intersect.
7. The graphs of all the conic sections are not the graphs of a function.
8. If F_1 and F_2 are the two foci of an ellipse and P is a point on the ellipse, then $d(P,F_1)+d(P,F_2)=2a$ where a is the length of the semimajor axis of the ellipse.
9. The eccentricity of a hyperbola is always greater than 1.

Review Exercises

In Exercises 1 to 12, find the foci and vertices of each conic. If the conic is a hyperbola, find the asymptotes. Sketch the graph.

1. $x^2-y^2=4$.	2. $y^2=16x$.
3. $x^2+4y^2-6x+8y-3=0$.	4. $3x^2-4y^2+12x-24y-36=0$.
5. $3x-4y^2+8y+2=0$.	6. $3x+2y^2-4y-7=0$.
7. $9x^2+4y^2+36x-8y+4=0$.	8. $11x^2-25y^2-44x-50y-256=0$.
9. $4x^2-9y^2-8x+12y-144=0$.	10. $9x^2+16y^2+36x-16y-104=0$.
11. $4x^2+28x+32y+81=0$.	12. $x^2-6x-9y+27=0$.

In Exercises 13 to 20, find the equation in standard form of the conic that satisfies the given conditions.

13. Ellipse with vertices at (7, 3) and (-3, 3); length of minor axis 8.

14. Hyperbola with vertices at (4, 1) and (-2, 1); eccentricity $\frac{4}{3}$.

15. Hyperbola with foci (-5, 2) and (1, 2); length of transverse axis 4.

16. Parabola with focus (2, -3) and directrix $x = 6$.

17. Parabola with vertex (0, -2) and passing through the point (3, 4).

18. Ellipse with eccentricity 2/3 and foci (-4, -1) and (0, -1).

19. Hyperbola with vertices $(\pm 6;0)$ and asymptotes whose equations are $y = \pm\frac{1}{9}x$.

20. Parabola passing through the points (1, 0), (2, 1), and (0, 1) with axis of symmetry parallel to the y-axis.

21. Find the equation of the parabola traced by a point $P(x;y)$ that moves so that the distance between $P(x;y)$ and the line $x = 2$ equals the distance between $P(x;y)$ and the point (-2, 3).

22. Find the equation of the parabola traced by a point $P(x;y)$ that moves so that the distance between $P(x;y)$ and the line $y = 1$ equals the distance between $P(x;y)$ and the point (-1, 2).

23. Find the equation of the ellipse traced by a point $P(x;y)$ that moves so that the sum of its distances to (-3, 1) and (5, 1) is 10.

24. Find the equation of the ellipse traced by a point $P(x;y)$ that moves so that the sum of its distances to (3,5) and (3, -1) is 8.

25. Find the equation of the hyperbola traced by a point $P(x;y)$ that moves so that the difference of its distances to (-1, -1) and (-1, 7) is 6.

26. Find the equation of the hyperbola traced by a point $P(x;y)$ that moves so that the difference of its distances to (4, -5) and (4, 7) is 8.

27. Assuming the orbit of the earth around the sun is a perfect ellipse with the sun at a focus, 92.9 million miles as the length of the semimajor axis, and eccentricity of the orbit 0.017, find the greatest distance between the earth and the sun.

28. Find the equation of the ellipse that passes through the origin and has foci (-1, 1) and $(1;1)$.

29. Find the equation of the hyperbola that passes through (0, 0) and has foci (-1, 0) and $(9;0)$.

30. A ball is released at an angle of 60° with the ground and at a height of 4 feet with a speed of 66 feet per second. Assuming no air resistance, the path of the ball is a parabola whose equation is $y = -0{,}17x^2 + 1{,}732x + 4$. Find the distance the ball travels and the maximum height attained by the ball. (Hint: To find the distance the ball travels let $y = 0$ and solve for x. To find the maximum height of the ball, find the vertex of the parabola.)

Literature

1 Aufmann R. and ets. College algebra. 1990. pp. 553.
2 Stein Sh. Calculus and analytic geometry. 1987. pp.1061.
3 Stewart J., Redlin L., Watson S. Precalculus Mathematics for Calculus. 2009. pp. 1062.

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